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Some Maximal Inequalities for Quadratic Forms of Negative Superadditive Dependence r.v.'s

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Abstract

Let $\{X_n; n \geq 1\}$ be a sequence of negative superadditive dependence random variables with $E(X_n^2) < \infty$, for all $n \geq 1$. Denote $T_n = \sum_{1\leq i<j\leq n} X_iX_j$ and $Q_n = \sum_{1\leq i<j\leq n} a_{ij}X_iX_j$, where $\{a_{ij}; 1 \leq i < j \leq n\}$ is an array of real numbers. We derive two maximal inequalities for quadratic forms T_n and Q_n, then, using these inequalities we obtain strong law of large numbers and the rate of almost sure convergence under some suitable conditions on $E(X_n^2)$ and $\{a_{ij}; 1 \leq i < j \leq n\}$.

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1 Introduction and Preliminaries

Let $\{X_n; n \geq 1\}$ be a sequence of independent identically distributed random variables (r.v.'s). Consider the following quadratic forms (Q.F.'s)

$$Q_n = \sum_{1\leq i<j\leq n} a_{ij}X_iX_j.$$

In particular, if $a_{ij} = 1$ for all $i \neq j$, then we define

$$T_n = \sum_{1\leq i<j\leq n} X_iX_j.$$

There are many authors who devote the study to maximal inequalities and almost sure (a.s.) convergence of sequence $\{T_n; n > 1\}$. For instance, Cuzich et al.(1995) and Zhang(1996) proved the sufficient and necessary conditions for the strong law of large numbers of sequence $\{T_n; n > 1\}$, Gadidov(1998) gave two Rosenthal-type inequalities of sums of products for independent and identically distributed random variables, Shanchao(2006) improved the corresponding ones in Gadidov(1998). Also, Whittle(1960, 1964) showed inequalities for moments of Q_n similar to the Rosenthal inequalities for sums of independent random variables and proved a central limit theorem under the additional condition that the array $\{a_{ij}; 1 \leq i < j \leq n\}$ have a special quasi-diagonal structure. Varberg(1966) obtained conditions of convergence of Q_n in quadratic mean and almost surely, de Jong(1987) discussed central limit theorem for quadratic forms and references therein.

This paper, organize as follows: in section 2, we prove the maximal inequality for $T_n = \sum_{1\leq i<j\leq n} X_iX_j$ and $Q_n = \sum_{1\leq i<j\leq n} a_{ij}X_iX_j$, where $\{X_n; n \geq 1\}$ is a sequence of nonnegative and negative superadditive dependence random variables. The strong law of large numbers and the rate of almost sure convergence of sequences $\{T_n; n > 1\}$ and $\{Q_n; n > 1\}$ derived in section 3. In section 4, we present some examples that satisfy conditions that are mentioned for convergence of sequence $\{T_n; n > 1\}$.
The following definitions, lemmas and theorems are needed in the next sections.

Definition 1. *([10]) A function $\phi : \Re^m \to \Re$ is called superadditive if $\phi(\boldsymbol{x} \vee \boldsymbol{y}) + \phi(\boldsymbol{x} \wedge \boldsymbol{y}) \geq \phi(\boldsymbol{x}) + \phi(\boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in \Re^m$, where $\vee$ is for componentwise maximum and $\wedge$ is for componentwise minimum.*

Lemma 1. *([10]) If ϕ has continuous second partial derivatives, then the superadditivity of ϕ is equivalent to $\partial^2\phi/\partial x_i \partial x_j \geq 0$, $1 \leq i \neq j \leq m$.*

Definition 2. *([7]) A random vector $\mathbf{X} = (X_1, X_2, \dots, X_m)$ is said to be negatively superadditive dependent (NSD) if*

$$E\phi(X_1, X_2, \dots, X_m) \leq E\phi(X_1^*, X_2^*, \dots, X_m^*), \tag{1}$$

where $X_1^*, X_2^*, \dots, X_m^*$ are independent with $X_i \stackrel{d}{=} X_i^*$ for each i and ϕ is a superadditive function such that the expectations in (2) exist.

Definition 3. *([8]) Random variables $X_1, X_2, \dots, X_k$ are said to be negatively associated (NA) if for every pair of disjoint subsets A_1 and A_2 of $\{1, 2, \dots, k\}$*

$$Cov\{f_1(X_i, i \in A_1), f_2(X_j, j \in A_2) \leq 0, \tag{2}$$

whenever f_1 and f_2 are increasing. Clearly, (3) holds if both f_1 and f_2 are decreasing.

Lemma 2. *([8]) Increasing functions defined on disjoint subsets of a set of negatively associated random variables are negatively associated random variables.*

Lemma 3. *([2]) Let $\{X_n; n \geq 1\}$ be a sequence of negatively associated random variables. Then $\{X_n; n \geq 1\}$ is a sequence of negative superadditive dependence random variables.*

Theorem 1. *If $\{(X_n, \mathcal{F}_n); n \geq 1\}$ is a nonnegative submartingale, then:*
(i)([6]) For all $p > 1$, we have

$$E\left(\max_{0\leq k\leq n} X_k\right)^p \leq \left(\frac{p}{p-1}\right)^p E(X_n^p).$$

(ii)([1]) For any $\varepsilon > 0$,

$$P\left[\max_{1\leq k\leq n} X_k > 2\varepsilon\right] \leq P[X_1 > \varepsilon] + \varepsilon^{-1}E^{1/2}(X_n^2)P^{1/2}[X_n > \varepsilon].$$

Lemma 4. *([9])The sequence $\{X_n; n \geq 1\}$ converges almost surely if and only if for every $\varepsilon > 0$,*

$$\lim_{n\to\infty} P\left[\sup_{k\geq 1} |X_{n+k} - X_n| > \varepsilon\right] = 0.$$

2 The Maximal Inequalities

In this section, we prove two maximal inequalities and extend Kolomogorov's convergence criterion of strong law of large numbers for negative superadditive dependence random variables. Without loss of generality, we suppose that $\{X_n; n \geq 1\}$ is a sequence of nonnegative random variables. Because, if assumption of nonnegativity random variables is removed, then by $|X_n| = X^+ + X^-$ and

$$\begin{aligned} |T_n| &\leq \sum_{1\leq i<j\leq n} |X_i X_j| = \sum_{1\leq i<j\leq n} |X_i||X_j| \\ &= \sum_{1\leq i<j\leq n} X_i^+ X_j^+ + \sum_{1\leq i<j\leq n} X_i^+ X_j^- + \sum_{1\leq i<j\leq n} X_i^- X_j^+ + \sum_{1\leq i<j\leq n} X_i^- X_j^- \end{aligned}$$

where $X^+ = \max\{X, 0\}$ and $X^- = \max\{0, -X\}$, all of results are valid.

Theorem 2. *Let $\{X_n; n \geq 1\}$ be a sequence of nonnegative and negative superadditive dependence random variables with $E(X_j^2) < \infty$, $j \geq 1$, then for every $\varepsilon > 0$,*

$$P\left[\max_{2\leq k\leq n} T_k > \varepsilon\right] \leq \frac{4}{\varepsilon^2}\left(\sum_{j=2}^{n} \sqrt{E(X_j^2)} \sum_{i=1}^{j-1} \sqrt{E(X_i^2)}\right)^2 .$$

Proof. The negativity of sequence $\{X_n; n \geq 1\}$ and $T_{n+1} = T_n + X_{n+1}\left(\sum_{i=1}^{n} X_i\right)$, for all $n \geq 2$, imply that the sequence $\{(T_n, \mathcal{F}_n); n \geq 1\}$ is a nonnegative submartingale, where $\mathcal{F}_n = \sigma(X_1, X_2, \ldots, X_n)$ is the smallest sigma filed produced by $X_1, X_2, \ldots, X_n$. Moreover, by Lemma 1, for all $n \geq 2$ the function $\varphi(x_1, x_2, \ldots, x_n) = \left(\sum_{1\leq i<j\leq n} x_i x_j\right)^2$ is superadditive.

Set $S_{j-1}^* = \sum_{i=1}^{j-1} X_i^*$, then,

$$\begin{aligned} E(T_n^*)^2 &= E\left(\sum_{j=2}^{n} X_j^* S_{j-1}^*\right)^2 = E\left(\sum_{j=2}^{n} X_j^{*2} {S_{j-1}^*}^2 + \sum_{2\leq i\neq j\leq n} X_i^* S_{i-1}^* X_j^* S_{j-1}^*\right) \\ &= \sum_{j=2}^{n} E(X_j^2) \sum_{i=1}^{j-1} E(X_i^2) + \sum_{j=2}^{n} E(X_j^2) \sum_{1\leq l\neq k\leq j-1} E(X_l)E(X_k) \\ &\quad + \sum_{2\leq i\neq j\leq n} E(X_i)E(X_j) \sum_{l=1}^{i-1} E(X_l) \sum_{k=1}^{j-1} E(X_k) \\ &\leq \left(\sum_{j=2}^{n} \sqrt{E(X_j^2)} \sum_{i=1}^{j-1} \sqrt{E(X_i^2)}\right)^2 . \end{aligned}$$

So, applying Markov's inequality and Theorem 1(i), we have for $p = 2$,

$$\begin{aligned} P\left[\max_{2\leq k\leq n} T_k > \varepsilon\right] &\leq \frac{1}{\varepsilon^2} E\left(\max_{2\leq k\leq n} T_k\right)^2 \\ &\leq \frac{4}{\varepsilon^2}\left(\sum_{j=2}^{n} \sqrt{E(X_j^2)} \sum_{i=1}^{j-1} \sqrt{E(X_i^2)}\right)^2 . \end{aligned}$$

This completes the proof. □

Theorem 3. *Let $\{X_n; n \geq 1\}$ be a sequence of nonnegative and negative superadditive dependence random variables with $E(X_j^2) < \infty$, $j \geq 1$, then for any $\varepsilon > 0$,*

$$P\left[\max_{2\leq k\leq n} T_k > 2\varepsilon\right] \leq \frac{1}{\varepsilon}E(X_1X_2) + \frac{1}{\varepsilon^2}\left(\sum_{j=2}^{n}\sqrt{E(X_j^2)}\sum_{i=1}^{j-1}\sqrt{E(X_i^2)}\right)^2.$$

Proof. Using Theorem 1(ii) and Markov's inequality, we have

$$\begin{aligned}
P\left[\max_{2\leq k\leq n} T_k > 2\varepsilon\right] &\leq \frac{1}{\varepsilon}E(T_2) + \frac{1}{\varepsilon}E^{1/2}(T_n^2)P^{1/2}[T_n > \varepsilon] \\
&\leq \frac{1}{\varepsilon}E(T_2) + \frac{1}{\varepsilon}E^{1/2}(T_n^2)\left(\frac{E(T_n^2)}{\varepsilon^2}\right)^{\frac{1}{2}} \\
&\leq \frac{1}{\varepsilon}E(X_1X_2) + \frac{1}{\varepsilon^2}E(T_n^2) \\
&\leq \frac{1}{\varepsilon}E(X_1X_2) + \frac{1}{\varepsilon^2}E(T_n^{*2}) \\
&\leq \frac{1}{\varepsilon}E(X_1X_2) + \frac{1}{\varepsilon^2}\left(\sum_{j=2}^{n}\sqrt{E(X_j^2)}\sum_{i=1}^{j-1}\sqrt{E(X_i^2)}\right)^2.
\end{aligned}$$

□

Theorem 4. *Let $\{X_n; n \geq 1\}$ be a sequence of nonnegative and negative superadditive dependence random variables. Assume that $\{a_{ij}; 1 \leq i < j \leq n\}$ be an array of real numbers, then for given $\varepsilon > 0$,*

$$P\left[\max_{2\leq k\leq n} Q_k > \varepsilon\right] \leq \frac{4}{\varepsilon^2}\left(\sum_{j=2}^{n}\sqrt{E(X_j^2)}\sum_{i=1}^{j-1}|a_{ij}|\sqrt{E(X_i^2)}\right)^2.$$

Where $Q_n = \sum_{1\leq i<j\leq n} a_{ij}X_iX_j$.

Proof. It is easy to show that $\{(Q_n, \mathcal{F}_n); n > 1\}$ is a nonnegative submartingale and the function $\psi(x_1, x_2, \ldots, x_n) = \left(\sum_{1\leq i<j\leq n} a_{ij}x_ix_j\right)^2$ is superadditive, for all $n \geq 2$. Moreover,

$$\begin{aligned}
E(Q_n^*)^2 &= E\left(\sum_{j=2}^{n}X_j^*\sum_{i=1}^{j-1}a_{ij}X_i^*\right)^2 \\
&= \sum_{j=2}^{n}E(X_j^2)\sum_{i=1}^{j-1}a_{ij}^2E(X_i^2) + \sum_{j=2}^{n}E(X_j^2)\sum_{1\leq l\neq k\leq j-1}a_{lj}a_{kj}E(X_l)E(X_k) \\
&\quad + \sum_{2\leq i\neq j\leq n}E(X_i)E(X_j)\sum_{l=1}^{i-1}a_{li}E(X_l)\sum_{k=1}^{j-1}a_{kj}E(X_k) \\
&\leq \left(\sum_{j=2}^{n}\sqrt{E(X_j^2)}\sum_{i=1}^{j-1}|a_{ij}|\sqrt{E(X_i^2)}\right)^2.
\end{aligned}$$

Therefore, similar to Theorem 2, this completes the proof. □

3 Almost Sure Convergence

In this section, using the result of section 2, we investigate strong law of large numbers and rate of almost sure convergence of quadratic forms T_n and Q_n.

Theorem 5. *Let $\{X_n; n \geq 1\}$ be a sequence of nonnegative and negative superadditive dependence random variables with $E(X_j^2) < \infty$, $j \geq 1$. If $\sum_{j=2}^{\infty} \sqrt{E(X_j^2)} \sum_{i=1}^{j-1} \sqrt{E(X_i^2)} < \infty$, then the series $T_n = \sum_{1 \leq i < j \leq n} X_i X_j$ converges a.s. as $n \to \infty$.*

Proof. By applying Lemma 4 and Theorem 2, for any $\varepsilon > 0$,

$$\begin{aligned} P\left[\sup_{k \geq 1} |T_{n+k} - T_n| > \varepsilon\right] &= \lim_{m \to \infty} P\left[\sup_{1 \leq k \leq m} |T_{n+k} - T_n| > \varepsilon\right] \\ &\leq \frac{4}{\varepsilon^2} \left(\sum_{j=n+1}^{\infty} \sqrt{E(X_j^2)} \sum_{i=1}^{j-1} \sqrt{E(X_i^2)} \right)^2. \end{aligned}$$

Since $\sum_{j=2}^{\infty} \sqrt{E(X_j^2)} \sum_{i=1}^{j-1} \sqrt{E(X_i^2)} < \infty$, it follows that

$$\lim_{n \to \infty} P\left[\sup_{k \geq 1} |T_{n+k} - T_n| > \varepsilon\right] = 0, \quad \textit{for all } \varepsilon > 0,$$

this completes the proof. □

Corollary 1. *Under the assumptions of Theorem 5, if $E(X_i^2) = O(i^{-\alpha})$, for some $\alpha > 2$, then $\{T_n; n > 1\}$ converges a.s. as $n \to \infty$.*

Theorem 6. *Let $\{X_n; n \geq 1\}$ be a sequence of nonnegative and negative superadditive dependence random variables with $E(X_j^2) < \infty$, $j \geq 1$. Suppose $\sum_{j=2}^{\infty} \frac{1}{b_j} \sqrt{E(X_j^2)} \sum_{i=1}^{j-1} \sqrt{E(X_i^2)} < \infty$, then*

$$\frac{1}{b_n} \sum_{1 \leq i < j \leq n} X_i X_j \xrightarrow{a.s.} 0. \tag{3}$$

Where $\{b_n; n \geq 1\}$ is a sequence of positive increasing real numbers such that $b_n \to \infty$ as $n \to \infty$.

Proof. If $\sum_{j=2}^{\infty} \frac{1}{b_j} \sqrt{E(X_j^2)} \sum_{i=1}^{j-1} \sqrt{E(X_i^2)} < \infty$, similar to proof of Theorem 4, we can show the series $\sum_{1 \leq i < j \leq n} \frac{1}{b_j} X_i X_j$ converges a.s. Therefore by Kronecker's lemma, the proof is completed. □

Corollary 2. *Under the assumptions of Theorem 6, (i) if $E(X_i^2) = O(i^{-\alpha})$, for some $0 < \alpha \leq 2$ and $(b_n)^{-1} = O(n^{-\beta})$, such that $\beta > 2 - \alpha$, then (4) holds,*
(ii) if $E(X_i^2) = O(i^{\alpha})$, for some $\alpha > -2$ and $(b_n)^{-1} = O(n^{-\beta})$, such that $\beta > 2 + \alpha$, then (4) holds.

Corollary 3. *Under the assumptions of Theorem 4, if $E(X_i^2) = O(i^{-\alpha})$ and $a_{ij} = O(i^{-\beta} j^{-\gamma})$, for some α, β and γ such that $\alpha/2 + \beta > 1$ and $\alpha + \beta + \gamma > 2$, $Q_n = \sum_{1 \leq i < j \leq n} a_{ij} X_i X_j$ converges a.s. as $n \to \infty$.*

Corollary 4. *Let $\{X_n; n \geq 1\}$ be a sequence of nonnegative and negative superadditive dependence random variables with $E(X_j^2) < \infty$; $j \geq 1$. If for some $\beta > 0$, $\sum_{n=2}^{\infty} n^{\beta-2} \left(\sum_{j=2}^{n} \sqrt{E(X_j^2)} \sum_{i=1}^{j-1} \sqrt{E(X_i^2)} \right)^2 < \infty$, then for every $\varepsilon > 0$ and some $\beta > 0$*

$$\sum_{n=1}^{\infty} n^{\beta} P \left[\max_{2 \leq k \leq n} T_k > n\varepsilon \right] < \infty.$$

Corollary 5. *Let $\{X_n; n \geq 1\}$ be a sequence of nonnegative and negative superadditive dependence random variables and suppose that $\{a_{ij}; 1 \leq i < j \leq n\}$ be an array of real numbers. If*

$$\sum_{n=2}^{\infty} n^{\beta-2} \left(\sum_{j=2}^{n} \sqrt{E(X_j^2)} \sum_{i=1}^{j-1} |a_{ij}| \sqrt{E(X_i^2)} \right)^2 < \infty,$$

for some $\beta > 0$, then we have

$$\sum_{n=2}^{\infty} n^{\beta-2} P \left[\max_{2 \leq k \leq n} Q_k > \varepsilon \right] < \infty, \qquad \text{for all } \varepsilon > 0.$$

4 Examples

The following examples satisfy the conditions of our results.

Example 1. *Let $\{X_n; n \geq 1\}$ be a sequence of negative superadditive dependence random variables with margins exponential distributions and $E(X_n) = 1/n^{\alpha}$ for all $n \geq 1$ and $\alpha > 2$. Then $\sum_{j=2}^{\infty} \sqrt{E(X_j^2)} \sum_{i=1}^{j-1} \sqrt{E(X_i^2)} = 2 \sum_{j=2}^{\infty} \frac{1}{j^{\alpha}} \sum_{i=1}^{j-1} \frac{1}{i^{\alpha}} < \infty$, Therefore Corollary 2 implies that $\sum_{1 \leq i < j \leq n} X_i X_j$ converges a.s. as $n \to \infty$.*

Example 2. *Let $(X_{k,i})_{i=1,2,\ldots,k}$, $k = 1, 2, \ldots$ be a triangular array of negatively associated random variables that are uniformly distributed on $(0,1)$. We define $Y_k = \prod_{i=1}^{k} X_{k,i}$. By Lemmas 2 and 3, it is clear that $Y_1, Y_2, \ldots$ are negative superadditive dependence random variables. So*

$$E(Y_k^2) \leq \prod_{i=1}^{k} E(X_{k,i}^2) = \left(\frac{1}{3} \right)^k,$$

and $\sum_{j=2}^{\infty} \sqrt{E(Y_j^2)} \sum_{i=1}^{j-1} \sqrt{E(Y_i^2)} = \sum_{j=2}^{\infty} \left(\frac{1}{\sqrt{3}} \right)^j \sum_{i=1}^{j-1} \left(\frac{1}{\sqrt{3}} \right)^i < \infty$, then $T_n = \sum_{1 \leq i < j \leq n} Y_i Y_j$ converges a.s. as $n \to \infty$.

Example 3. *Let $\{X_n; n \geq 1\}$ be a sequence of negative superadditive dependence r.v.'s with the following marginal probability distributions, $P[X_n = 0] = 1 - \frac{1}{n^{\alpha}}$ and $P[X_n = n] = \frac{1}{n^{\alpha}}$, for all $n \geq 1$, $\alpha > 4$. Then,*

$$\begin{aligned} \sum_{j=2}^{\infty} \sqrt{E(X_j^2)} \sum_{i=1}^{j-1} \sqrt{E(X_i^2)} &= \sum_{j=2}^{\infty} \frac{1}{j^{\alpha/2-1}} \sum_{i=1}^{j-1} \frac{1}{i^{\alpha/2-1}} \\ &\leq \sum_{j=1}^{\infty} \frac{1}{j^{\alpha/2-1}} . \frac{1}{j^{\alpha/2-2}} < \infty \end{aligned}$$

Therefore $\sum_{1 \leq i < j \leq n} X_i X_j$ converges a.s. as $n \to \infty$.

Remark 1. *If $\{X_n; n \geq 1\}$ is a sequence of independent random variables with $E(X_j^2) < \infty$, $j \geq 1$, then all of results in sections 2 and 3 hold.*

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