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An Algorithm for Constructing Nonsmooth Lyapunov Functions for Continuous Nonlinear Time Invariant Systems

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Abstract: This paper presents an algorithm based on the Generalized Lyapunov Theorem (GLT) for constructing nonsmooth Lyapunov Function (LF) for nonlinear time invariant continuous dynamical systems which can be differentiable almost every-where. A new method is firstly defined that a neighborhood of the equilibrium point (origin) is partitioned into several regions by means of the coordinate hyperplans (axes) and system state equations (nullclines); hence, the number of regions is a function of number of system states. Then, this method selects a LF in each region by original nonlinear model of system, based on the several proposed analytical Notes. These Notes select LF's and solve continuity problem of them on the boundaries of regions in more cases. The existing methods that use piecewise model of system in each region for constructing piecewise LF are approximate and computational, but, the defined method is completely exact and analytic. The different steps of this method are proposed by means of a non-iterative algorithm for constructing a nonsmooth continuous Generalized Lyapunov Function (GLF) in whole neighborhood of the origin. The ability of this algorithm is demonstrated via a few examples for constructing LF and analyzing system stability.
Keywords: Stability analysis, Continuous nonlinear dynamical systems, Generalized Lyapunov theorem, Nonsmooth continuous Lyapunov functions.

چکیده: در این مقاله الگوریتمی بر اساس قضیه لیاپانوف تعمیم یافته، جهت ساخت توابع لیاپانوف ناهموار برای سیستمهای دینامیکی غیر خطی پیوسته نامتغیر بازمان که می تواند در برخی نقاط نامشتق پذیر باشد، ارائه می شود. ابتدا روشی جدید معرفی می شود که توسط آن، همسایگی نقطه تعادل (مبدا) توسط محورها و معادلات حالت سیستم به چندین ناحیه تقسیم می گردد، بنابراین تعداد نواحی تقسیم بندی شده تابعی از تعداد متغیرهای حالت است. سپس این روش با استفاده از مدل اصلی غیرخطی سیستم در هر ناحیه یک تابع لیاپانوف پیدا می کند، که انتخاب این توابع لیاپانوف و پیوسته نمودن آنها بر روی مرزهای نواحی مبتنی بر چند نکته تحلیلی می باشد. برخلاف سایر روشهای موجود که جهت ساخت توابع لیاپانوف تکه ای در هر ناحیه از مدل تکه ای سیستم استفاده می کنند و تقریبی و محاسباتی هستند، این روش کاملاً دقیق و تحلیلی است. مراحل ساخت یک تابع لیاپانوف تعمیم یافته پیوسته و ناهموار برای کل همسایگی مبدا بر اساس روش معرفی شده در قالب یک الگوریتم غیر تکرار شونده ارایه شده و توانایی آن جهت ساخت توابع لیاپانوف و تحلیل پایداری سیستمها با چند مثال نشان داده شده است.

کلمات کلیدی: تحلیل پایداری،سیستم دینامیکی غیرخطی پیوسته، قضیه لیاپانوف تعمیم یافته، تابع لیاپانوف تعمیم یافته پیوسته ناهموار

1- Introduction

Lyapunov theorem is used for system stability analysis, which is an important issue in nonlinear dynamical systems theory. A main advantage of this theorem is reduction of system stability analysis with several dimensional equations, to the study of a LF with one-dimensional equation. There is no a systematic approach to choose LF for any nonlinear system, and the choice of LF is not unique.

Several nonsmooth Lyapunov stability theorems are defined in the articles. These theorems can be classified in two main categories. The first category determines the generalized derivative of a nonsmooth LF on its nonsmooth surfaces, via differential inclusion or similar approaches. In these theorems, the important step is to verify the generalized derivative of a nonsmooth LF on its nonsmooth surfaces. For example, [5] defined a nonsmooth Lyapunov stability theorem for a class of

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nonsmooth Lipschitz continuous LF's using Filipov's differential inclusion and Clarke's generalized gradient. Based on the latter paper, [6] constructed the LF's for several complicated systems.

The second category of nonsmooth Lyapunov stability theorems does not determine the generalized derivative of a nonsmooth LF on its nonsmooth surfaces. These theorems analyze system stability without calculation of gradient vector to the system solutions. Several of these theorems are mentioned below.

[1] proved a GLT for nonlinear dynamical systems, in which, LF can be discontinuous except for the origin, so, all regularity assumptions are removed for the system dynamics and LF's. Our algorithm in this paper is proposed using this GLT.

equations, whose right hand side [illegible] not Lipschitz continuous, in general. For such systems, stability cannot be characterized in general by means of smooth LF's.

[8] defined another theorem for constructing weak GLF for time invariant continuous systems. In nonsmooth Lyapunov stability theorems, the LF's can be nonsmooth except for the origin. Therefore, based on these theorems, nonsmooth LF's can be constructed for both continuous and discontinuous systems.

A number of articles have dealt with the continuity type for LF, for example; [2] proved for nonlinear systems, which are at least continuous, that the existence of a continuous LF does not imply the existence of a locally Lipschitz continuous LF, and also the existence of a Lipschitz continuous LF doesn't imply the existence of continuously differentiable LF.

The nonlinear systems can be analyzed by partitioning the state space into several divisions. By this method, firstly, in each division a Piecewise Model (PM) of the original nonlinear system is selected, and using it, a LF is constructed in each region. After that, the constructed LF's under special conditions are combined, and a piecewise LF for the PM of the whole system is obtained. This method should finally prove this obtained piecewise LF is useful for stability analysis of original nonlinear system.

Various applications of this method have been reported in the literature; for example; [9] obtained a switching LF for a class of nonlinear continuous systems. It approximates a PM by a switching fuzzy model in each quadrant. The stability is analyzed via a derived Piecewise Quadratic (PWQ) LF for each region. The parameters of quadratic matrix are solved by Linear Matrix Inequalities (LMI).

Johansson and Rantzer defined a method for time invariant nonlinear systems with Piecewise Affine (PWA) dynamic model [3], [4]. In this method around the origin is divided into some polyhedral cells with pair-wise disjoint interior, then, a PWQ LF is computed in each of them. The search for a PWQ LF is formulated as a convex optimization problem in terms of LMI.

[10] defined a construction method of PWQ LF for a simplified Piecewise Linear (PWL) model of the original nonlinear system. This method divided around the origin into a lot of simplices, then, computes a PWQ LF in each division by means of [illegible]d.

[11] proposed an algorithm for constructing a PWA LF for nonlinear continuous time invariant ordinary differential equations in a family of simplices by linear programming.

[12] considered a parametric PWL model of nonlinear system, over a simplicial partitions in an area around the equilibrium point. It constructed a PWL LF using linear programming methods.

[13] computed global LF for nonlinear systems by means of radial basis functions.

All PM methods in above, have these disadvantages; they are approximate and computational, also, the result of the system analysis depends on the state space partitioning. To obtain sufficient resolution in the analysis, it is often necessary to refine an initial partition. Such refinements can be targeted towards increasing the accuracy of the model, or towards increasing the flexibility of the LF computations.

This paper describes a non-iterative algorithm, which is introduced for constructing nonsmooth continuous GLF for nonlinear time invariant continuous systems that can be differentiable almost every-where. The proposed algorithm is based on a GLT in [1].

This algorithm has three main stages. In the first stage, it defines a method in accordance to PM method for dividing neighborhood of the origin into several regions by means of coordinate hyperplans (axes) and state equations (nullclines), therefore, in this method the number of regions is a function of number of system states.

In the second stage, this method constructs a LF in each region by means of the original nonlinear model of system and several analytical Notes. Unlike the existing PM methods, that use an approximate model of nonlinear system, this method is completely exact. Also, the existing PM methods are computational, but this method is analytic.

In the final stage, it combines selected LF's and constructs a nonsmooth continuous GLF with a condensed formula based on a proved theorem.

The restrictions of this algorithm are; original selection of LF's for regions, and then, continuity of LF's on boundaries of regions, hence, the Notes are proposed to solve these restrictions in more cases.

In the next section, the mathematical framework for this paper is given. A new method for partitioning the neighborhood of the origin into several regions is presented in section 3. Section 4 explains construction of GLF. A proposed algorithm for obtaining GLF is defined in section 5. The capability of this algorithm is illustrated, when employed on two examples, in section 6.

2- Mathematical framework

Consider a nonlinear time invariant continuous dynamical system (1), which can be differentiable almost every-where.

$$\dot{x}=f(x(t)),\ t\geq 0,\ x(t)\in D\subseteq R^n,\ f:D\to R^n,\ f(0)=0\in D \quad (1)$$

where D is an open set and $x:T\subseteq R\to D$ is said to be a solution to (1) on the time interval T, providing $x(t)$ satisfies (1) for all $t\in T$. f is such that the solution $x(t)$ is well defined on $T=[0,\infty)$, that is, assume, for every $y\in D$ there is a unique solution $x(t)$ of (1) on T, such that $x(0)=y$, and all the solutions $x(t), t\geq 0$ are continuous functions of the initial conditions $x_0=x(0)\in D$ [1]. GLF is lower semi-continuous and differentiable almost every-where. Two definitions and a theorem are recalled from [1], below.

Definition 1 [1]: A function $V:D\to R$ is lower semi-continuous on D, if for every sequence $\{x_n\}_{n=0}^{\infty}\subset D$ such that, $\lim_{n\to\infty}x_n=x$, then $V(x)\leq \liminf_{n\to\infty}V(x_n)$.

Definition 2 [1]: A lower semi-continuous, positive definite function $V(x)$, which is continuous at the origin, and satisfies $V(x(t))\leq V(x(\tau))$ for all $t\geq\tau\geq 0$ is called a GLF.

Theorem 1 [1]: Consider the nonlinear dynamical system (1) and let, $x(t), t\geq 0$, denotes the solution to (1). Assume that, there exists a lower semi-continuous, positive-definite function $V:D\to R$ such that $V(x)$ is continuous at the origin and $V(x(t))\leq V(x(\tau))$ for all $t\geq\tau\geq 0$. Then the zero solution $x(t)\equiv 0$ is Lyapunov stable.

3. Partition method (definition of region)

Let, the coordinate hyperplans (axes) $x_i=0$, and nullclines $\dot{x}_i=f_i=0$, $i\in\{1,2,\ldots,n\}$, partition an open set $D\subset R^n$ in a neighborhood of the origin into several regions, where each region is denoted by R_j, $j\in\{1,2,\ldots,m\}$. Obviously, a common boundary of two neighboring regions is a coordinate hyperplan or a nullcline. If a nullcline is along a coordinate hyperplan, then the coordinate hyperplan is considered. The common vertex of all regions is the origin. Each region has n common boundaries $S_{ji}=R_j\cap R_{ji}$, which are the coordinate hyperplans or nullclines with its neighboring regions R_{ji}.

4. Construction of GLF

For constructing GLF for (1), a smooth LF is selected in each region; hence, each LF is non-increasing within its corresponding region. Moreover, if all neighboring LF's be equal on their common boundary, therefore, the condition $V(x(t))\leq V(x(\tau))$ for all $t\geq\tau\geq 0$ is satisfied, so, one can use theorem 1 for constructing GLF.

Assume, $v_j(x)$ in R_j satisfies (2),(3).

$$\forall j, v_j(x):R_j\subset D\subset R^n\to R, v_j(0)=0 \quad (2)$$

$$\forall x\in B_j=B\cap R_j:(v_j(x)>0\ \text{for}\ x\neq 0)\ \text{and}\ \dot{v}_j(x)\leq 0 \quad (3)$$

$$\forall x\in B_j=B\cap R_j:(v_j(x)>0\ \text{for}\ x\neq 0)\ \text{and}\ \dot{v}_j(x)<0 \quad (3')$$

Where B is an open set in neighborhood of the origin and $\forall j\ \ 0\in B_j\subset B\subset D$.

$$\forall x\in S_{jk}\cap B: v_j(x)=v_k(x) \quad (4)$$

If (4) is satisfied on all common boundaries of regions, then all neighboring LF's are continuous on their common boundary S_{jk}, in B.

Definition 3: A proposed parametric LF is a time-invariant smooth function $v_j(x)$, in R_j, that satisfy (2) and (3).

At first, $v_j(x)$ is often selected for more regions as $v_j(x)=\alpha|h_l(x)|, \alpha\in R^+$, where, $h_l, l=1,2$ is a time-invariant smooth function in R_j.

Definition 4: A proper LF is a proposed parametric LF which satisfies (4) at some common boundaries of its region.

Definition 5: A special LF is a proper LF which satisfies (4) on all common boundaries of its region, and its parameters are identified.

Definition 6: An orthant is the n-dimensional generalization of the two dimensional quadrant and three dimensional octant.

To construct GLF, the following steps must be carried out: firstly, proposed LF's are chosen for more regions. Secondly, using these proposed LF's, proper LF's are constructed. Special LF's are obtained by means of the proper LF's, and finally, a GLF is defined using the special LF's. Each kind of boundaries, which are the coordinate hyperplans, nullclines or both of them, will provided different relationships for LF's. An algorithm is proposed for constructing GLF.

5. Proposed algorithm

Please, trace each step with its corresponding step in examples, to illustrate algorithm.

Divide a neighborhood of the origin into several regions by the coordinate hyperplans and nullclines. Select the lowest order of all LF's equal together to satisfy (4) on the coordinate hyperplans, else, the continuity of GLF is not provided on them.

Select proper LF for regions, which are on either side of the nullclines that aren't along the coordinate hyperplans by the next Note.

Note 1: Let, a nullcline $S_{jk}:\dot{x}_i=0$, be the common boundary of R_j and R_k and it isn't along the coordinate hyperplans. Let, $|f_{i1}(x)|$ and $|f_{i2}(x)|$ be proposed LF's in R_j and R_k, respectively, such that, $f_{i1}(x)-f_{i2}(x)=f_i(x)=\dot{x}_i$ (5)

then, $v_j(x)=\alpha_{jk}|f_{i1}(x)|$ and $v_k(x)=\alpha_{jk}|f_{i2}(x)|, \alpha_{jk}\in R^+$ are proper LF's for R_j and R_k, because, $f_{i1}(x)-f_{i2}(x)=f_i(x)=\dot{x}_i$ and $\forall x\in S_{jk}\cap B\Rightarrow \dot{x}_i=f_i(x)=0$

so, $\forall x\in S_{jk}\cap B\Rightarrow f_{i1}(x)-f_{i2}(x)=0\Rightarrow|f_{i1}(x)|=|f_{i2}(x)|$, thus, $v_j(x)=v_k(x)$ and holds (4) true.

The previous step offers n LF's for a region whose all boundaries are nullclines which are not along the coordinate hyperplans. For such a region, compare the offered LF's with LF's of its neighboring regions, and then, for this region select a LF equal to one of the LF's of its neighboring regions.

Select proposed LF for a region whose boundaries are only coordinate hyperplans, by the next Note.

Note 2: Suppose, R_j is a region whose boundaries are the coordinate hyperplans $S_{ji}: x_i=0$, this region is an orthant. Let, LF's of all neighboring regions of R_j, v_{ji}, are selected by the previous steps. Therefore, each $v_{ji}(x)\big|_{x_i=0}$ is a LF for $S_{ji}: x_i=0$ in B.

To satisfy (4), $v_j(x)\big|_{x_i=0}=v_{ji}(x)\big|_{x_i=0}$ must be satisfied for all $S_{ji}: x_i=0$ in B. It imply that, (6) can satisfy (4) on all common boundaries of R_j by adding some statements with each $v_{ji}(x)\big|_{x_i=0}$ or a selection of appropriate parameters for LF's in the next step.

$$v_j(x)=\sum_{i=1}^{n} b_{ji}v_{ji}(x)\big|_{x_i=0}, b_{ji}\in R^+ \tag{6}$$

(when, $n=2$, if $b_{j1}=b_{j2}=1$, then $v_j(x)\big|_{x_1=0}=v_{j1}(x)\big|_{x_1=0}$, $v_j(x)\big|_{x_2=0}=v_{j2}(x)\big|_{x_2=0}$, hence, $v_j(x)=v_{j1}(x)\big|_{x_1=0}+v_{j2}(x)\big|_{x_2=0}$ satisfies (4) in R_j.) Moreover, if (6) satisfies (3) in R_j, then, (6) is a proper LF on this orthant.

If the following Note is satisfied on all coordinate hyperplans, then parameters of LF's and special LF's are specified

Note 3: Let, $R_j\subset Q_J$ and $R_k\subset Q_K$ be two neighboring regions, where Q_J and Q_K denote orthants and $S_{jk}:x_i=0$ is their common boundary. All LF's in Q_J and Q_K are already selected in previous steps.

Assume, $d_{jk}(x)=d_j(x)-d_k(x)$ (7)

Such that, $d_j(x)=v_j(x)\big|_{x_i=0}, d_k(x)=v_k(x)\big|_{x_i=0}$.

If a selection of appropriate parameters for $v_j(x)$ and $v_k(x)$ satisfies $d_{jk}(x)=0$, then these proper LF's are continuous on $S_{jk}: x_i=0$, and holds (4) true.

Else if, the lowest order of $v_j(x)$ and $v_k(x)$ is deleted in $d_{jk}(x)$ by a selection of appropriate parameters for them, then, the value of $d_{jk}(x)$ is smaller than the value of each LF in neighborhood of the origin (note that, the lowest order of all LF's for system must be equal).

Therefore, by adding $d_{jk}(x)$ with all LF's in Q_K (or $-d_{jk}(x)$ with all LF's in Q_J), new LF's in Q_J and Q_K are constructed, as the new constructed $v_j(x)$ and $v_k(x)$ are continuous on $S_{jk}: x_i=0$, and holds (4) true.

If this Note is repeated on all coordinate hyperplans, then, parameters of LF's and all special LF's may be identified.

If the special LF's for all regions are identified, then, construct a nonsmooth continuous GLF for system (1) by following Note.

Note 4: A nonsmooth continuous function is constructed by combination of the special LF's, $v_j(x), j\in\{1,\ldots,m\}$

$$V(x)=\sum_{j=1}^{m} v_j(x)\Psi_j(x) \qquad \Psi_j(x)=\begin{cases}1 & x\in B_j\\ 0 & x\notin B_j\end{cases} \tag{8}$$

where $\Psi_j(x)$ is a characteristic function.

(9) defines derivative of (8) almost every where.

$$\dot V_f(x)=\sum_{j=1}^{m} \dot v_j(x)\Psi_j(x) \qquad a.e. \tag{9}$$

According to theorem 2, $V(x)$ in (8) is a GLF, and the origin is (asymptotically) stable.

Theorem 2: Consider the nonlinear dynamical system (1). Let, $B\subset D$ be an open set, $0\in \mathring{B}$ where is the interior of B and $\mathring{B}_j$ be the interior of B_j, $0\in B_j\subset R_j$. Suppose, D is divided by the coordinate hyperplans and nullclines of system into several regions R_j.

If $V(x)$ in (8) which is constructed by the special LF's, satisfies (2) and (3) within all regions in B, and also, if it satisfies (4) on all common boundaries of the regions in B, then, (8) is a GLF for (1), and the origin is Lyapunov stable. Moreover, If all special LF's satisfy (3') within their regions, then, origin is asymptotically stable.

Proof:

* Since, $V(x)$ satisfies (2), $\forall j, v_j(0)=0 \Rightarrow V(0)=0$, therefore, $V(x)$ is continuous at the origin.

* Since, $V(x)$ satisfies (3),

$\forall x\in B_j\subset B: V(x)=v_j(x)>0 \text{ for } x\neq 0$, hence, $V(x)$ is a positive-definite function, also, $\forall x\in B_j, \dot v_j(x)\le 0$, but, $\forall x\in S_{jk}-\{0\}$, $V(x)$ is non-differentiable in general, therefore, $\dot V_f(x)$ isn't often defined on the boundaries. Thus, $\forall x\in \mathring{B}_j\subset B: \dot V_f(x)=\dot v_j(x)\le 0$, i.e. $V(x)$ is differentiable and non-increasing within all regions in B.

* Since, $\forall x\in S_{jk}\cap B: V(x)=v_j(x)=v_k(x)$ in (4), thus, $V(x(t))\le V(x(\tau))$ for all $t\ge\tau\ge 0$ for any $x_0\in B$.

* Since, $V(x)$ has a zero minimum value in B, for every sequence $\{x_n\}_{n=0}^{\infty}\subset B$, so, $\lim_{n\to\infty} x_n=0$. Hence, $\inf_{n\to\infty} V(x_n)$ exists and $\liminf_{n\to\infty} V(x_n)\ge V(0)=0$, i.e. $V(x)$ is a lower semi-continuous function.

* According to theorem 1, since, function $V: D\to R$ is lower semi-continuous, positive-definite and continuous at the origin, and moreover, $V(x(t))\le V(x(\tau))$ for all $t\ge\tau\ge 0$, (8) is a GLF for (1) in B and $x(t)\equiv 0$ is Lyapunov stable.

* Moreover, If each special LF's satisfies (3') within its region $\forall x\in \mathring{B}_j\subset B: \dot V_f(x)=\dot v_j(x)<0$, it means that $V(x)$ is decreasing within all regions in B, therefore, $\dot V_f(x)<0$ *a.e.* along the system solutions in B.

* In addition, the special LF's satisfy (4) on all common boundaries, $\forall x\in S_{jk}\cap B: V(x)=v_j(x)=v_k(x)$, so, $V(x(t))<V(x(\tau))$ for all $t\ge\tau\ge 0$ for any $x_0\in B$.

* $V(x)$ in (8) has a zero minimum value in B, if $t\to+\infty \Rightarrow V(x(t))\to 0$, that it means the origin is asymptotically stable.

(18)

6- Examples

In this section, the GLF is constructed for two systems. The stability of the origin in these examples is approved by simulation with MATLAB software. These examples demonstrate ability of the proposed algorithm for system stability analysis.

Example 1:

$$\begin{cases} \dot{x}_1 = f_1(x) = x_2 - x_1 \tan(x_2^2) \\ \dot{x}_2 = f_2(x) = (\mathrm{sgn}(x_1) - 2)x_1 - 4sat(x_2) + \sin^2(x_1) \end{cases}$$
$$-0.2 \le sat(x) \le 0.2$$

Figure 1 shows a neighborhood of the origin for this continuous system. The simulation in figure 2 shows that the system is stable. The proposed algorithm is implemented for constructing GLF.

The neighborhood of the origin is divided by the axes and nullclines into 6 regions.

In the second quadrant, on $\dot{x}_2 = 0$,

$4x_2 - \sin^2(x_1) = -3x_1$, therefore,

$$v_2(x) = \alpha_{23}|f_{21}(x)| = \alpha_{23}(4x_2 - \sin^2(x_1)),$$
$$v_3(x) = \alpha_{23}|f_{22}(x)| = -3\alpha_{23}x_1 .$$

In the fourth quadrant, on $\dot{x}_2 = 0$,

$-4x_2 + \sin^2(x_1) = x_1$, therefore,

$$v_5(x) = \alpha_{56}|f_{21}(x)| = \alpha_{56}(-4x_2 + \sin^2(x_1)),$$
$$v_6(x) = \alpha_{56}|f_{22}(x)| = \alpha_{56}x_1 .$$

No region exists with nullcline boundaries.

Consider the first and third quadrants of this two dimensional system:

In the first quadrant,

$$v_2(x)|_{x_1=0} = 4\alpha_{23}x_2 ,\ v_6(x)|_{x_2=0} = \alpha_{56}x_1 ,$$

so, $v_1(x) = \alpha_{56}x_1 + 4\alpha_{23}x_2$, and holds (4). After checking, we find that, (3), is satisfied by it in R_1, hence, it's a proper LF.

In the third quadrant,

$$v_3(x)|_{x_2=0} = -3\alpha_{23}x_1 ,\ v_5(x)|_{x_1=0} = -4\alpha_{56}x_2 ,$$

so, $v_4(x) = -3\alpha_{23}x_1 - 4\alpha_{56}x_2$, satisfies (4).

Similarly, since, this function satisfies (3) in R_3, it's a proper LF for this region.

Since, system is two dimensional; the continuity of LF's on all axes is provided in the previous step. Thus, $\forall \alpha_{23}, \alpha_{56} \in R^+$, the special LF's are specified. By assumption, $\alpha_{23} = \alpha_{56} = 1$,

$$v_1(x) = x_1 + 4x_2 ,\ v_2(x) = 4x_2 - \sin^2(x_1) ,$$
$$v_3(x) = -3x_1 ,\ v_4(x) = -3x_1 - 4x_2 ,$$
$$v_5(x) = -4x_2 + \sin^2(x_1), v_6(x) = x_1 .$$

$$V(x) = \sum_{j=1}^{6} v_j(x)\Psi_j(x)$$

$$\dot{V}_f(x) = \sum_{j=1}^{6} \dot{v}_j(x)\Psi_j(x) < 0 \quad a.e.$$

It is a GLF for the system and the origin is asymptotically stable.

Example 2:

$$\begin{cases} \dot{x}_1 = f_1 = -3x_1 + |x_1| + \dfrac{|x_2|}{1+x_1^2} \\ \dot{x}_2 = f_2 = \dfrac{x_1}{1+x_1x_2} - x_2 \end{cases}$$

Two functions $f_1(.)$ and $f_2(.)$ are continuous, but, $f_1(.)$ is non-differentiable on the axes. Two figures 3, 4 show the neighborhood of the origin for this stable system and its phase plan, respectively. The proposed algorithm is used for constructing GLF for the system.

The neighborhood of the origin is divided by the axes and nullclines into 8 regions.

In the first quadrant, $\dot{x}_2 = 0 \Rightarrow x_1 = x_2 + x_1x_2^2$

$$v_1(x) = \alpha_{12}|f_{21}(x)| = \alpha_{12}(x_1),$$
$$v_2(x) = \alpha_{12}|f_{22}(x)| = \alpha_{12}(x_2 + x_1x_2^2).$$

In the first quadrant, $\dot{x}_1 = 0 \Rightarrow 2x_1 = x_2 - 2x_1^3$

$$v_2(x) = \alpha_{23}|f_{11}(x)| = 2\alpha_{23}x_1,$$
$$v_3(x) = \alpha_{23}|f_{12}(x)| = \alpha_{23}(x_2 - 2x_1^3).$$

In the third quadrant, $\dot{x}_2 = 0 \Rightarrow x_1 = x_2 + x_1x_2^2$

$$v_5(x) = \alpha_{56}|f_{21}(x)| = \alpha_{56}(-x_1),$$

$v_6(x) = \alpha_{56}|f_{22}(x)| = \alpha_{56}(-x_2 - x_1x_2^2)$.

In the fourth quadrant, $\dot{x}_1 = 0 \Rightarrow -x_1 = x_2 + 2x_1^3$

$v_7(x) = \alpha_{78}|f_{11}(x)| = 0.5\alpha_{78}(-x_2 - 2x_1^3)$,
$v_8(x) = \alpha_{78}|f_{12}(x)| = \alpha_{78}x_1$.

The previous step offered two LF for R_2, if $v_1(x)=v_2(x)=2\alpha_{23}x_1=\alpha_{12}x_1$, then, (4) is satisfied on their common boundary.

Since, the system is two dimensional, $v_4(x) = v_3(x)|_{x_1=0} + v_5(x)|_{x_2=0} = \alpha_{23}x_2 - \alpha_{56}x_1$ satisfies (4) on x_2^+ and x_1^-. Moreover, after checking, we get that, (3), is satisfied by it in R_4, hence, it's a proper LF for this region.

For continuity of LF's on x_1^+, x_2^-,

$d_1(x) = 2\alpha_{23}x_1, d_8(x) = \alpha_{78}x_1$, if,

$\alpha_{23} = 0.5\alpha_{78} \Rightarrow d_{18}(x) = 0$.

$d_6(x) = -\alpha_{56}x_2, d_7(x) = -0.5\alpha_{78}x_2$,
if, $\alpha_{56} = 0.5\alpha_{78} \Rightarrow d_{67}(x) = 0$.

Therefore, $0.5\alpha_{78} = \alpha_{56} = \alpha_{23}$, by assumption, $\alpha_{23} = 1$, the special LF's of the regions are identified.

$v_1(x) = v_2(x) = v_8(x) = 2x_1, v_3(x) = x_2 - 2x_1^3$,
$v_4(x) = x_2 - x_1, v_5(x) = -x_1, v_6(x) = -x_2 - x_1x_2^2$
$v_7(x) = -x_2 - 2x_1^3$.

$$V(x) = \sum_{j=1}^{8} v_j(x)\Psi_j(x), \quad \dot{V}_f(x) = \sum_{j=1}^{8} \dot{v}_j(x)\Psi_j(x) < 0 \quad a.e.$$

$V(x)$ is a GLF for this system and the origin is asymptotically stable.

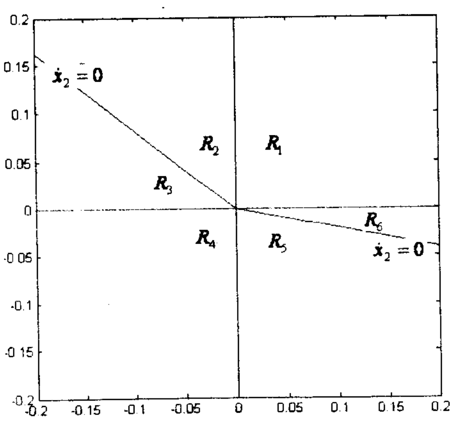

Figure 1: The regions for example 1

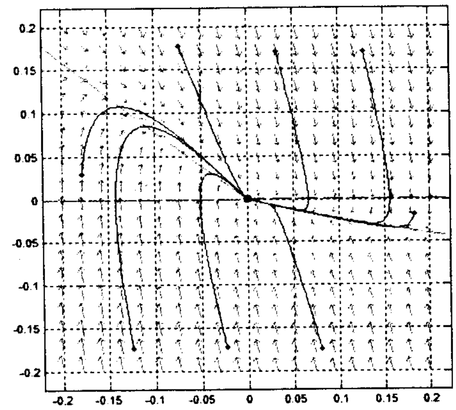
Figure 2: Simulation of example 1

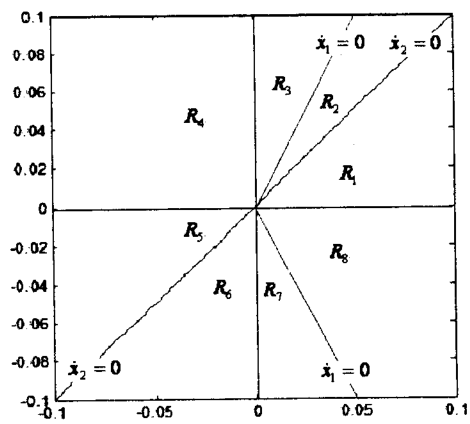

Figure 3: The regions for example 2

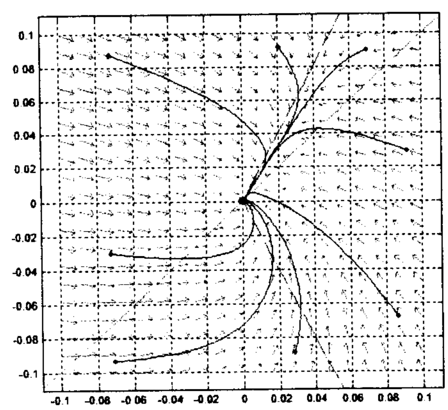
Figure 4: Simulation of example 2

7. Conclusion

In this paper, a non-iterative algorithm was proposed for constructing Generalized Lyapunov Function for nonlinear time invariant system which can be differentiable almost every-where, such that, the system solutions be well defined. The proposed algorithm was based on the Generalized Lyapunov theorem, hence, it didn't require calculation of the generalized derivative of nonsmooth LF's on their nonsmooth surfaces.

Unlike the methods that for constructing piecewise LF, used approximate piecewise model of system in each region, the defined method used original nonlinear model of system, hence, this method was exact. Furthermore, these other methods are computational and more detailed analysis comes to the cost of increased computations, but, this method was analytic.

The steps of algorithm were defined by means of several proposed Notes, which select LF with attention to kind of boundaries of each region.

According to the algorithm, a GLF for the whole system was constructed by a condensed formula. The capability of the algorithm was demonstrated by successful construction of GLF's for two nonsmooth examples.

The main restrictions of this algorithm were original selection of LF's for regions, and then, continuity problem of LF's on their common boundaries. The Notes are proposed to solve these restrictions in many cases.

In the next researches, one can suggest these subjects; can this algorithm obtain GLF for every stable continuous system? Is there a systematic approach for selection of LF's and continuity of them on the boundaries?

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