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A NUMERICAL STUDY OF FLOW AND HEAT TRANSFER BETWEEN TWO ROTATING SPHERES WITH CONSTANT ANGULAR VELOCITIES

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ABSTRACT: Solution of steady-state motion and heat transfer of a viscous incompressible fluid contained between two concentric spheres maintained at different temperatures and rotating about a common axis with different constant angular velocities is considered. These solutions are obtained by choosing the appropriate similarity parameters. These similarity parameters change the system of partial differential governing equations into a system of ordinary differential equations along with appropriate boundary and initial conditions which can be solved numerically with a lot less complexities. Gravity has been neglected therefore forced convection is the heat transfer phenomenon. The ratio of $Gr/\mathrm{Re}^2 \ll 1$, (Gr = Grashof number, Re =Reynolds number), which is the case for a big range of engine oils supports this simplification. It can be shown that the magnitude of the effect of the density variations on the rotational acceleration terms such as Coriolis and centrifugal is a function of the ratio of the gravitational acceleration to the rotational acceleration of the inner sphere. Therefore neglecting the gravitational acceleration in our study seems reasonable.
All the past solutions for the flow problem are direct numerical attempts which uses a lot of computational time. The resulting flow pattern, temperature distribution, and heat transfer characteristics are presented for the various cases. Same results as previous works are obtained for Navier-Stokes and energy equations but with less computational complexities.

1-Introduction

Similarity solution for motion of an incompressible viscous fluid and its heat transfer in a rotating spherical annuli is considered numerically when the spheres are concentric and their angular velocities about a common axis of rotation is constant. Such motions may be described in terms of a pair of coupled non-linear partial differential equations in three independent variables and the energy equation is linear when velocity field is known. Gravity has been neglected therefore forced convection is the heat transfer phenomenon. The ratio of $Gr/\mathrm{Re}^2 \ll 1$, (Gr = Grashof number, Re=Reynolds number), which is the case for a big range of engine oils supports this simplification. It can be shown that the magnitude of the effect of the density variations on the rotational acceleration terms such as Coriolis and centrifugal is a function of the ratio of the gravitational acceleration to the rotational acceleration of the inner sphere, Bar-Yoseph (1993). Therefore neglecting the gravitational acceleration in our study seems reasonable.

Available theoretical work concerning such problems is primarily of a boundary-layer or singular-perturbation character considered by Howart [1951], Proudman [1956], Lord & Bowden [1963], Fox [1964], Greenspan [1964], Carrier [1966] and Stewartson [1966]. The first numerical study of time-dependent viscous flow between two rotating spheres has been presented by Pearson [1967] in which the cases of one (or both) spheres is given an impulsive change in angular velocity starting from a state of either rest or uniform rotation. Munson and Joseph [1971] have considered the case of steady motion of a viscous fluid between concentric rotating spheres using perturbation techniques for small values of Reynolds number and a Legendre polynomial expansion for larger values of Reynolds numbers. In both studies above the viscous dissipation terms have been neglected. Thermal convection in rotating spherical annuli has been considered by Douglass, Munson and Shaughnessy [1978] in which the steady forced convection of a viscous fluid contained between two concentric spheres which are maintained at different temperatures and rotate about a common axis with different angular velocities is studied. Approximate solutions to the governing equations are obtained in terms of a regular perturbation solution valid for small Reynolds number and a modified Galerkian solution for moderate Reynolds numbers. Viscous dissipation is neglected in their study and all the fluid properties are assumed constant. A study of viscous flow in oscillatory spherical annuli has been done by Munson and Douglass [1979] in which a perturbation solution valid for slow oscillation rates is presented and compared with experimental results. Another interesting work is the study of the axially symmetric motion of an incompressible viscous fluid between two concentric rotating spheres done by Gagliardi et al. [1990]. This work involves the study of the steady state and transient motion of a system consisting of an incompressible, Newtonian fluid in an annulus between two concentric, rotating, rigid spheres. The primary purpose of their research is to study the use of an approximate analytical method for analyzing the transient motion of the fluid in the annulus and spheres which are started suddenly due to the action of prescribed torques and also the study of Jen-Kang Yang et al. [1989] and the finite element study by Ni and Negro [1994]. These problems include the case where one or both spheres rotate with prescribed constant angular velocities and the case in which one sphere rotates due to the action of an applied constant or impulsive torque. Also, Bar-Yoseph et al. [1993] consider the problem of mixed-convection of rotating fluids in spherical annuli in which they focus on the formation of various secondary flow patterns in the meridional plane using the Galerkin finite element method. The thermal effects on axisymmetric vortex breakdown in a spherical gap is also has been considered by Arkadyev et al. [1993] in which the influence of a temperature field on the vortex breakdown phenomenon is examined using a finite element formulation. The physical system considered is the spherical annulus between two concentric spheres with radii ratio 1:2 which is filled with a Boussinesq fluid and the outer sphere is stationary and hot while the inner sphere rotates and is at a lower temperature. The other work to mention is the study of axisymmetric vortex breakdown for generalized Newtonian fluid contained between rotating spheres by Bar-Yoseph el al. [1996] with the purpose of providing a more complete understanding of the secondary flow structure of dilute suspensions in rotating systems. The physical system considered is the spherical annulus between two concentric spheres, radii 1:2 which is filled with a Boussinesq generalized Newtonian fluid and the walls of the spherical annulus are held at uniform but different temperatures. A weak penalty finite element formulation is also used in this problem. Besides, there are many studies considering natural convection. These are including: Laminar natural convection about an isothermally heated sphere at small Grashof number by, Fendell [1968], Natural convection between two concentric spheres-transition towards a multicellular flow by, Caltagirone et al. [1979], Natural convection between concentric spheres at low Rayleigh numbers by, Mack et al. [1968], Natural convection between concentric spheres by, Garg [1992], Transient natural convection heat transfer between concentric spheres by, Chu et al. [1993], Transient natural convection heat transfer between concentric and vertically eccentric spheres by, Chiu et al. [1996], and Transient natural convection heat transfer of fluids with variable viscosity between concentric and vertically eccentric spheres by, Wu et al. [2004].

The state-of-the-art on similarity solution of flow and heat transfer problems is abundant in the literature. Some recent ones can be named as analytical solutions for unsteady free convection in porous media by Magyari [2004], new analytical solutions of a well-known boundary value problem in fluid mechanics by Pop et al. [2003], new solutions for flow in a channel with porous walls and/or non-rigid walls by Zaturska et al. [2003], mixed convection along a vertical surface: similarity solutions for uniform flow by Merkin and Pop [2002], diffusion equation coupled to Burgers' by Miyazaki [1987], a class of exact solutions of the Navier-Stokes equations by Kambe [1986], axisymmetric stagnation flow on a cylinder by Wang [1974], heat transfer in an axisymmetric stagnation flow on a cylinder, by Gorla [1976], unsteady laminar axisymmetric stagnation flow over a circular cylinder, by Gorla [1977], nonsimilar axisymmetric stagnation flow on a moving cylinder, by Gorla [1978], transient response behavior of an axisymmetric stagnation flow on a circular cylinder due to time-dependent free stream velocity, by Gorla [1978], unsteady viscous flow in the vicinity of an axisymmetric stagnation-point on a cylinder, by Gorla [1979], shear flow over a rotating plate, by Wang [1989], radial stagnation flow on a rotating cylinder with uniform transpiration, by Cunning et al. [1998], suppression of turbulence in wall-bounded flows by high-frequency spanwise oscillations, by Jung et al. [1992], axisymmetric stagnation flow towards a moving plate, by Wang [1973], axisymmetric stagnation-point flow impinging on a transversely oscillating plate with suction, by Weidman [1997], axisymmetric stagnation-point flow and heat transfer of a viscous fluid on a moving cylinder with time-dependent axial velocity and uniform transpiration by Saleh and Rahimi [2004], axisymmetric stagnation-point flow and heat transfer of a viscous fluid on a rotating cylinder with time-dependent angular velocity and uniform transpiration, by Rahimi and Saleh [2007], and Similarity solution of unaxisymmetric heat transfer in stagnation-point flow on a cylinder with simultaneous axial and rotational movements, by Rahimi and Saleh [2007].

Similarity solution in study of steady-state motion and heat transfer of an incompressible viscous fluid filling the annuli of two concentric spheres rotating with constant angular velocities has not been considered in the literature. In the present study a numerical solution of similarity equations of steady-state momentum and energy equations are solved for viscous flow between two concentric rotating spheres maintained at different temperatures and rotating with constant angular velocities. Aside from energy equation, same results as existing in the literature for Navier-Stokes equations are obtained but with less computational complexities. Such a rotating containers are used in engineering designs like centrifuges and fluid gyroscopes and also are important in geophysics. Other applications of the geometric configuration used in this problem are in meteorological instrumentations where such apparatus and equipments are used to obtain quantitative information about the weather.

2- Problem Formulation

The geometry of the spherical annulus considered is indicated in Fig. 1. A Newtonian, viscous incompressible fluid fills the gap between the inner and outer spheres which are of radii R_i and R_o and with constant surface temperatures T_i and T_o and rotate about a common axis with angular velocities Ω_i and Ω_o, respectively. The components of velocity in directions r, θ, and ϕ are v_r, v_θ, and v_ϕ, respectively. These velocity components for incompressible flow and in meridian plane satisfy the continuity equation and are related to stream function ψ and angular momentum function Ω in the following manner:

$$v_r = \frac{\psi_\theta}{r^2 \sin\theta}, \qquad v_\theta = \frac{-\psi_r}{r \ \sin\theta}, \qquad v_\phi = \frac{\Omega}{r \ \sin\theta} \tag{1}$$

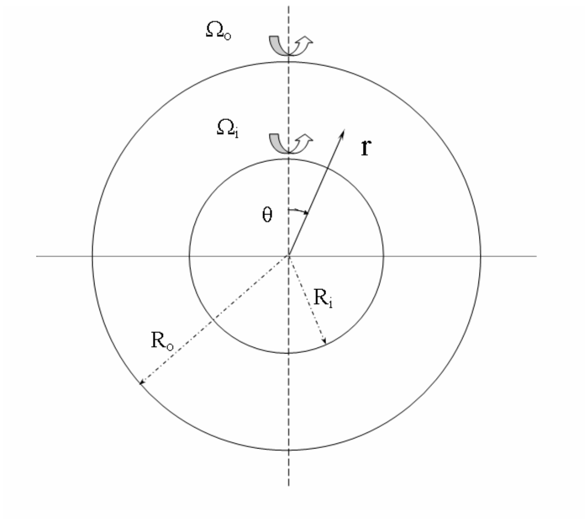


Figure 1. Spherical annulus.

Since the flow is assumed to be independent of the longitude, ϕ, the non-dimensional Navier-Stokes equations and energy equation can be written in terms of the stream function and the angular velocity function as follows:

$$\frac{\psi_\theta \Omega_r - \psi_r \Omega_\theta}{r^2 \sin\theta} = \frac{1}{(\text{Re})} D^2 \Omega \tag{2}$$

$$\frac{2\Omega}{r^3 \sin^2\theta}[\Omega_r r\cos\theta - \Omega_\theta \sin\theta] - \frac{1}{r^2 \sin\theta}[\psi_r (D^2\psi)_\theta - \psi_\theta (D^2\psi)_r] +$$

$$+ \frac{2D^2\psi}{r^3 \sin^2\theta}[\psi_r r\cos\theta - \psi_\theta \sin\theta] = \frac{1}{(\text{Re})} D^4 \psi \tag{3}$$

$$v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r}\frac{\partial T}{\partial \theta} = \frac{1}{(Pe)}[\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial T}{\partial \theta}] + (Ek)(Dissipation\ terms) \tag{4}$$

in which,

$$Dissipation\ terms = \{2[(\frac{\partial v_r}{\partial r})^2 + (\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r})^2 + (\frac{v_r}{r} + \frac{v_\theta}{r}\cot\theta)^2] + [r\frac{\partial}{\partial r}(\frac{v_\theta}{r}) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}]^2 + [\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}(\frac{v_\phi}{\sin\theta})]^2 + [r\frac{\partial}{\partial r}(\frac{v_\phi}{r})]^2\} \quad (5)$$

and the non-dimensional quantities Reynolds number (Re), Prandtl number (Pr), Peclet number (Pe), and Eckert number (Ek) are defined as:

$$\mathrm{Re} = \frac{\Omega_o R_o^2}{\nu},\ \mathrm{Pr} = \nu/\alpha,\quad Pe = \mathrm{Re}.\mathrm{Pr} = \frac{\Omega_o R_o^2}{\alpha},\quad Ek = \frac{\nu\Omega_o}{c_P(T_o - T_i)} \quad (6)$$

The following non-dimensional parameters have been used in the above equations and then the asterisks have been omitted:

$$r^* = \frac{r}{R_o},\quad \psi^* = \frac{\psi}{R_o^3\Omega_o},\quad \Omega^* = \frac{\Omega}{R_o^2\Omega_o},\quad T^* = \frac{T - T_i}{T_o - T_i} \quad (7)$$

The non-dimensional boundary and initial conditions for the above governing equations are:

$$\begin{aligned}
&\theta = 0 \rightarrow \{\psi = 0,\quad D^2\psi = 0,\quad \Omega = 0\},\quad \frac{\partial T}{\partial \theta} = 0\\
&\theta = \pi \rightarrow \{\psi = 0,\quad D^2\psi = 0,\quad \Omega = 0\},\quad \frac{\partial T}{\partial \theta} = 0\\
&r = \frac{R_i}{R_o} = b \rightarrow \begin{cases}\psi = 0,\quad \psi_{,r} = 0,\quad \Omega = \Omega_{io} b^2 \sin^2\theta\\ T = o\end{cases}\\
&r = 1 \rightarrow \begin{cases}\psi = 0,\quad \psi_{,r} = 0,\quad \Omega = \sin^2\theta\\ T = 1\end{cases}
\end{aligned} \quad (8)$$

where,

$$\begin{aligned}
&D^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} - \frac{\cot\theta}{r^2}\frac{\partial}{\partial \theta}\\
&\Omega_{io} = \Omega_i / \Omega_o
\end{aligned} \quad (9)$$

These governing equations along with their related boundary conditions are to be changed into differential equations with related appropriate boundary conditions by using similarity parameters and then to be solved numerically in the next sections.

3- Similarity variables

Using the following similarity parameters, one independent variable can be omitted:

$$\eta = \frac{(r - b)\sin^2\theta}{1 - r + b\sin^2\theta} \quad (10)$$

$$\begin{cases} \Omega = \sin^2\theta F(\eta) \\ \psi = \sin^2\theta G(\eta) \\ T(r,\theta) = H(\eta) \end{cases} \tag{11}$$

The momentum and energy equations become:

$$c_1(F'G - G'F) = \frac{1}{\mathrm{Re}}(c_2F'' + c_3F' + c_4F) \tag{12}$$

$$(d_1F' + d_2F)F + (d_3G' + d_5G'')G' + (d_4G' + d_6G + d_7G''' + d_8G'')G = \frac{1}{\mathrm{Re}}(d_9G' + \\ + d_{10}G + d_{11}G''' + d_{12}G'' + d_{13}G'''') \tag{13}$$

$$(e_1v_r + e_2v_\theta)H' = \frac{1}{Pe}(f_1H'' + f_2H') + Ek.(dissipation\ terms) \tag{14}$$

where the c's and d's coefficients are functions of r and θ. The Maple software has been used to do all the algebra and produce coefficients in a compact form which have been presented in the Appendix. The appropriate boundary conditions are:

$$\eta = 0 \rightarrow \quad G(\eta) = 0, \quad G'(\eta) = 0, \quad F(\eta) = \Omega_{io}b^2, \quad H = 0 \\ \eta = \frac{1}{b} - 1 \rightarrow G(\eta) = 0, G'(\eta) = 0, F(\eta) = 1, H = 1 \tag{15}$$

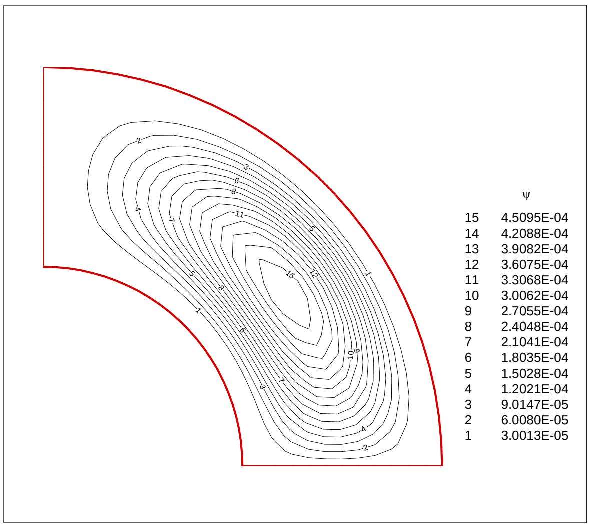
The differential equations (12)-(14) along with boundary conditions (16) constitute a closed form system which is solved numerically with less complexities compared to the initial system of partial differential equations. Note that formulation of the problem this way enables one to find the functions $\psi(r,\theta)$ and $\Omega(r,\theta)$ at any desired point within the flow field independently without having to solve for the whole region.

4- Computational Procedure

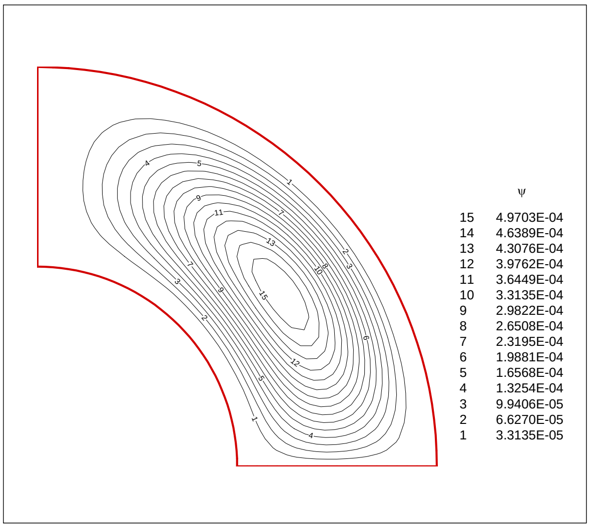
The flow Equations (12) and (13) are not coupled with the energy equation (14) and therefore need to be solved before the latter can be solved. These non-linear flow equations are solved numerically using finite difference approximations. A quasi-linearization technique is first applied to replace the non-linear terms at a linear stage with the corrections incorporated in subsequent iterative steps until convergence. Then, Crank-Nicolson algorithm is used to replace the different terms by their second-order central difference approximations, Press et al. [1997]. An iterative scheme is used to solve the quasi-linear system of difference equations. An initial guess is chosen and the iterations are continued till convergence within prescribed accuracy. Finally, the resulting block tri-diagonal system was solved using generalized Thomas' algorithm. The energy Equation (14) is a linear second-order ordinary differential equation with variable coefficients which are known from the solution of the flow equations. This equation is solved numerically using central differences for the derivatives and Thomas' algorithm for the solution of the set of discretized equations. Convergence is assumed when the ratio of every one of quantities for the last two approximations differed from unity by less than

10^{-5} at all values of independent variable. A mesh independence study has been demonstrated in Figures 2 and 3. In this mesh-study, the conditions of flow and heat transfer fields are: Re =10, Pr =10, Ek=0, and Ω_{io}=0. As it can be seen, the difference between the contours of ψ function for the coarse grid (case (a), with mesh size 25*12) and the fine grid (case (b) with mesh size 40*20) is almost large (about 12%), but the difference between case (c) (with grid size 45*25) and case (d) (with grid size 50*25) is really negligible (less than 0.03%). Hence the numerical solution is mesh-independent for cases c or d and even b. For the results presented in our solution, a 50*25 mesh grid has been selected though a 40*20 mesh would have been fine. The mesh sizes mentioned above are in θ x r directions. The contours of temperature has also been drawn for mesh sizes from case (a), 25*12 to case (d), 50*25 in Fig. 3. In this case no significant differences between these cases can be seen and that is because the energy equation is linear and its solution has much less complexities compared with momentum equation.

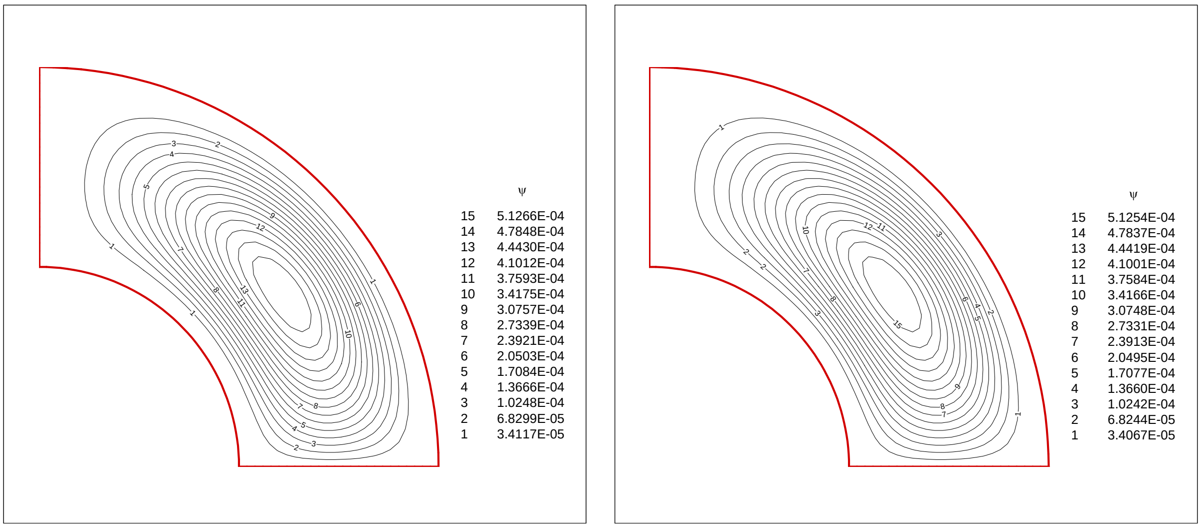
The final results obtained in each case are exactly the results of the work of Pearson [1967] and Munson et al. [1971] for the Navier-Stokes equations and energy equation. In our study these results have been obtained with a lot less computational complexities since they have been reached by solving an ordinary differential system of equations.



a

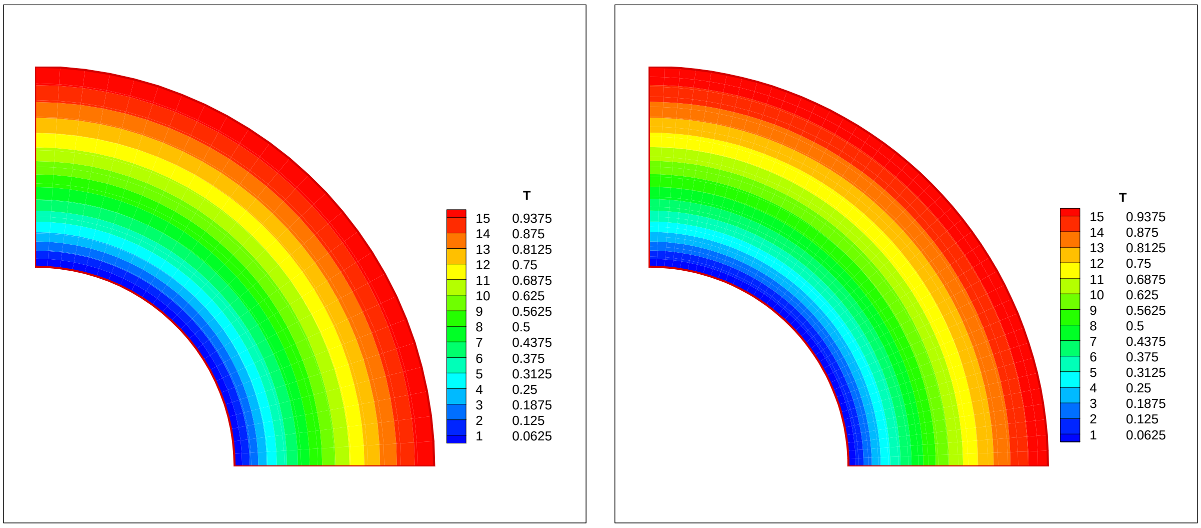


b

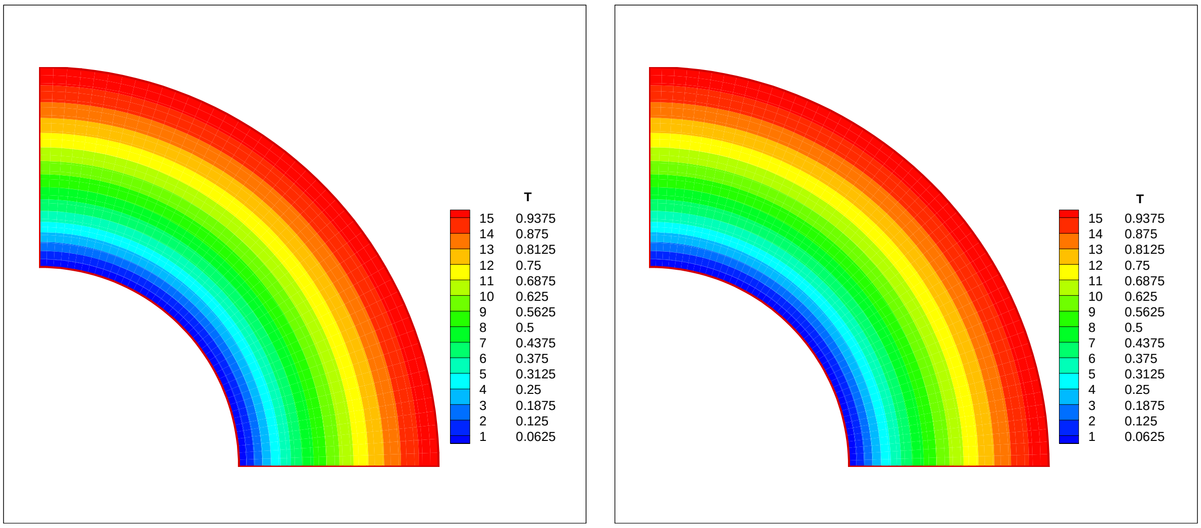


c d

Fig. 2: contours of stream function for various mesh-size grid



a b



c d

Fig. 3: contours of temperature for various mesh-size grid

5- Presentation of Results

If the bounding spherical surfaces were stationary, there would be no fluid motion and the temperature distribution would simply be conduction distribution. Any rotation of the bounding spheres sets up a primary flow (ω) around the axis of rotation. This relative motion induces an unbalanced centrifugal force field which drives the secondary flows (ψ) in the meridian plane. Thus, if the bounding spheres are of unequal temperatures, this secondary flow produces forced convection within the annulus, resulting in a temperature distribution which is different from the pure conduction distribution. The relative magnitudes of the secondary flow and forced convection effects depend upon the parameters involved, including those concerning the geometry of the flow and those concerning the dynamics of the flow such as $\Omega_{io} = \Omega_i / \Omega_o$, $b = R_i / R_o$, Prandtl number and Reynolds number. These secondary flows known as vortex have clockwise or counterclockwise motion depending upon whether the outer sphere or the inner sphere is dominant, as far as the secondary flow is concerned. Results for temperature fields are presented when the outer sphere is hotter than the inner one. The cases considered here are constant angular velocities and presentations are only at selected Ω_{io}.

The velocity fields and temperature distribution for the particular case of constant angular velocity, $\Omega_{io} = -3.0$ (negative sign indicates rotation in different directions) are presented in Figure 4 for Reynolds number $\mathrm{Re} = 50$, $\mathrm{Pr} = 10$, and Ek =0. As it is expected, the ψ contours show that the annulus space is under the effect of both spheres. The vortex close to the inner sphere is dominating the flow field since it is rotating three times faster than the outer one. The same type of dominating effect is shown in Fig. 4b for Ω function. In terms of velocity vectors, Fig. 4c is displaying the same effect. Fig. 4d is presenting the temperature field and showing that there is large delay in heat transfer because of the rotation of the inner sphere.

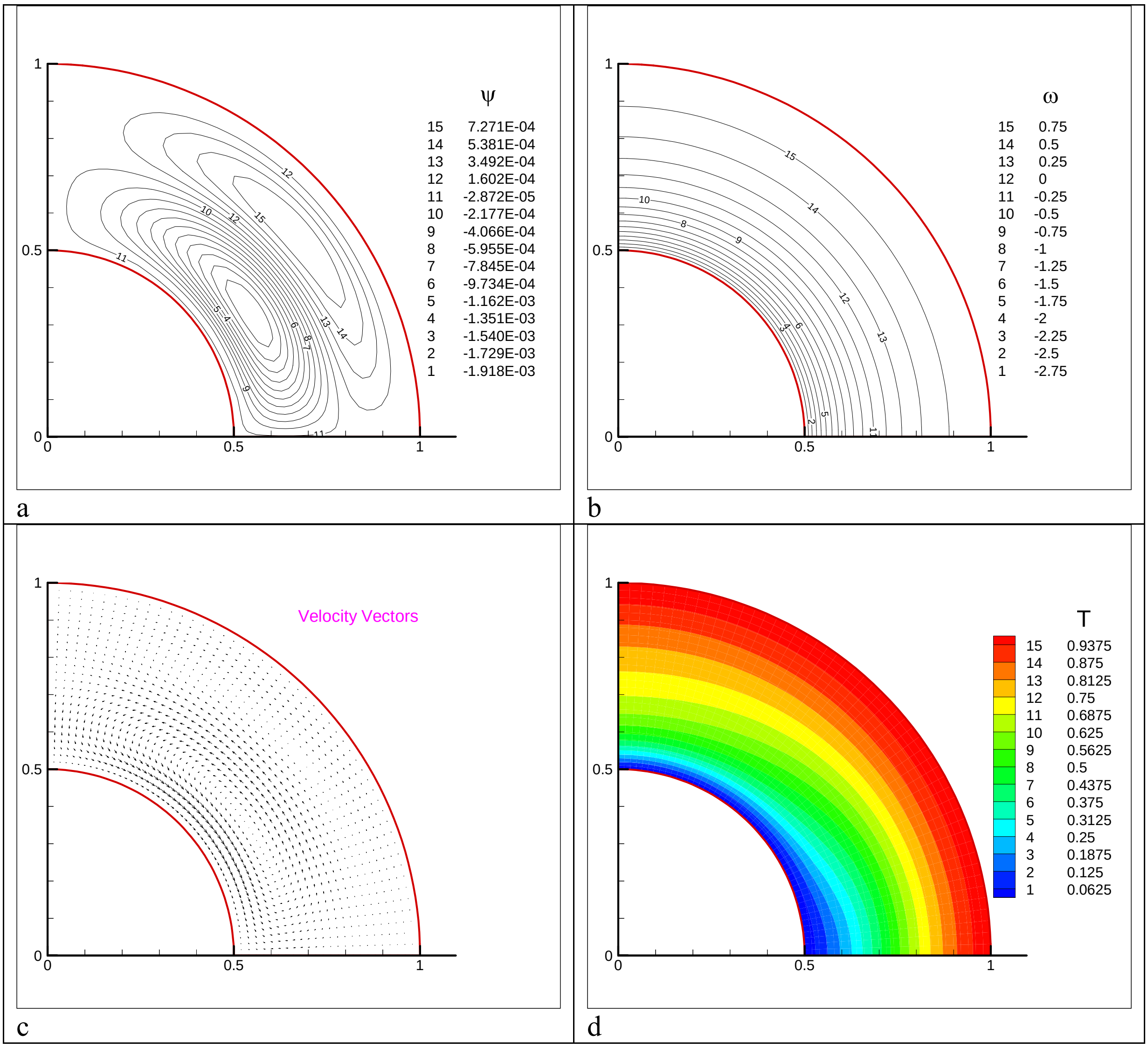


Fig- 4 : **Flow and heat patterns** for Re=50, Pr =10, Ek=0, Ω_{io} = -3

Figures 5-7 are the same situations but for the case $\Omega_{io} = -1$ and for Re=100, Pr=1, and Ek=0; for the case Ω_{io} = -2 and Re=250, Pr=10, and Ek=0; and for the case of Ω_{io} = -1 and Re=500, Pr=1, and Ek=0. The effect of Reynolds number can be seen obviously in these figures in comparison with each other and also compared with Figure 4. A detailed physical discussion regarding the flow field and heat transfer characteristics can be presented using these figures.

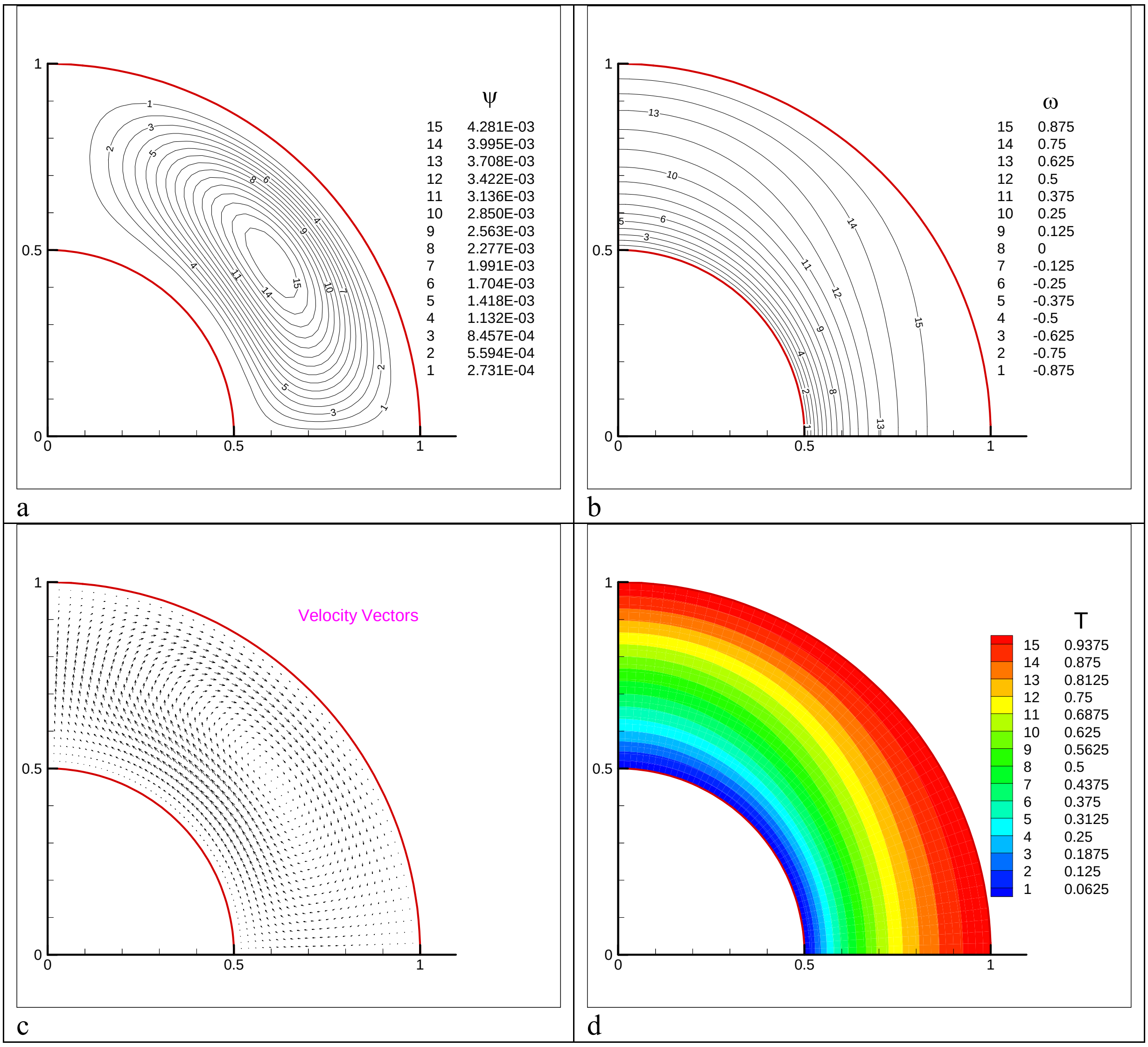


Fig- 5 : **Flow and heat patterns** for Re=100, Pr =1, Ek=0, Ω_{io} = -1

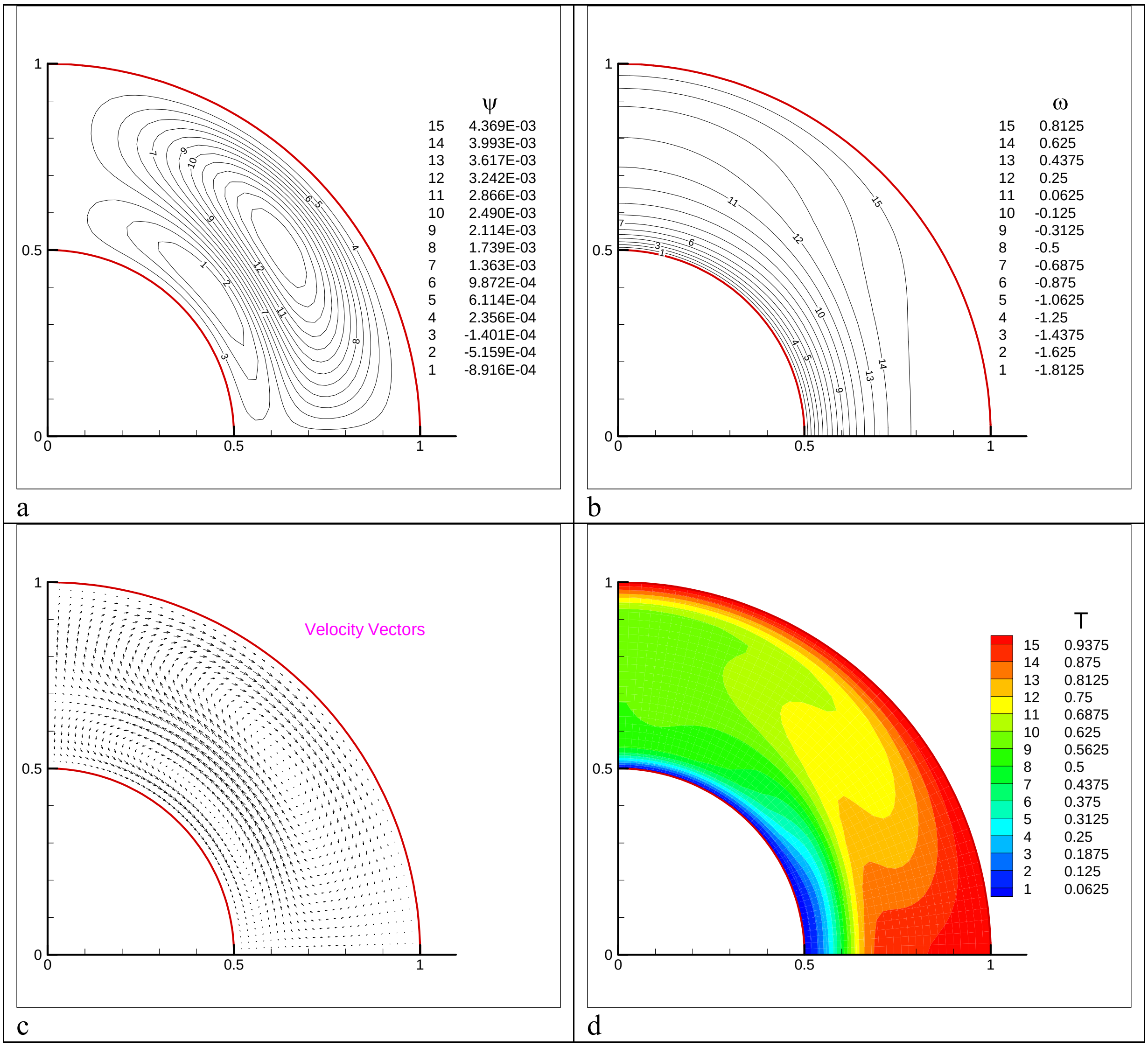


Fig- 6 : **Flow and heat patterns** for Re=250, Pr =10,Ek=0, Ω_{io} = -2

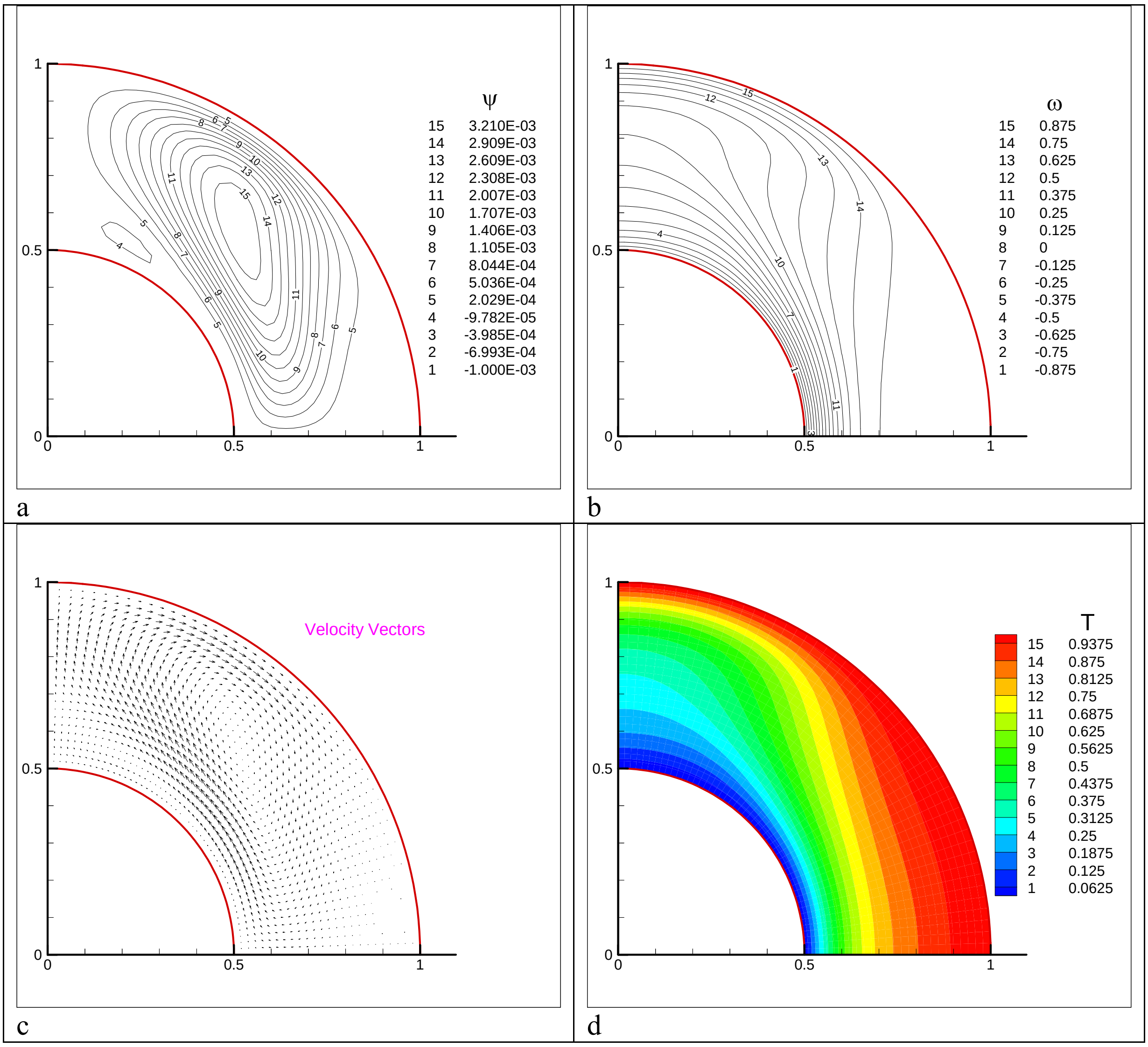


Fig- 7: **Flow and heat patterns** for Re=500, Pr =1, Ek=0, Ω_{io} = -1

6- Conclusions

A similarity solution for the problem of flow and heat transfer of a viscous incompressible fluid within a rotating spherical annulus has been investigated when the spheres have constant angular velocities. The results, aside from energy equation, are the same as the previous works, [Pearson (1967) and Munson et al. (1971)], for Navier-Stokes equations but with less computational complexities. Besides, formulation of the problem this way enables one to find the functions $\psi(r,\theta)$ and $\Omega(r,\theta)$ at any desired point within the flow field independently without having to solve for the whole region.

Appendix

The Maple software has been used to do all the algebra and produce coefficients in Equations 12-14 in a compact form which have been presented in the following:

$$\lambda = (-1 + r - b\sin^2\theta)$$

$$\gamma = (-1 + r - b + b\cos^2\theta)$$

$$c_1 = 2\cos\theta\sin^2\theta(1 - b\cos^2\theta)\lambda^2$$

$$c_2 = \sin^2\theta\{-b^2r^2\cos^6\theta + b(2+b)r^2\cos^4\theta + [4r^4 - 8(1+b)r^3 + (3+14b+4b^2)r^2 \\ - 8b(1+b)r + 4b^2]\cos^2\theta + r^2\}$$

$$c_3 = -2\lambda\{b[2r^2 - (1+b)r + b]\cos^4\theta + [5r^3 - (8b+11)r^2 + (2b^2+12b+5)r - 5b - 2b^2]\cos^2\theta \\ - r^3 + (3+2b)r^2 - (1+3b+b^2)r + b(1+b)\}$$

$$c_4 = -2\lambda^4$$

$$d_1 = 2r^2\lambda^6\sin^2\theta\cos\theta\{br\sin^2\theta + 2r^2 - (3b+1)r + 2b\}$$

$$d_2 = -4r^2\cos\theta\lambda^8$$

$$d_3 = 4\sin^2\theta\lambda^3\{[2r^3b^2 - b^2(1+b)r^2 + rb^3]\cos^6\theta + [-2b(b+4)r^3 + b(3b+5-b^2)r^2 + \\ + b^2(5b-1)r - 4b^3]\cos^4\theta - [20r^5 - (65+47b)r^4 + (63+30b^2+129b)r^3 - (124b+5b^3 \\ + 20 + 79b^2)r^2 + b(13b^2+40+70b)r - 20b^2 - 8b^3]\cos^2\theta + 4r^5 - (21+11b)r^4 + \\ + (23+10b^2+37b)r^3 - (25b^2+8+37b+3b^3)r^2 + b(19b+12+7b^2)r - 4b^2(1+b)\}\cos\theta$$

$$d_4 = -4\lambda^4\{rb^3\cos^8\theta - [6r^2b^2 - 4b^2(1+b)r + 8b^3]\cos^6\theta - [22r^3b - 2b(21b+22)r^2 + \\ + b(70b+18b^2+23)r - 32b^2 - 24b^3]\cos^4\theta - [24r^4 - (65b+71)r^3 + (62+154b+58b^2)r^2 \\ - (112b+104b^2+20b^3+18)r + 48b^2 + 24b^3 + 24b]\cos^2\theta + 8r^4 - (23b+27)r^3 + \\ + (22+54b+22b^2)r^2 - (38b^2+6+37b+7b^3)r + 8b^3 + 8b + 18b^2\}\cos\theta$$

$$d_5 = -4\cos\theta\sin^4\theta\lambda^2[r^2b^2(3r^2 - 2(1+b)r + b)]\cos^6\theta + [4r^5b - b(11b+14)r^4 + \\ + 2b(4+3b^2+11b)r^3 - b^2(10+9b)r^2 + 4rb^3]\cos^4\theta + [-8r^6 + (24+22b)r^5 - \\ - (62b+21+20b^2)r^4 + (64b+60b^2+6+6b^3)r^2 + b^2(24+22b)r - 8b^3]\cos^2\theta \\ - 2r^5 + (1+4b)r^4 - 2b(b+3)r^3 + b(3+4b)r^2 - 2rb^2)$$

$$d_6 = 16\gamma^8\cos\theta$$

$$d_7 = 2r\lambda^2 \cos\theta \sin^4\theta (b\cos^2\theta - 1)\{b^2 r^2 \cos^6\theta - b(b+2)r^2\cos^4\theta - [4r^4 - 8(1+b)r^3 + \\ + (4b^2 + 3 + 14b)r^2 - 8b(1+b)r + 4b^2]\cos^2\theta - r^2\}$$

$$d_8 = 4\sin^2\theta\lambda^3\{r^2 b^3 \cos^8\theta + [5r^3 b^2 - b^2(3b+4)r^2 + rb^3]\cos^6\theta + [5r^4 b - b(7b+15)r^3 \\ + b(5+2b-b^2)r^2 + b^2(10b+6)r - 8b^3]\cos^4\theta + [-8r^5 + (19b+15)r^4 - (14b^2 + 39b \\ + 2)r^3 - (2 - 3b^3 - 41b^2 - 31b)r^2 - b(11b^2 + 7 + 37b)r + 8b^2 + 8b^3]\cos^2\theta + r^4 - \\ - (2b+6)r^3 + (b^2 + 4b + 2)r^2 - b(b+1)r\}\cos\theta$$

$$d_9 = 8r\lambda^3\{\cos^{10}\theta rb^4 + b^3[4r^2 - 4(1+b)r - b]\cos^8\theta + [6r^3 b^2 - b^2(15b+18)r^2 + \\ + b^2(6b^2 + 9 + 14b)r + 4b^4 + 4b^3]\cos^6\theta + [-14r^4 b + (63b + 28b^2)r^3 - (78b + 3b^3 + \\ + 102b^2)r^2 + (26b + 30b^3 + 120b^2 - 4b^4)r - 41b^2 - 32b^3 - 6b^4]\cos^4\theta + [10r^5 - \\ - (49 + 17b)r^4 - (2b^2 - 48b - 90)r^3 - (60b - 11b^3 - 30b^2 + 64)r^2 - (38b^3 - b^4 + 43b^2 - \\ - 16 - 42b)r + 14b^2 - 10b + 4b^4 + 28b^3]\cos^2\theta - 2r^5 + (7b+9)r^4 - (22 + 8b^2 + 27b)r^3 \\ + (16 + 34b + 3b^3 + 18b^2)r^2 - (16b + 4 + 2b^3 + 14b^2)r + 3b^2 + 2b - b^4\}$$

$$d_{10} = -8r\gamma^8$$

$$d_{11} = -4r\lambda\sin^2\theta\{[-4r^4 b^3 + (b^4 + b^3)r^3 - r^2 b^4]\cos^{10}\theta - [17r^5 b^2 - (30b^3 + 37b^2)r^4 + \\ + (7b^4 + 11b^2 + 31b^3)r^3 - (3b^4 + 7b^3)r^2 - 4rb^4]\cos^8\theta - [4r^6 b - (46b + 42b^2)r^5 - \\ + (82b + 154b^2 + 64b^3)r^4 - (129b^3 + 141b^2 + 23b^4 + 31b)r^3 + (39b^4 + 96b^3 + 47b^2)r^2 \\ - (28b^3 + 28b^4)r + 12b^4]\cos^6\theta + [36r^7 - (116b + 152)r^6 + (119b^2 + 432b + 223)r^5 + \\ - (131 + 614b + 419b^2 + 34b^3)r^4 + (638b^2 + 147b^3 - 5b^4 + 419b + 27)r^3 - (233b^3 \quad - b^4 \\ + 441b^2 + 103b)r^2 + (156b^3 + 4b^4 + 112b^2)r - 36b^3]\cos^4\theta - [12r^7 - (56 + 48b)r^6 + \\ + (78 + 190b + 72b^2)r^5 - (248b^2 + 48b^3 + 262b + 30)r^4 + (150b^3 + 335b^2 + 12b^4 + 2 + \\ + 159b)r^3 - (36b^4 + 30b + 199b^2 + 178b^3)r^2 + (36b^4 + 88b^3 + 40b^2)r - 12b^3(1+b)]\cos^2\theta \\ - r^5 + (2b+5)r^4 - (1 + 3b + b^2)r^3 + b(b+1)r^2\}$$

$$d_{12} = 4r\lambda^2\{-68r^3b + (38b+2)r^2 + (-28b^2 - 8b)r + [-4r^3b^3 + (4b^3 + 4b^4)r^2 + 2rb^4]\cos^{10}\theta$$
$$+[-15r^4b^2 + (24b^2 + 18b^3)r^3 + (8b^3 - 3b^4 - 3b^2)r^2 + (-20b^3 - 14b^4)r - b^4]\cos^8\theta$$
$$+[-22r^5b + +(132b + 112b^2)r^4 + (-114b^3 - 378b^2 - 198b)r^3 + (366b^2 + 16b^4 + 70b$$
$$+234b^3)r^2 + (-210b^3 - 12b^4 - 110b^2)r + 24b^4 + 74b^3]\cos^6\theta + [71r^6 + (-168b - 320)r^5$$
$$+(594b + 55b^2 + 510)r^4 + (-840b - 326 - 118b^2 + 82b^3)r^3 + (251b^2 - 190b^3 - 31b^4$$
$$+74 + 610b)r^2 + (-256b^2 + 52b^4 + 166b^3 - 160b)r - 42b^4 - 46b^3 + 71b^2]\cos^4\theta$$
$$+[-50r^6 + (240 + 154b)r^5 + (-584b - 146b^2 - 400)r^4 + (818b + 244 + 30b^3 + 434b^2)r^3 +$$
$$+(-86b^3 - 526b - 536b^2 - 52 + 12b^4)r^2 + (-22b^4 + 90b^3 + 298b^2 + 120b)r - 50b^2 - 34b^3$$
$$+16b^4]\cos^2\theta - 16r^5 + (34 + 50b)r^4 + (-14 - 58b^2)r^3 + (30b^3 + 66b^2)r^2 + (-26b^3$$
$$-6b^4)r + 3b^2 + 6b^3 + 3b^4 - \cos^{12}\theta\, r^2b^4 + 3r^6 - 12r^3b^3 + 18r^4b^2 + 3r^2b^4 - 12r^5b\}$$

$$d_{13} = -r\sin^4\theta\{b^2r^2\cos^6\theta - (2b + b^2)r^2\cos^4\theta - [4r^4 - 8(1+b)r^3 + (3 + 14b + 4b^2)r^2$$
$$-8b(1+b)r + 4b^2]\cos^2\theta - r^2\}^2$$

$$e_1 = \frac{(1 - b\cos^2\theta)\sin^2\theta}{\lambda^2}$$

$$e_2 = \frac{(r-b)(1-r)\sin 2\theta}{r\lambda^2}$$

$$f_1 = \frac{r^2\sin^4\theta(1 - b\cos^2\theta)^2 + \sin^2 2\theta[(r-b)(1-r)]^2}{r^2\lambda^4}$$

$$f_2 = \frac{2\sin^2\theta(1 - b\cos^2\theta)(1 + b\sin^2\theta)}{r\lambda^3} + \frac{2(r-b)(1-r)[(\lambda + 2b\sin^2\theta)\cos 2\theta - b\sin^4\theta]}{r^2\lambda^3}$$
$$+\frac{2(r-b)(1-r)\cos^2\theta}{r^2\lambda^2}$$

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