Some Properties of the Schur Multiplier with Algebraic Topological Approach

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Abstract: In this talk, using a relationship between the Schur multiplier of a group $G$, the fundamental group, and the second homology group of the Eilenberg-MacLane space of $G$, we present new proofs for some famous properties of the Schur multiplier and it’s structure for the free product and the direct product, with an algebraic topological approach.

1 Introduction and Preliminaries

Let $F/R$ be a free presentation of a group $G$, then the Schur multiplier of $G$ is defined to be

$$M(G) = (R \cap F')/[R,F]$$

(see [2] for further details). There are some well-known facts about $M(G)$. By a theorem of Schur [2], with respect to the presentation of $G$, there exists an upper bound for the number of the generators of $M(G)$. In particular, if $G$ is a cyclic group, $M(G)$ is trivial [2]. A theorem of C. Miller [2], says that

$$M(G_1 * G_2) \cong M(G_1) \oplus M(G_2).$$

Also Schur Theorem [2] asserts that

$$M(G_1 \times G_2) \cong M(G_1) \oplus M(G_2) \oplus G_1 \otimes G_2.$$

In this talk, we are going to give some new proofs in algebraic topological methods for the above results. We suppose that the reader is familiar with some well-known notions in algebraic topology such as homotopy groups, homology groups, CW-spaces, and basic notion in group theory.

In order to find a suitable relation between the Schur multiplier and some famous notions of algebraic topology, we mention the following notes.
**Theorem 1.1** ( [6] ) For any group $G$, there exists a CW-complex $X$ with 
\[ \pi_1(X) \cong G \text{ and } \pi_n(X) = 1 \text{ for all } n \geq 2. \]

The space $X$ is called an Eilenberg-MacLane space of $G$.

**Theorem 1.2** (Hopf Formula [1]) If $K$ is a CW-complex with $\pi_1(K) = G$ and $F/R$ is a presentation of $G$ then 
\[ \frac{H_2(K)}{h_2(\pi_2(K))} \cong \frac{R \cap F'}{[R, F]}, \]

where $H_2(K)$ is the second homology group of $K$ and $h_2$ is the corresponding Hurewicz map. (see Theorem 1.4)

**Corollary 1.3** For any group $G$ and its Eilenberg-MacLane space, $K$ say, we have 
\[ \pi_1(K) \cong G \text{ and } H_2(K) \cong M(G). \]

**Remark** ( [4], [6] ) For any CW-complex $K$ with $k_i$ many $i$-cells, $d(H_i(K)) \leq k_i$. In addition, the number of 2-cells in an Eilenberg-MacLane space, obtained from a presentation of a group $G$, is equal to the number of its relators.

**Theorem 1.4** ([3], [5]) For any numbers $n, m \in \mathbb{N}$, there exists a CW-complex $L(n, m)$, called Lens space, with 
\[ \pi_1(L(n, m)) \cong \mathbb{Z}_n \text{ and } H_2(L(n, m)) = 1. \]

**Theorem 1.5** (Mayer-Vietoris Sequence [4]) For any two subspaces $X_1$ and $X_2$ of space $X$, with $X = X_1 \cup X_2$, there is an exact sequence as follows 
\[ \cdots \to H_n(X_1 \cap X_2) \to H_n(X_1) \oplus H_n(X_2) \to H_n(X) \to H_{n-1}(X_1 \cap X_2) \to \cdots. \]

**Theorem 1.6** (Hurwicz Theorem [4]) For any path connected space $X$, there exists an isomorphism called Hurewicz map, as follows 
\[ \frac{\pi_1(X, x_0)}{(\pi_1(X, x_0))'} \cong H_1(G). \]

**Theorem 1.7** (Kunneth Formula [4]) For any pair of topological spaces $X$ and $Y$ and for every integer $n \geq 0$, we have the following relation between their homology groups 
\[ H_n(X \times Y) \cong \sum_{i+j=n} H_i(X) \otimes H_j(Y) \oplus \sum_{p+q=n-1} \text{Tor}(H_p(X), H_q(Y)). \]
2 Main Results

Theorem 2.1 For any group \( G \) and it's arbitrary presentation with \( k \) relations, we have an upper bound for the number of the generators of it's Schur multiplier, as follows

\[
d(M(G)) \leq k
\]

Proof. First, suppose that \( K \) is the Eilenberg-MacLane space of \( G \), then using the remark 1, the number of 2-cells in the complex \( K \) equals exactly to \( k \) and consequently \( d(H_2(K)) \leq k \). Finally by Hopf Formula for the Eilenberg-MacLane space \( K \), \( M(G) \cong H_2(K) \) and so the result holds. \( \Box \)

Theorem 2.2 The Schur multiplier of any cyclic group \( G \) is trivial.

Proof. If \( G \) is an infinite cyclic group, we can consider a circle \( S^1 \) as it's Eilenberg-MacLane space ( \( S^1 \) is a CW-complex whose fundamental group is infinite cyclic and it's higher homotopy groups are trivial [4]). Hence using the Hopf isomorphism \( M(G) \cong H_2(S^1) \) and the fact of \( S^1 \) whose second homology group is trivial, in this case, the result is satisfied.

Also for a finite cyclic group \( G \) of order \( n \), using the theorem 1.2, we have the Lens space \( L(n,1) \) as an Eilenberg-MacLane space of \( G \). So by a similar argument to the above and Theorem 1.2, the proof is completed. \( \Box \)

Theorem 2.3 For any two groups \( G_1 \) and \( G_2 \), we have the isomorphism

\[
M(G_1 \ast G_2) \cong M(G_1) \oplus M(G_1).
\]

Proof. First, using the theorem 1.1, let \( K_1 \) and \( K_2 \) be the Eilenberg-MacLane spaces of \( G_1 \) and \( G_2 \), respectively. By Van-Kampen Theorem for the fundamental group of wedge space, \( \pi_1(K_1 \vee K_2) \cong \pi_1(K_1) \ast \pi_1(K_2) \). Also using the definition, \( \pi_n(K_1 \vee K_2) = 1 \), for all \( n \geq 2 \). Hence the wedge spaces \( K_1 \vee K_2 \) can be considered as an Eilenberg-MacLane spaces of \( G_1 \ast G_2 \) and with respect to the Hopf Theorem, we have

\[
M(G_1) \cong H_2(K_1), \ M(G_2) \cong H_2(K_2)
\]

\& \( M(G_1 \ast G_2) \cong H_2(K_1 \vee K_2) \).

Finally, by using the Mayer-Vietories sequence for the wedge space \( K_1 \vee K_2 \), we conclude the following isomorphism which completes the proof

\[
H_2(K_1 \vee K_2) \cong H_2(K_1) \oplus H_2(K_2). \Box
\]
THEOREM 2.4 For any two groups $G_1$ and $G_2$, we have the relation

$$M(G_1 \times G_2) \cong M(G_1) \oplus M(G_2) \oplus (G_1)_{ab} \otimes (G_2)_{ab}.$$ 

Proof. Similar to the previous proof, suppose that $K_1$ and $K_2$ are the Eilenberg-MacLane spaces of $G_1$ and $G_2$, respectively. Using one of the properties of the homotopy functor $\pi_n$ to preserve the direct product, we have $\pi_1(K_1 \times K_2) \cong \pi_1(K_1) \times \pi_1(K_2)$ and $\pi_n(K_1 \times K_2) = 1$, for all $n \geq 1$.

So the space $K_1 \times K_2$ is an Eilenberg-MacLane spaces of $G_1 \times G_2$. Hence by Hopf isomorphism, we have $M(G_i) \cong H_2(K_i)$, for $i = 1, 2$, and

$$M(G_1 \times G_2) \cong H_2(K_1 \times K_2).$$

Also using Kunneth Formula, the properties of the functor Tor and tensor product, and the fact of $H_0(X)$ which is isomorphic to the infinite cyclic group $\mathbb{Z}$ where $X$ is a path connected space (note that Eilenberg-MacLane spaces are path connected), we have the following relation between the first and the second homology groups,

$$H_2(K_1 \times K_2) \cong H_2(K_1) \oplus H_2(K_2) \oplus H_1(K_1) \otimes H_1(K_2)$$

Finally, by Hurewicz isomorphisms $H_1(K_i) \cong (\pi_1(K_i))_{ab} = (G_i)_{ab}$, for $i = 1, 2$, we conclude the result of the theorem. $\square$

References