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REGULAR ARTICLE

Complete convergence of weighted sums under negative dependence

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Abstract In this paper, we study the complete convergence of weighted sums. In fact, we extend the result of Amini and Bozorgnia (J Appl Math Stoch Anal 16(2):121–126, 2003) on unweighted average to a weighted average under mild conditions.

Keywords Negatively dependent · Complete convergence · Weighted sums

Mathematics Subject Classification (2000) 60F15

1 Introduction

In many statistical applications, we may suppose that the variables be independent. But in real studies, this assumption is not true. So, it is of interest of many statisticians to extend this condition to the dependent cases. One of these dependent structures is negative dependence. Informally speaking, random variables are said to be negatively dependent, if they have the following property: if any one subset of the variables is "high" then other (disjoint) subsets of the variables are "low". Such variables arise frequently in the analysis of algorithms, for which a stream of random bits influences either the input or the execution of the algorithm [see for more information and

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examples, Adler et al. (1995), Dietzfelbinger and Meyer auf der Heide (1993), and Panconesi and Srinivasan (1992)].

Some convergence theorems for weighted sums of independent sequence of generalized Gaussian random variables has been studied by Chow (1966). Studying complete convergence of estimators is stochastically more convincing than convergence in probability and even convergence in distribution. So in this paper, we study this type convergence of the random variables. The concept of complete convergence was first introduced by Hsu and Robbins (1947). A sequence of random variables $\{X_n, n \geq 1\}$ is said converge completely to θ if

$$\sum_{n=1}^{\infty} P[|X_n - \theta| > \varepsilon] < \infty, \quad \text{for every} \quad \varepsilon > 0.$$

Hsu and Robbins (1947) proved that the sequence of arithmetic means of independent and identically distributed (iid) random variables converges completely to the expected value if variance of the summands is finite. Their result may be formulated as follows.

Theorem 1.1 (Hsu and Robbins 1947) *If $\{X, X_n, n \geq 1\}$ are iid random variables, then $\frac{1}{n}\sum_{k=1}^{n} X_k$ converges to 0 completely if only if $E[X] = 0$ and $E[X^2] < \infty$.*

This result has been generalized and extended in several directions [see Rohatgi (1971), Hu et al. (1989) and Gut (1992) among others]. Recently, the strong convergence of weighted sums for the case of independent random variables has been discussed by Wu (1999). Hu and Volodin (2000) and Hu et al. (2003) proved the complete convergence theorem for arrays of independent random variables and Amini and Bozorgnia (2003) studied complete convergence of the sequence $\frac{1}{n}\sum_{k=1}^{n} X_k$, via exponential bounds in the case of negatively dependent and identically random variables. In this paper, we study complete convergence of weighted sums $T_n = \sum_{k=1}^{n} a_{nk} X_k$ where $\{X_n, n \geq 1\}$ is a sequence of negatively dependent and identically distributed random variables and $a_{nk}, n \geq 1, k \geq 1$ is an array of real numbers where $a_{nk} = 0$ if $k > n$, $|a_{nk}| \leq C A_n$ for $A_n = \sum_{k=1}^{\infty} a_{nk}^2$ and some $0 < C < \infty$. The material in this note is closely related to what proposed by Stout (1974) and Chow (1966).

To prove the main result we need to the following definition and lemmas.

Definition 1.1 The random variables $X_1, \ldots, X_n$ are said to be negatively dependent (ND) if we have

$$P\left[\bigcap_{j=1}^{n} (X_j \leq x_j)\right] \leq \prod_{j=1}^{n} P[X_j \leq x_j],$$

and

$$P\left[\bigcap_{j=1}^{n} (X_j > x_j)\right] \leq \prod_{j=1}^{n} P[X_j > x_j],$$

for all $x_1, \dots, x_n \in R$. An infinite sequence $\{X_n, n \geq 1\}$ is said to be ND if every finite subset $X_{i_1}, \dots, X_{i_n}$ is ND.

Lemma 1.1 (Bozorgnia et al. 1996) *Let $\{X_n, n \geq 1\}$ be a sequence of ND random variables and $\{f_n, n \geq 1\}$ be a sequence of Borel functions all of which are monotone increasing (or all are monotone decreasing). Then $\{f_n(X_n), n \geq 1\}$ is a sequence of ND random variables.*

Lemma 1.2 (Bozorgnia et al. 1996) *Let $X_1, \dots, X_n$ be a finite sequence of ND random variables and $t_1, \dots, t_n$ be all nonnegative (or nonpositive) then*

$$E\left[e^{\sum_{i=1}^{n} t_i X_i}\right] \leq \prod_{i=1}^{n} E\left[e^{t_i X_i}\right].$$

2 Strong convergence of weighted sums

Let $\{X_n, n \geq 1\}$ be a sequence of ND random variables and $a_{nk}, n \geq 1, k \geq 1$ be an array of real numbers and $A_n = \sum_{k=1}^{\infty} a_{nk}^2$. The following theorem is a extension of Hsu and Rabbin's theorem.

Theorem 2.1 *Let $\{X_n, n \geq 1\}$ be a sequence of ND and identically distributed random variables with $EX_1 = 0$. Let $a_{nk} = 0$ if $k > n$, and $|a_{nk}| \leq CA_n$ for some $0 < C < \infty$, $n, k = 1, 2, \dots$. If for some $0 < \alpha \leq 1$, $A_n \leq Cn^{-\alpha}$ and $E|X_1|^{2/\alpha} \leq C$, then*

$$\sum_{n=1}^{\infty} P[|T_n| \geq \varepsilon] < \infty \tag{1}$$

for every $\varepsilon > 0$, where $T_n = \sum_{k=1}^{n} a_{nk} X_k$.

Proof We can assume that $C \geq 1$ and $A_n > 0$ for each n. For $0 < \beta < \alpha$ and $N = 2, 3, \dots$, we define
 if $a_{nk} \geq 0$

$$X_k^{'} = X_k I_{[X_k \leq n^\beta]},$$
$$Y_k = X_k I_{[X_k \leq n^\beta]} + n^\beta I_{[X_k > n^\beta]},$$
$$X_k^{''} = X_k I_{[X_k \geq \varepsilon n^\alpha/(NC^2)]},$$

and if $a_{nk} < 0$

$$X_k^{'} = X_k I_{[X_k \geq -n^\beta]},$$
$$Y_k = X_k I_{[X_k \geq -n^\beta]} - n^\beta I_{[X_k < -n^\beta]},$$
$$X_k^{''} = X_k I_{[X_k \leq -\varepsilon n^\alpha/(NC^2)]}.$$

And put

$$
\begin{aligned}
X_k^{'''} &= X_k - X_k^{'} - X_k^{''},\\
T_n^{'} &= \sum_{k=1}^{n} a_{nk} X_k^{'}, \quad T_n^{''} = \sum_{k=1}^{n} a_{nk} X_k^{''}, \quad T_n^{'''} = \sum_{k=1}^{n} a_{nk} X_k^{'''},\\
U_n &= \sum_{k=1}^{n} a_{nk} Y_k, \quad U_n^{(1)} = \sum_{k=1}^{n} a_{nk} Y_k I_{[B^c]}, \quad U_n^{(2)} = \sum_{k=1}^{n} a_{nk} Y_k I_{[B]},
\end{aligned}
$$

where $B = \{a_{nk} \geq 0\}$ and I_A represents the characteristic function of the set A. If a random variable $X \leq 1$ a.e., then obviously $Ee^X \leq e^{EX+EX^2}$. Let $0 < t < C^{-1}n^{-\beta}$. Then $ta_{nk}Y_k/A_n \leq 1$ and $E(a_{nk}Y_k) \leq 0$. Hence

$$
\begin{aligned}
E\exp\{ta_{nk}X_k^{'}/A_n\} &\leq E\exp\{ta_{nk}Y_k/A_n\}\\
&\leq \exp\{t^2a_{nk}^2A_n^{-2}EY_k^2\}\\
&\leq \exp\{t^2a_{nk}^2A_n^{-2}EX_k^2\}\\
&\leq \exp\{Ct^2a_{nk}^2A_n^{-2}\}.
\end{aligned}
$$

Since $\{X_n, n \geq 1\}$ is a sequence of ND random variables, by Cauchy Schwartz's inequality, Lemmas 1.1 and 1.2 we have

$$
\begin{aligned}
E\exp\{tT_n^{'}/A_n\} &\leq E\exp\{tU_n/A_n\}\\
&\leq [E\exp\{2tU_n^{(1)}/A_n\}.E\exp\{2tU_n^{(2)}/A_n\}]^{1/2}\\
&\leq \exp\{t^2K/A_n\},
\end{aligned}
$$

where K = 2C. Therefore

$$
\begin{aligned}
P[T_n^{'} \geq \varepsilon] &= P[tT_n^{'}A_n^{-1} \geq t\varepsilon A_n^{-1}]\\
&\leq \exp\{-t(\varepsilon - tK)/A_n\}.
\end{aligned}
$$

If put $t = n^{-\beta}K^{-1}$ for sufficiently large n, we have

$$
P[T_n^{'} \geq \varepsilon] \leq \exp\{-2\varepsilon n^{\alpha-\beta}/K^2\}.
$$

Since $0 < \beta < \alpha$,

$$
\sum_{n=1}^{\infty} P[T_n^{'} \geq \varepsilon] < \infty. \tag{2}
$$

On the other hand, $\{X_n, n \geq 1\}$ is a sequence of identically distributed random variables. So, we have

$$\begin{aligned} P[T_n^{''} \geq \varepsilon] &\leq nP[|X_1| \geq \varepsilon n^{\alpha} N^{-1} C^{-2}] \\ &= nP[(NC^2|X_1|/\varepsilon)^{\alpha^{-1}} \geq n]. \end{aligned}$$

Since $E|X_1|^{2/\alpha} < \infty$, N and C are fixed constants and using Markov's inequality

$$\sum_{n=1}^{\infty} P[T_n^{''} \geq \varepsilon] < \infty. \tag{3}$$

$a_{nk} X_k^{'''} \leq \varepsilon/N$ and $T_n^{'''} \geq \varepsilon$ implies that there are at least N non-zero $X_i^{'''}$ for $i = 1, 2, \ldots, n$. Hence

$$P[T_n^{'''} \geq \varepsilon] \leq \binom{n}{N} P[|X_1| > n^{\beta}, \ldots, |X_N| > n^{\beta}].$$

Now, we define $A_i = [|X_i| > n^{\beta}]$. Thus for any $i, 2 \leq i \leq N$, there exists nonnegative real number as L_i such that

$$P[A_i | A_1, \ldots, A_{i-1}] = L_i P[A_1].$$

Hence if we put $L = \max_{2 \leq i \leq N} L_i$, we will have

$$\begin{aligned} P\left[A_1 \bigcap A_2 \bigcap \cdots \bigcap A_N\right] &= P[A_1]P[A_2|A_1] \ldots P[A_N|A_1, \ldots, A_{N-1}]. \\ &\leq L^N P[A_1]^N \\ P[T_n^{'''} \geq \varepsilon] &\leq \binom{n}{N} L^N P[|X_1| > n^{\beta}]^N \\ &\leq L^N \binom{n}{N} P[|X_1|^{\alpha^{-1}} > n^{\beta/\alpha}]^N. \end{aligned}$$

By Markov's inequality,

$$\begin{aligned} P[T_n^{'''} \geq \varepsilon] &\leq L^N \binom{n}{N} (Cn^{-2\beta/\alpha})^N \\ &\leq Mn^{(1-2\beta/\alpha)N}, \end{aligned}$$

for all large n, where M is a finite constant, depending only on N and C, choose $\beta = 2\alpha/3$ and $N = 6$, then for large n, $P[T_n^{'''} \geq \varepsilon] \leq Mn^{-2}$. Therefore

$$\sum_{n=1}^{\infty} P[T_n^{'''} \geq \varepsilon] < \infty. \tag{4}$$

From (2)–(4), we have

$$\sum_{n=1}^{\infty} P[T_n \geq \varepsilon] < \infty.$$

By symmetry, $\{-X_n, n \geq 1\}$ is also ND. So, we obtain $\sum_{n=1}^{\infty} P[T_n \leq -\varepsilon] < \infty$. Hence the proof is complete. □

Remark 2.1 It may be noted that the proof of Theorem 2.1 is similar to that of Theorem 4.1.3 in Stout (1974, p. 226).

According to the previous theorem, we obtain the following result.

Corollary 2.1 *Let $\{X_n, n \geq 1\}$ be a sequence of ND and identically distributed random variables such that $E[X_1^4] < \infty$ and $E[X_1] = 0$. By using Theorem 2.1 and assuming $\alpha = 1/2$ and $a_{nk} = n^{-1/2}(\log n)^{-1/2-\beta}$ for $1 \leq k \leq n$ and each $\beta > 0$,*

$$\sum_{k=1}^{n} X_k/[n^{1/2}(\log n)^{1/2+\beta}]$$

converges completely to zero.

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