Asymptotic Behavior of Product of Two Heavy-Tailed Dependent Random Variables

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Abstract

Let $X$ and $Y$ be positive weakly negatively dependent (WND) random variables with finite expectations and continuous distribution functions $F$ and $G$ with heavy tails, respectively. The asymptotic behavior of the tail of distribution of $XY$ is studied and some closure properties under some suitable conditions on $F(x) = 1 - F(x)$ and $G(x) = 1 - G(x)$ are provided. Moreover, subexponentiality of $XY$ when $X$ and $Y$ are WND random variables is derived.

MSC2000: 60E05, 60F99

keywords: Weakly negative dependent, Heavy-tailed, Asymptotic behavior.

1 Introduction

The subexponentiality of distribution of $XY$ when $X$ and $Y$ are independent heavy tailed random variables with distribution functions $F$ and $G$ respectively, has been studied by Cline and Samorodnitsky[3]. They proved that, if $F$ belongs to the class of subexponential distributions ($S$), denoted by $F \in S$ under some suitable conditions on $F(x) = 1 - F(x)$ and $G(x) = 1 - G(x)$, then the distribution of $XY$ belongs to S. Tang [9] by removing conditions of Cline and Samorodnitsky and adding a mild condition to the distribution $F$, extended these results. Following the works of Tang [9] and Cline and Samorodnitsky[3], we will study asymptotic behavior of the tail distribution of $XY$, when $X$ and $Y$ are WND random variables with finite expectations and continuous distribution functions $F$ and $G$, respectively. In fact, we prove, if $F$ and $G$ belong to classes $L$ or $D$, under some suitable conditions on $F$ and $G$, then the distribution of $XY$ also belongs to the $L$ or $D$. Finally, we derive subexponentiality of $XY$ when $X$ and $Y$ are WND random variables.

An important class of heavy tailed distributions is $D$, which consists of all distributions with dominated variation. By definition, a distribution function $F$ belongs to the class $D$, if $\lim \sup \frac{F(xy) / F(x)}{y} < \infty$, holds for some $0 < y < 1$ as $x \to \infty$. A wider class of heavy tailed distributions is $L$, which consists of all distributions with long tailed distributions. By definition, a distribution function belongs to $L$, if $\lim \frac{F(x+y)}{F(x)} = 1$ holds for any $y \in \mathbb{R}$, as $x \to \infty$. For a distribution $F$ with $F(x) > 0$ for all $x \geq 0$,
the lower Matuszewska index of the function \( F \) is defined as follow,

\[
J_\alpha(F) = \sup \left\{ \frac{F^*(v)}{\log v} \right\} \quad \text{with} \quad F^*(v) = \limsup_{x \to \infty} \frac{F(tx)}{F(x)} \quad \text{for} \quad v > 1.
\]

It is easy to see that, the condition \( 0 < J_\alpha(F) \leq \infty \) is equivalent to condition \( F^*(v) < 1 \), for some \( v > 1 \). For details of the lower Matuszewska indices see Bingham et al[1] (chapter 2.1) and for further discussions and applications see Cline and Samorodnitsky [3] and Tang and Tistlshvili [10]. Throughout this paper all distribution functions are defined on \([0, \infty)\) and \( f(x) \sim g(x) \) means that \( \lim f(x)/g(x) = 1 \) as \( x \to \infty \). We denote the tail of distribution of \( F \) by \( F(x) = 1 - F(x) \) and distribution of product of \( XY \) by \( H \), say \( H(t) = P(XY \leq t) \). The Weakly Negative Dependence (WND), which is introduced as follows, is a kind of dependence which has some good and simple property that allows us to prove some useful results.

**Definition 1.1.** The random variables \( X \) and \( Y \) are said Weakly Negatively Dependent (WND) if there exists some \( C > 1 \) such that, \( f(x,y) \leq C f_1(x) f_2(y) \) where \( f(x,y) \), \( f_1(x) \) and \( f_2(y) \) are joint density and marginal densities of \( X \) and \( Y \), respectively.

**Remark 1.2.** Let \( X \) and \( Y \) be two WND random variables with distribution functions \( F \) and \( G \) respectively. Then it is easy to show that,

1. For every \( x, y \in \mathbb{R} \) we have \( F_{XY}(x, y) \leq C F(x) G(y) \).
2. For all positive values \( x \), \( P(X + Y > x) \leq C \int_0^x F(z - u) dG(u) \).
3. If \( h_1(.) \) and \( h_2(.) \) are monotone measurable functions then \( h_1(X) \) and \( h_2(Y) \) are WND. In particular, it is valid when \( h_i(x) = c_i x, i = 1, 2 \) where \( c_i \in \mathbb{R} \).

## 2 Main results

In this section, we study the asymptotic behaviors and some closure properties of classes D and \( L \) for product of two random variables with heavy tail distribution functions.

**Theorem 2.1.** Let \( X \) and \( Y \) be two WND random variables with distribution functions \( F \) and \( G \), respectively. Suppose that \( F \) and \( G \) belong to \( D \) and \( 0 < J_\alpha(G) < \infty \). If there exists some \( 0 < p < J_\alpha(G) \) for which \( E(X^{-p}) < \infty \), then \( H \in D \), where \( H(x) = P(XY > x) \).

**Proof.** Since we study the asymptotic behavior of tail of the distribution functions for sufficiently large positive value of \( x \), hence, without loss of generality we assume that \( x > 1 \). We prove the theorem in three parts

1. If \( X > 1 \) a.s. and \( Y > 1 \) a.s., then we have

\[
\tilde{H}(x) = P(XY > x) = P(\ln X + \ln Y > \ln x) = P(\ln X + \ln Y > \ln x; \ln X < \ln x) + P(\ln X > \ln x) + P(\ln Y > \ln x; \ln Y < \ln x) = P(\ln Y > \ln x; \ln Y < \ln x).
\]

On the other hand, it is easy to see that for any positive values \( a, b, c \) and \( d \),

\[
\frac{a + b}{c + d} \leq \max \left\{ \frac{a}{c}, \frac{b}{d} \right\}.
\]

So, for some \( 0 < t < 1 \), we get

\[
\frac{\tilde{H}(tx)}{\tilde{H}(x)} \leq \max \{ I_1, I_2 \}.
\]
where,
\[
I_1 = \left[ \frac{P(X > tz) - P(X > x, Y > tz)}{P(X > x) - P(X > x, Y > x)} \right] \leq \frac{P(X > x)}{\frac{C.P(X > x)P(Y > x)}{P(X > x)}} = [I_3 + I_4]^{-1}. 
\]

The inequality directly follows from Remark 1.2. Then, by \( F \in D \), we have, \( 0 < \lim_{x \to \infty} I_5 < 1 \) and \( \lim_{x \to \infty} I_1 = 0 \), therefore, \( \limsup_{x \to \infty} I_1 < \infty \). Moreover,
\[
I_2 = \frac{P(Y > tx)}{P(Y > x)} + P(XY > tx; Y < tx; X < tx) + P(XY > x; Y < x; X < x) \leq \frac{P(Y > tx)}{P(Y > x)} + C. \int_t^{tx} \frac{G(tx/u) - G(tx)}{G(x)} dF(u) = \frac{P(Y > tx)}{P(Y > x)} + I_5. 
\]

Where the last equality follows from remark 1.2. Now, by Tong [9] and the second statement of Proposition 2.2.1 of Bingham et al. [1], we conclude that for each \( 0 < p < \mathcal{E}(G) \), there exist positive constants \( C' \) and \( x_0 \) such that the inequality \( G(y)/G(x) \leq C'(y/x)^p \), holds uniformly for \( y \leq x \), or equivalently, that the inequality
\[
\frac{G(tx/u)}{G(x)} \leq C'(t/x)^p, \quad x \leq t \leq x_0, \quad u \leq x \leq t. 
\]

holds uniformly for \( x_0 \leq \frac{tx}{u} \leq x \) for \( t \leq u \leq \frac{tx}{x_0} \). So we have
\[
\int_t^{tx} \frac{G(tx/u)}{G(x)} dF(u) \leq C'. \int_t^{tx} (t/u)^p dF(u) \leq M_t t^p E(X^{-p}) < \infty. \tag{24}
\]

Where, \( M_t = C.C' \). Now, if \( x_0 \leq 1 \), then, by (24) we can write
\[
I_0 \leq C. \int_t^{tx} \frac{G(tx/u)}{G(x)} dF(u) \leq C. \int_t^{tx} \frac{G(tx/u)}{G(x)} dF(u) < \infty. 
\]

If \( x_0 > 1 \), we have
\[
I_0 \leq C. \int_t^{tx} \frac{G(tx/u)}{G(x)} dF(u) = C. \int_t^{tx} \frac{G(tx/u)}{G(x)} dF(u) + C. \int_t^{tx} \frac{G(tx/u)}{G(x)} dF(u) \]
\[
= C. \int_t^{tx} \frac{G(tx/u)}{G(x)} dF(u) + I_5. 
\]

On the other hand, we have
\[
\limsup_{x \to \infty} I_6 = \limsup_{x \to \infty} \frac{G(k)}{G(uk/x)} \frac{tx}{k^2} f(tx/k) dk = 0. \tag{25}
\]

Where \( k = tx/u \) and \( f \) is density function of \( X \). The last equality follows from \( G \in D \) and the fact that \( \lim_{x \to \infty} tx/k^2 = 0 \). Therefore, using (24) and (25) we can conclude that \( \limsup_{x \to \infty} I_5 < \infty \) and then \( \limsup_{x \to \infty} I_2 < \infty \). So
\[
\limsup_{x \to \infty} \frac{H(tx)}{H(x)} < \infty. 
\]

ii. If \( X > 1 \) a.s. and \( 0 < Y < 1 \) a.s., then for some \( 0 < t < 1 \) and for all \( x > 1 \), Remark 1.2 and \( F \in D \) imply that
\[
\limsup_{x \to \infty} \frac{H(tx)}{H(x)} \leq \limsup_{x \to \infty} \frac{P(XY > tx)}{P(X > x)} \leq \limsup_{x \to \infty} C. \int_0^1 \frac{F(tx/u)}{F(x)} dF(u) < \infty. 
\]
iii. The case $0 < X < 1$ a.s. and $Y > 1$ a.s. is similar to (ii).

Using (2.2), and last three section we have

$$
\lim_{x \to \infty} \frac{h(tx)}{H(x)} \leq \lim_{x \to \infty} \sup \left\{ \frac{P(XY > tx; X > 1; Y > 1)}{P(XY > x; X > 1; Y > 1)} \right\}
$$

This completes the proof.

\[\square\]

**Theorem 2.2.** Let $X$ and $Y$ be two WND random variables with distribution functions $F$ and $G$, respectively. Suppose that $F, G \in D$ and $E(X) < \infty \ (E(Y) < \infty)$. If $F(x) = O(G(x))$ then $H \in D$.

**Proof.** The approach of the proof is similar to the proof of Theorem 2.1 just we need to change the relation (2.5). In the new situation, for all $x > 0$ and $0 < t < 1; h > 0$, we have

$$
\frac{P(XY > tx; Y < tx; X < x)}{P(Y > x)} \leq C \int_0^x \frac{G(ty/u)}{G(x)} dF(u)
$$

$$
\leq \sum_{m=0}^{N_0} \int_{nh}^{(m+1)h} \frac{G(ty/u)}{G(x)} dF(u) \leq \sum_{n=0}^{N_0} \frac{1}{G(x)} \sum_{m=0}^{N_0} \left( \frac{tx}{(n+1)h} \right) \left[ F(nh) - F((n+1)h) \right]
$$

$$
\leq \frac{G(ty/h)F(0)}{G(x)} + \sum_{n=1}^{N_0} \bar{F}(nh) \left[ \frac{G(ty/(n+1)h)) - G(ty/nh))}{G(x)} \right] = K_1 + K_2. \tag{2.6}
$$

Where $N_0 = [tx/h]$. Since $G \in D$, we have $\lim_{x \to \infty} K_1 < \infty$. On the other hand, by Theorem ..., we know for any $t, x > 0$ and $h > 0$ there exist some $n_0 \in N$ such that for every $n > n_0$, $nh > tx$. So we have

$$
K_2 = \sum_{n=1}^{N_0} \bar{F}(nh) \left[ \frac{G(ty/(n+1)h)) - G(ty/nh))}{G(x)} \right] = K_3 + K_4. \tag{2.7}
$$

Now we get

$$
\lim_{x \to \infty} K_3 \leq \sum_{n=1}^{N_0} \bar{F}(nh) \left[ \frac{\bar{G}(t(n+1)h)) - \bar{G}(ty/nh))}{\bar{G}(x)} \right] \leq M_2, \sum_{n=1}^{N_0} \bar{F}(nh) < \infty.
$$

Where the second inequality follows by

$$
M_2 = \lim_{x \to \infty} \left[ \frac{\bar{G}(t(n+1)h)) - \bar{G}(ty/nh))}{\bar{G}(x)} \right] < \infty. \quad \text{(by } G \in D)\]

Furthermore, we have

$$
\lim_{x \to \infty} K_4 \leq \lim_{x \to \infty} \sum_{n=n_0}^{N_0} \frac{\bar{G}(tx)}{\bar{G}(x)} \left[ G(ty/(n+1)h)) - G(tx/nh)) \right]
$$

$$
\leq M_2, \lim_{x \to \infty} \frac{\bar{G}(tx)}{\bar{G}(x)} \left[ G(ty/(n_0+1)h)) - G(tx/n_0h)) \right] < \infty. \tag{2.8}
$$

Where, last inequality follows from $G \in D$ and $M_2 = \lim_{x \to \infty} \bar{G}(tx)/\bar{G}(tx)$. Now by substituting (2.7), (2.8) and (2.8) in (2.6), proof completes. \[\square\]
Corollary 2.3. Let $X_{11}, \ldots, X_{n}$ be WND random variables with common distribution function $F \in D$. If $E(X) < \infty$ then $P_n = \prod_{i=1}^{n} X_i \in D$.

Theorem 2.4. Let $X$ and $Y$ be two WND random variables with distribution functions $F$ and $G$, respectively. If $F, G \in D \cap L$ and $G^*(t) < 1 \ (F^*(t) < 1)$, then $H \in D \cap L$.

Proof. By the same approach as used in the proof of Theorem 2.1, we have:

i. If $X > 1$ a.s and $Y > 1$ a.s., then for any $u > 0$ and for all $x > 1$, applying (2.1) and (2.2), we have

$$J_1 = \frac{P(X > x - u) - P(X > x - u; Y > x - u)}{P(X > x) - P(X > x; Y > x)} \leq \frac{P(X > x - u)}{P(X > x)} - C P(X > x) P(Y > x).$$

Since $F \in L$, then $\lim_{x \to \infty} J_1 \leq 1$. For $J_2$ we have

$$J_2 \leq \frac{1}{G(x)} \left[ \tilde{G}(x - u) + P(XY > x - u; X < x - u; Y < x - u) \right] = J_3 + J_4,$$

where $J_3 = \tilde{G}(x - u)/G(x)$ and

$$J_4 \leq C \int_{0}^{\infty} \frac{G((x - u)/t) - G(x - u)}{G(x)} dF(u).$$

Now, for each $t > 0$, $G \in L$ and $\tilde{G}^*(t) < 1$ we have $\lim_{x \to \infty} J_3 = 1$. Also, by Remark 1.2,

$$\lim_{x \to \infty} J_4 \leq C \int_{0}^{\infty} \limsup_{x \to \infty} \frac{I_{(0, x - u)}(G((x - u)/t) - G(x - u))}{G(x)} dF(t) \leq 0.$$

So, for each $u > 0$,

$$\lim_{x \to \infty} \frac{H(x - u)}{H(x)} = 1. \tag{2.9}$$

ii. If $X > 1$ a.s. and $0 < Y < 1$ a.s., then

$$1 \leq \frac{P(XY > x - u)}{P(XY > x)} = 1 \frac{P(x < XY < x)}{P(XY > x)} = 1 + J_5.$$

By Remark 1.2 and $F \in D \cap L$ we have,

$$\lim_{x \to \infty} J_5 \leq \lim_{x \to \infty} \frac{P(x < XY < x)}{P(X > x)} \leq \lim_{x \to \infty} C \int_{0}^{1} \frac{\tilde{F}((x - u)/t) - \tilde{F}(x)/t)}{\tilde{F}(x)} dG(t) = 0.$$

Hence,

$$\lim_{x \to \infty} \frac{H(x - u)}{H(x)} = 1. \tag{2.10}$$

iii. If $Y > 1$ a.s. and $0 < X < 1$ a.s., then, similar to (ii) we can obtain (2.10). Combining (2.2), (2.9) and (2.10), we derive

$$1 \leq \lim_{x \to \infty} \frac{H(x - u)}{H(x)} = \lim_{x \to \infty} \max \left\{ \frac{P(XY > x - u; X > 1; Y > 1)}{P(XY > x; X > 1; Y > 1)}, \frac{P(XY > x - u; X < 1; Y > 1)}{P(XY > x; X < 1; Y > 1)}, \frac{P(XY > x - u; X > 1; Y < 1)}{P(XY > x; X > 1; Y < 1)} \right\} \leq 1$$

This completes the proof. \qed
Theorem 2.5. Let $Y_1$ and $Y_2$ be two WND random variables with common distribution function $F \in L$ and $E(Y) < \infty$. Suppose that $X_1$ and $X_2$ are two independent random variables which are independent of $Y_1$ and $Y_2$, with distribution functions $F_1$ and $F_2$, respectively, then

$$P(X_1Y_1 + X_2Y_2 > x) \sim P(X_1Y_1 > x) + P(X_2Y_2 > x) \quad \text{as} \quad x \to \infty.$$ 

Proof. For every $x > 0$, we have

$$P(X_1Y_1 + X_2Y_2 > x) = \int_0^\infty \int_0^\infty P(x_1Y_1 + x_2Y_2 > x|X_1 = x_1, X_2 = x_2)dF_1(x_1)dF_2(x_2)$$

$$\sim \int_0^\infty \int_0^\infty [P(x_1Y_1 > x) + P(x_2Y_2 > x)]dF_1(x_1)dF_2(x_2) \quad (\text{as} \quad x \to \infty)$$

$$= P(X_1Y_1 > x) + P(X_2Y_2 > x).$$

The asymptotic relation follows by Theorem 2 of Ranjbar, et al.[8], and this completes the proof. \qed

Conclusions: All Theorems and Lemmas are valid for $C = 1$, as a matter of fact, the independence structure is special case of our work.

References


