Dynamics Modeling of 3-PSP Parallel Manipulator Using Natural Orthogonal Complement Method (NOC)

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Abstract

This paper presents the Dynamics modeling for a 3-dof special 3-PSP parallel manipulator using Natural Orthogonal method. The 3-PSP mechanism consists of two rigid bodies, a movable platform (formed like a star) and a fixed base that are connected to each other by means of three PSP legs. The manipulator could be driven with three independent actuators. Here we have decided to connect each motor to a ball screw that acts as the actuated prismatic joint for each leg. It could be concluded that the other twelve (nine revolute for three spherical and three upper prismatic) joints are functioning as passive joints. By differentiating the main structural coupled constraint equations to active and passive joints the Joint Orthogonal Complement matrix should be driven. This matrix derives all the passive joints velocities in the terms of active joint velocities. Considering the twist of each rigid body and the Joint Orthogonal Complement, the Natural Orthogonal matrix will be resulted. The use of this matrix will assist us to derive the equations of motion for this manipulator. The derived equations could be either used in Inverse or Direct simulations of the dynamics for this manipulator.

Keywords: Dynamics analysis, 3-PSP, Parallel Manipulator, Natural Orthogonal Complement (NOC) method.

Introduction

The dynamics of multi-body mechanical systems is a research area with applications in many engineering fields such as aerospace, vehicles, robotics, etc [1]. Also it is a classical subject and has been accomplished to a great extent [2-6]. its application to robotics is still a subject of current research of manipulator dynamics deals with the actuator forces/torques exerted on a manipulator and resulting motion of the manipulator. This usually referred to engineers as dynamics modeling.

Several fundamental formulations of equations of motion are used for the modeling, such as the Newton-Euler equations, the Euler-Lagrange equations, the Gibbs-Appel equations, Kane's equations, d'Alemberts principle, etc. Each formulation has its own advantages and disadvantages when applied to manipulators.

The commonly used formulation is the well-known Newton-Euler formulation introduced by Newton and Euler over two centuries ago. Early work applying the Newton-Euler equations to multi body system can be found [7]. In the Newton-Euler formulation, all forces, whether working or not associated with each rigid body, have to be examined. Six scalar equations govern the motion of each rigid body and hence, the total number of equations for the whole system will be very large. Since equations are associated with individual bodies and are simply related by constraint forces, they are particularly suitable for recursive computations if the system is of the serial or tree-type. This is why it has been widely accepted that the Newton-Euler equations can lead to the most efficient inverse dynamics algorithms. However, this fact is true only for serial or tree-type manipulators. For parallel manipulators, due to the presence of closed kinematic chains, recursiveness is lost, and hence, Newton-Euler equations can no longer be solved for constraint and external forces recursively.

Another commonly used formulation is the Euler-Lagrange formulation introduced by Lagrange in eighteen century. Early work using the Euler-Lagrange formulation in dynamics of multibody systems can be found in [2]. The Euler- Lagrange equations have several advantages. Such as the use of generalized coordinates instead of Cartesian coordinates, excluding nonworking forces and the derivation based on a scalar quantity-energy. Since the governing equations are associated with generalized coordinates and forces. They are very compact when applied to parallel manipulators. The Euler-Lagrange equations should include Lagrange multipliers because of the presence of dependent coordinates. Solving for Lagrange multipliers is equivalent to solving for nonworking constraint forces.

Another formulation which has been used by a few researchers is based on Kane's equations [8]. This method amount to d'Alernberts principle in Lagrangian form, namely using generalized coordinates and free of nonworking forces. When used to model spacecraft and non-holonomic systems, Kane's equations seem superior to other commonly used formulations [4,9]. In fact, Kane's equations have the advantages of both the Euler-Lagrange and the Newton-Euler equations. For example, Kane's equations are in Lagrangian form and, hence, have the advantages of the Lagrange equations, while the presence of the generalized inertia force eliminates tedious partial differentiation and facilitates the implementation of computational recursion [10]. It has been shown that Kane's equations can lead to algorithms as efficient as the Newton-Euler equations [9, 1 and 12].

Angeles and Lee introduced a formulation method called the method of the Natural Orthogonal Complement [13]. With this method, a set of Euler-Lagrange equations, free of constraint forces, is derived from the power equations of all individual links using the natural orthogonal complement of the coefficient matrix of the velocity constraint equations. Using this method, the equations of motion of a system can either be written in Kane's form (generalized inertia force in Cartesian spaces), which is suitable for recursive computations or in Euler-Lagrange's form (completely in joint space), which is suitable for integration. Moreover, it has been shown that this formulation method can lead to the most efficient algorithms for serial and parallel manipulators [10, 12, 14 and 15].

In this paper a 3-PSP parallel manipulator will be considered for deriving it's equations of motion using NOC method. Referring to [16] that brings a complete procedure of NOC method, this parallel manipulator with novel design will be processed for finding the equations of motion. The resulting dynamics model is in the form of Euler-Lagrange equations without involving Lagrangian multipliers.

Manipulator's structure

In this paper inverse dynamic of special kind of 3-PSP parallel manipulator is studied. The manipulator consists of three legs. Each leg is made of one actuated prismatic joint and a passive spherical joint which itself is paired with a passive prismatic joint. Industrial model of this manipulator type is shown in "Figure 1". To calculate the degrees of freedom of the system we must find the number of 1-dof joints and the number of movable rigid bodies [16]. This results in three degrees-of-freedom and two independent kinematic loops "Figure 2" and "Figure 3" [17].

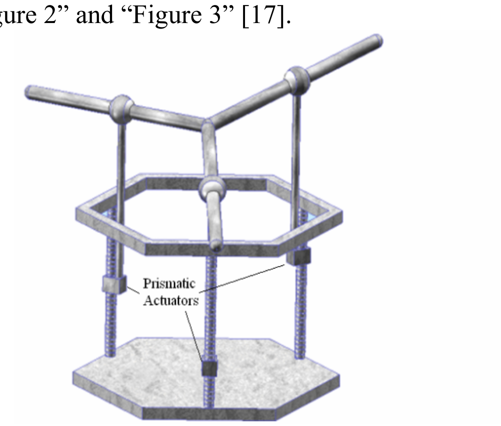

Figure 1: The special model of the 3-PSP structure

The movable platform is illustrated as a welded three bar in the form of a planar star. Legs of the star platform make 120 degrees with each other. The three degrees-of-freedom of the manipulator are manifested by changes in z-height of point P and changes in orientation of the moving star about x and y axes of frame {B} which is shown in "Figure 2". There is one actuated prismatic joint for each link and one passive spherical joint which is paired with a passive prismatic joint. The movable platform is illustrated as a welded three bar in the form of a planar star. Legs of the star platform make 120 degrees with each other. By extending or descending each actuated prismatic joint we could define position and orientation of a frame, {T}, attached to the center of the star platform with respect to fixed frame, {B}, attached to fixed base. The base platform is chosen to be an equilateral triangle. This creates a rather symmetrical shape for the 3-PSP robot. This symmetrical structure helps arrive at constraint equations that are more algebraically similar. Therefore, it will be easier to solve them manually. However, it is completely acceptable if the structure has no symmetry. Our solution method stated in this paper, can still be applied.

Because the prismatic actuators are welded to the fixed base, the manipulator could not have any rotation about z-axis of the base frame. The three degrees-of-freedom of the manipulator are manifested by changes in z-height of point P and changes in orientation of the moving star about x and y axes of frame {B}.

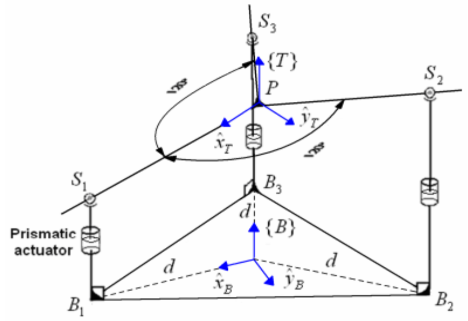

Figure 2: structure of special 3-PSP parallel manipulator

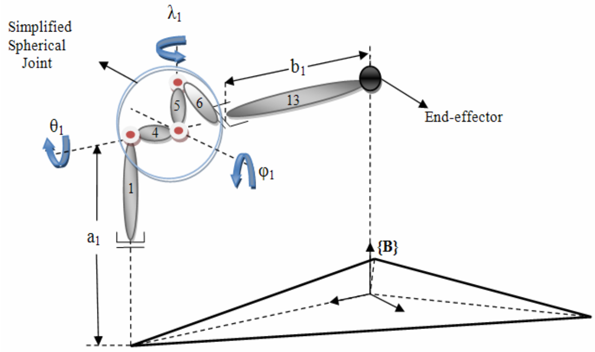

Figure 3: First leg's movable rigid bodies' specification

Manipulator's Kinematics

For the direct kinematics, unlike traditional methods which use constraint equations and numerical methods, a novel approach is used to formulate the direct kinematics problem. The approach uses relatively simpler geometric relations and results in a closed form solution with unique answer. Manipulator's structural properties also lead us to formulate nine coupled trigonometric constraint equations that are utilized in inverse kinematics analysis. Additionally, two relevant inverse kinematics formulations are investigated. The first formulation uses xyz coordinate of tool that leads to an exact solution with multiple answers. The second formulation uses orientation of the platform as well as its z coordinate which also leads to an exact solution with unique answer [17].

Manipulator's Dynamics

A joint coordinate is defined as the rotational or the translational displacement of a one-dof joint, which is used to describe the relative configuration of two successive bodies connected by this joint. We must introduce fifteen joint coordinates for each of the joints that are implemented in each legs of this manipulator. Except for open kinematic chains, joint coordinates, in general, are not all independent from each other. In other words, some of the joints have independent coordinates and the others have dependent coordinates. Accordingly, the joint-position vector q can be represented by,

$$q=\begin{Bmatrix} q_1 \\ q_2 \\ \vdots \\ q_r \end{Bmatrix}=\begin{Bmatrix} q^a \\ q^u \end{Bmatrix}, \qquad r=\{1,2,\cdots,15\}. \tag{1}$$

Since some of the joint coordinates are dependent upon the others, the joint coordinates are subjected to kinematic constraints. Based on the assumption that all kinematic pairs are holonomic, the aforementioned constraints can be expressed algebraically as:

$$\phi(q)=0 \tag{2}$$

Using the reference frames that are attached to each joint of the first leg we could derive corresponding constraint equations "Figure 4". The same pattern will be done for the other legs to obtain the main transformation matrices from base to the end-effector. The system has 3-dof and fifteen active and passive joints,

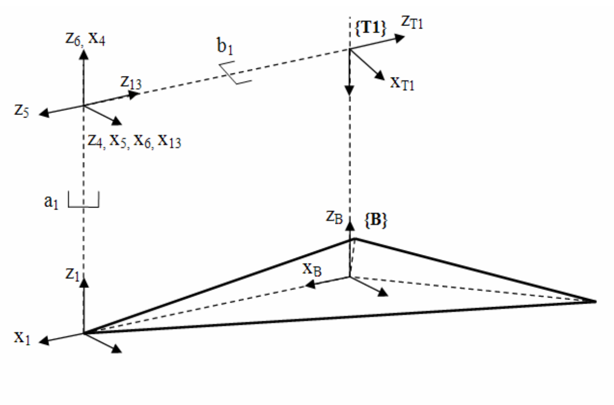

Figure 4: Positioning the main reference frames for a leg

Hence we need to derive twelve constraint equations to identify the connection of each joint with every existing rigid body. Using the Crag's notation for deriving the homogenous transformation matrices we can obtain a main transformation matrix for each leg [18],

$${}^{t}_{B}Ti=\begin{bmatrix} Ri & Pi \\ 0\ 0\ 0 & 1 \end{bmatrix}, \qquad i=\{1,2,3\}. \tag{3}$$

Where, Ri are the corresponding 3×3 rotation matrices and Pi are the related pure transformation vectors. Using the relations between each leg's Base frame and tool frame, we can conclude that,

$$\begin{cases} R1_{ii}=R2_{ii} \\ R1_{ii}=R3_{ii} \\ P1_{i4}=P2_{i4} \\ P1_{i4}=P3_{i4} \end{cases}, \qquad i=\{1,2,3\}. \tag{4}$$

As it is shown there are only twelve coupled trigonometric constraint equations that could be derived for this structure. The other important relation that was mentioned in advanced dynamics is the time derivatives of the constraint equations in the form bellow by defining,

$$\Phi=\frac{\partial\phi(q)}{\partial q} \tag{5}$$

And then we could proceed to,

$$\frac{\partial\phi(q)}{\partial t}=0 \Rightarrow \frac{\partial\phi(q)}{\partial q}\frac{\partial q}{\partial t}=\Phi\dot{q}=0 \tag{6}$$

Matrix Φ can be partitioned into two parts. One consisting of the columns of Φ which are associated with the independent joint velocities and the other consisting of the columns of Φ associated with the dependent joint velocities,

$$\Phi^a=\frac{\partial\phi(q)}{\partial q^a} \text{ and } \Phi^u=\frac{\partial\phi(q)}{\partial q^u} \tag{7}$$

The dimension of matrix Φ^a is 12×3 and that of Φ^u is 12×12 so it could be claimed that:

$$\frac{\partial\phi(q)}{\partial t}=\frac{\partial\phi(q)}{\partial q^a}\frac{\partial q^a}{\partial t}+\frac{\partial\phi(q)}{\partial q^u}\frac{\partial q^u}{\partial t}=\Phi^a\dot{q}^a+\Phi^u\dot{q}^u=0 \tag{8}$$

Hence,

$$\dot{q}^u=-(\Phi^u)^{-1}\Phi^a\dot{q}^a \tag{9}$$

It has been shown that the dependent joint velocities can always be expressed in terms of the independent joint velocities. From the joint velocity constraint equations and the relation between independent and dependent joint velocities we can derive the following relation [16],

$$\dot{q}=L\dot{q}^a \tag{10}$$

Where,

$$L=\begin{bmatrix} I \\ -(\Phi^u)^{-1}\Phi^a \end{bmatrix} \tag{11}$$

Where L is a 15×3 matrix and I is a 3×3 identity matrix.

Twist and Wrench vectors

In order to describe the velocity field the concept of twist is used,

$$t = \begin{bmatrix} \omega \\ v \end{bmatrix} \tag{12}$$

Where ω and v are the angular velocity vector and the translational velocity vector of the considered body. Since the translational velocity is associated with a particular point of the rigid body, the twist t is also associated with that point. This point is then called the reference point of the twist. Twists for the rigid bodies of the first leg are derived as below "Figure 3",

$$t_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ v_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \dot{a}_1 \end{bmatrix}, \quad t_4 = \begin{bmatrix} 0 \\ \omega_y \\ 0 \\ 0 \\ 0 \\ v_z \end{bmatrix} = \begin{bmatrix} 0 \\ \varphi_1 \\ 0 \\ 0 \\ 0 \\ \dot{a}_1 \end{bmatrix}, \quad t_5 = \begin{bmatrix} \omega_x \\ \omega_y \\ 0 \\ 0 \\ 0 \\ v_z \end{bmatrix} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\varphi}_1 \\ 0 \\ 0 \\ 0 \\ \dot{a}_1 \end{bmatrix},$$

$$t_6 = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \\ 0 \\ 0 \\ v_z \end{bmatrix} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\varphi}_1 \\ \dot{\lambda}_1 \\ 0 \\ 0 \\ \dot{a}_1 \end{bmatrix}, \quad t_{13} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\varphi}_1 \\ \dot{\lambda}_1 \\ \begin{bmatrix} \dot{\theta}_1 \\ \dot{\varphi}_1 \\ \dot{\lambda}_1 \end{bmatrix} \times \begin{bmatrix} P1_{14} - Ps_{14} \\ P1_{24} - Ps_{24} \\ P1_{34} - Ps_{34} \end{bmatrix} + R1 \begin{bmatrix} -\dot{b}1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{a}_1 \end{bmatrix} \end{bmatrix}$$

Where $\dot{\theta}_1$, $\dot{\varphi}_1$, $\dot{\lambda}_1$ are the rotational velocities about x , y, z accesses correspondingly. $P1_{i4}$ are the products of homogenous transformation matrices from the base frame to the last frame of the end-effector, referred to the first leg. Ps_{i4} are the products of homogenous transformation matrices from the base frame to the last frame of the spherical joint referred to the first leg.

There are two basic quantities in dynamics, namely force and moment. These two quantities are physically different. In order to unify the treatment for both, the concept of wrench is used. Wrench is defined as the 6-D vector containing both the force and the moment exerted on the mass center of a rigid body. Mathematically, it is defined as,

$$w = \begin{bmatrix} f \\ n \end{bmatrix} \tag{13}$$

Where f denotes a force applied at the mass centre and n denotes a moment acting onto the associated rigid body at hand. Note that this definition is compatible with the definition of twist. The inner product of twist and wrench yields the power transmitted by the wrench w to the rigid body moving at twist t. Like the concept of twist, the introduction of wrench encloses the forces and moments in one frame, which will ease dynamic formulation. As well, a force exerted on a rigid body can always be considered as an equivalent wrench exerted at the mass center of the rigid body.

A twist of a rigid body is a 6-D vector defined as,

$$t_i = \begin{bmatrix} \omega_i \\ \upsilon_i \end{bmatrix} = K_i \dot{q} \tag{14}$$

Where ω_i and υ_i are the angular velocity vector and the translational velocity vector of this body. The generalized twist vector will be expressed as,

$$t = [t_1 \quad t_2 \quad \cdots \quad t_r] \tag{15}$$

Hence, the overall twist of all the rigid bodies of the system is defined by,

$$t = K\dot{q} \tag{16}$$

Where t is a 78×1 vector and K is a 78×15 matrix. With the use of the twist vector of each rigid body, we can derive the main K matrix with the use of the equation (14). Using equations (10) and (16) we can find the Natural Orthogonal Complement,

$$t = KL\dot{q}^a \Rightarrow t = T\dot{q}^a \tag{17}$$

So,

$$T = KL \tag{18}$$

Where T denotes the 78×3 orthogonal complement matrix. The NOC is defined as a linear transformation which maps independent joint velocities into generalized twist of system. This type of orthogonal complement is very useful for dynamic simulation of parallel manipulators. In that case, the twist of the whole system is expressed in terms of the twist of the end-effector using such an orthogonal complement. With this method the simulation can be performed in Cartesian space. Hence, the time-consuming direct kinematics will not be involved.

Referring to [16], we can map the inertia wrench of the ith body as below,

$$W_i^* = \begin{bmatrix} n_i^* \\ f_i^* \end{bmatrix} = -M_i \dot{t}_i - \Omega_i M_i t_i \tag{19}$$

Where M_i is referred to each rigid body's mass matrix and Ω_i is referred to rigid body's rotation velocities matrix,

$$M_i = \begin{bmatrix} I_i & 0 \\ 0 & m_i 1 \end{bmatrix}, \quad \Omega_i = \begin{bmatrix} \omega_i \times 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{20}$$

Where I_i is 3 x 3 inertia tensor of the ith body about its mass center and 1 is the 3×3 identity matrix.

Dynamics modeling

The generalized driving force is a vector consisting of the driving forces and torques of all actuated joints:

$$\tau = [\tau_1^a \quad \tau_2^a \quad \cdots \quad \tau_n^a] \tag{21}$$

The inertia force and the moment can be calculated in the following form:

$$f_i^* = -m_i \ddot{c}_i$$
$$n_i^* = -\omega_i \times I_i \omega_i - I_i \dot{\omega}_i \quad (22)$$

The power terms consisting power supplied by the actuators (π^a), the power associated with the inertia (π^*), the power generated by gravity (π^g) and the power dissipated by friction (π^f) could be expressed here,

$$\pi^a = \dot{q}^a . \tau^a = (\dot{q}^a)^T \tau^a,$$
$$\pi^* = \sum_{i=1}^{r} t_i . w_i^* = \sum_{i=1}^{r} t_i^T w_i^* = t^T w^*,$$
$$\pi^g = \sum_{i=1}^{r} t_i . w_i^g = \sum_{i=1}^{r} t_i^T w_i^g = t^T w^g, \quad (23)$$
$$\pi^f = \sum_{i=1}^{r} t_i . w_i^f = \sum_{i=1}^{r} t_i^T w_i^f = t^T w^f.$$

The power equations of all individual links can be derived using the sum of the used and gained powers in the whole system. By using the NOC the generalized twist appearing in the above equations can be represented in the terms of the joint velocities, namely:

$$\pi^* = t^T \omega^* = (\dot{q}^a)^T T^T w^*,$$
$$\pi^g = t^T w^g = (\dot{q}^a)^T T^T w^g, \quad (24)$$
$$\pi^f = t^T w^f = (\dot{q}^a)^T T^T w^f.$$

With the consideration of the inertia wrenches on the system we can treat the dynamic system as a static system according to d' Alemberts principle. In a static system, the sum of all these powers should vanish,

$$\pi^a + \pi^* + \pi^f + \pi^g = 0 \quad (25)$$

In which the power generated by the joint constraint forces and constraint torques are not included because they vanish identically. Equation above can be rewritten in more detail as,

$$(\dot{q}^a)^T \tau^a + (\dot{q}^a)^T T^T w^* + (\dot{q}^a)^T T^T w^g + (\dot{q}^a)^T T^T w^f = 0 \quad (26)$$

All components of vector $\dot{q}^a$ are independent so, we derive,

$$\tau^a = -T^T w^* - T^T w^g - T^T w^f = -T^T (w^* + w^g + w^f) \quad (27)$$

This is the dynamics model applicable for any holonomic multibody system. Notice that the dynamics model given in equation above is already in the form of the solution of the inverse dynamics problem if the multi body system is a manipulator. In other words, to solve the inverse dynamics problem of a manipulator, one only needs to evaluate the right hand side of the above equation numerically.

In the dynamics model, the generalized inertia wrench w^* can be evaluated based on equation (19), which is straightforward if the twist and its time derivative of each body are known. The evaluation of the generalized gravity wrench is rather simple, as expressed bellow:

$$w^g \equiv \begin{bmatrix} w_1^g \\ w_2^g \\ \vdots \\ w_r^g \end{bmatrix} = \begin{bmatrix} 0 \\ m_1 g \\ 0 \\ m_2 g \\ \vdots \\ 0 \\ m_r g \end{bmatrix} \quad (28)$$

Where 0 denotes the 3-D zero vector and g is the gravity acceleration vector. Obviously, g is a 3-D constant vector in a base coordinate frame.

Finally, the evaluation of the friction wrench (w^f) could be neglected for simplicity. We achieve an alternative dynamics model namely,

$$\tau^a = -T^T w^* - T^T w^g = -T^T (w^* + w^g) \quad (29)$$

To obtain such a form, we just substitute the expression of the inertia wrench w^* as given, into the above mentioned dynamics model. After some algebraic manipulations, we obtain the following new form,

$$T^T M \dot{t} + T^T \Omega M t - T^T w^g = \tau^a \quad (30)$$

Where M and Ω are block-diagonal matrices, defined as:

$$M \equiv diag(M_1, M_2, \cdots, M_r),$$
$$\Omega \equiv diag(\Omega_1, \Omega_2, \cdots, \Omega_r). \quad (31)$$

Apparently, the dimension of the both matrices is 6m x 6m. For manipulator applications, the dynamic model of a system is usually represented in joint space. Substituting equations that are introduced previously one obtains the model in terms of the independent joint coordinates as follows:

$$T^T M T \ddot{q}^a + (T^T M \dot{T} + T^T \Omega M T) \dot{q}^a - T^T w^g = \tau^a \quad (32)$$

Where it could be simplify:

$$I \ddot{q}^a + C \dot{q}^a - \tau^g = \tau^a \quad (33)$$

Where;
I: n×n generalized inertia matrix which is symmetric and positive-defined,
C: n×n matrix producing the coriolis and centrifugal torques,
τ^g: n-dimensional vector of the independent joint torques induced by gravity.

Numerical example for inverse dynamics

Here is a numerical example that uses the derived equations of motion for this manipulator. It must be noted that the rigid bodies of the other two legs have the same characteristics that are illustrated in "Table 1".

Table1: Manipulator's initial conditions

Joint number	Joint character	Joint Position (m) / orientation (deg)
1	a_1	0.5
2	a_2	0.5
3	a_2	0.5
4	θ_1	0
5	φ_1	0
6	λ_1	0
7	θ_2	0
8	φ_2	0
9	λ_2	0
10	θ_3	0
11	φ_3	0
12	λ_3	0
13	b_1	0.18
14	b_2	0.18
15	b_3	0.18

Table2: first leg's movable rigid bodies' specifications

Rigid bodies' reference number	Mass (kg)	Inertia matrix (kg/m^2)
1	0.5	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$
4	0	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$
5	0	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$
6	0	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$
13	1	$\begin{bmatrix}5.655*10^{-3}&0&0\\0&5.655*10^{-3}&0\\0&0&1.131*10^{-2}\end{bmatrix}$

Table3: motors' trajectories

Motor number	Motors trajectory
1	$\theta_1 = 5/3\sin(5t)$
2	$\theta_2 = 5/6\sin(2t)$
3	$\theta_3 = 0$

With the simplified characteristics for the manipulator rigid bodies in "Table 1" and "Table 2" the required torques for a considered trajectory is illustrated through "Figure 3".

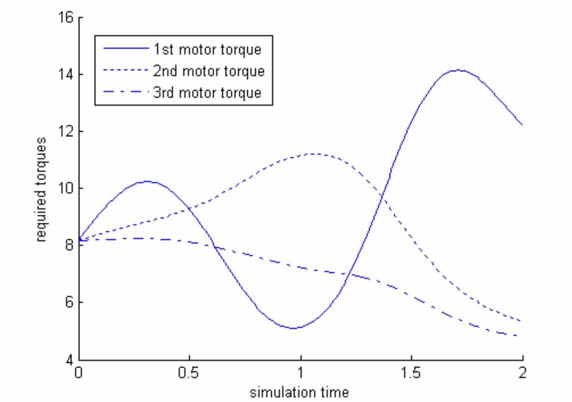

Figure 3: computed torques for a specified trajectory

The simulation time is 2 seconds and using the Matlab software and simulink toolbox we obtain the required torques for the desired trajectory.

Conclusion

This paper presents the Dynamics modeling for a 3-PSP parallel manipulator using Natural Orthogonal Complement. With this method, a set of Euler-Lagrange equations, free of constraint forces, is derived from the power equations of all individual links. It could be concluded that with the use of this method we can indirectly solve the direct and inverse kinematics of this manipulator. Moreover, the other exclusive advantage of this method is that there is no need to velocity or acceleration inversions. Finally a numerical example is designed as an illustration for checking the correctness of the obtained equations. Although this method has some problems dealing with big matrices algebraic operations, it could be used in continuous control procedures because of its fast and reliable computational capabilities.

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