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Strong Gaussian approximations of product-limit and quantile processes for truncated data under strong mixing

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ABSTRACT

In this paper, we consider the product-limit quantile estimator of an unknown quantile function under a truncated dependent model. This is a parallel problem to the estimation of the unknown distribution function by the product-limit estimator under the same model. Simultaneous strong Gaussian approximations of the product-limit process and normed product-limit quantile process are constructed with rate $O((\log n)^{-\lambda})$ for some $\lambda > 0$. The strong Gaussian approximation of the product-limit process is then applied to derive the law of the iterated logarithm for the product-limit process.

1. Introduction and preliminaries

In medical follow-up or in engineering life testing studies, one may not be able to observe the variable of interest, referred to hereafter as the lifetime. Among the different forms in which incomplete data appear, right censoring and left truncation are two common ones. Left truncation may occur if the time origin of the lifetime precedes the time origin of the study. Only subjects that fail after the start of the study are being followed, otherwise they are left truncated. Woodroofe (1985) reviews examples from astronomy and economy where such data may occur. In the left-truncation model, if the lifetime observations in the sample are assumed to be mutually independent, the nonparametric product-limit (PL) estimator of the survival function has been studied extensively by many authors during recent years, such as Woodroofe (1985), Chao and Lo (1988), Keiding and Gill (1990), Stute (1993) and others. However, it is not clear if the properties of the PL-estimator still hold when observations are dependent. Our focus in the present paper is to study large sample properties of the PL-estimator for the left-truncated data which exhibit some kind of dependence.

Let $\mathbf{X_1}, \mathbf{X_2}, \ldots, \mathbf{X_N}$ be a sequence of the lifetime variables which may not be mutually independent, but have a common unknown distribution function (d.f.) F with a density function $f = F'$. Let $\mathbf{T_1}, \mathbf{T_2}, \ldots, \mathbf{T_N}$ be a sequence of independent and identically distributed random variables (rv's) with continuous d.f. G; they are also assumed to be independent of the random variables $\mathbf{X_i}$'s. In the left-truncation model, $(\mathbf{X_i}, \mathbf{T_i})$ is observed only when $\mathbf{X_i} \geq \mathbf{T_i}$. Let $(X_1, T_1), \ldots, (X_n, T_n)$ be the actually observed sample (i.e., $X_i \geq T_i$), and put $\gamma := \mathbf{P}(\mathbf{T_1} \leq \mathbf{X_1}) > 0$, where $\mathbf{P}$ is the absolute probability (related to the N-sample). Note that n itself is a rv and that γ can be estimated by n/N (although this estimator cannot be calculated since N is unknown). Assume, without loss of generality, that $\mathbf{X_i}$ and $\mathbf{T_i}$ are nonnegative random variables, $i = 1, \ldots, N$. For any d.f. L denotes the left and right endpoints of its support by $a_L = \inf\{x : L(x) > 0\}$ and $b_L = \sup\{x : L(x) < 1\}$, respectively. Then under the current model, as discussed by Woodroofe (1985), we assume that $a_G \leq a_F$ and $b_G \leq b_F$. Define

$$C(x) = \mathbf{P}(\mathbf{T_1} \leq x \leq \mathbf{X_1} | \mathbf{T_1} \leq \mathbf{X_1}) = \mathbb{P}(T_1 \leq x \leq X_1) = \gamma^{-1} G(x)(1 - F(x)), \tag{1.1}$$

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where $\mathbb{P}(.) = \mathbf{P}(.|n)$ is the conditional probability (related to the n-sample) and consider its empirical estimate

$$C_n(x) = n^{-1}\sum_{i=1}^{n} I(T_i \le x \le X_i), \tag{1.2}$$

where $I(.)$ is the indicator function. Then the PL-estimator $\widehat{F}_n$ of F is given by

$$\widehat{F}_n(x) = 1 - \prod_{X_i \le x}\left(1 - \frac{1}{nC_n(X_i)}\right). \tag{1.3}$$

The cumulative hazard function $\Lambda(x)$ is defined by

$$\Lambda(x) = \int_0^x \frac{\mathrm{d}F(u)}{1 - F(u)}. \tag{1.4}$$

Let

$$F^*(x) = \mathbf{P}(\mathbf{X_1} \le x|\mathbf{T_1} \le \mathbf{X_1}) = \mathbb{P}(X_1 \le x) = \gamma^{-1}\int_0^x G(u)\mathrm{d}F(u), \tag{1.5}$$

be the d.f. of the observed lifetimes. Its empirical estimator is given by

$$F_n^*(x) = n^{-1}\sum_{i=1}^{n} I(X_i \le x).$$

On the other hand, the d.f. of the observed T_i's is given by

$$G^*(x) = \mathbf{P}(\mathbf{T_1} \le x|\mathbf{T_1} \le \mathbf{X_1}) = \mathbb{P}(T_1 \le x) = \gamma^{-1}\int_0^\infty G(x \wedge u)\mathrm{d}F(u),$$

and is estimated by

$$G_n^*(x) = n^{-1}\sum_{i=1}^{n} I(T_i \le x).$$

It then follows from (1.1) and (1.2) that

$$C(x) = G^*(x) - F^*(x), \qquad C_n(x) = G_n^*(x) - F_n^*(x-). \tag{1.6}$$

Finally (1.1), (1.4) and (1.5) give

$$\Lambda(x) = \int_0^x \frac{\mathrm{d}F^*(u)}{C(u)}.$$

Hence, a natural estimator of Λ is given by

$$\widehat{\Lambda}_n(x) = \int_0^x \frac{\mathrm{d}F_n^*(u)}{C_n(u)} = \sum_{i=1}^{n} \frac{I(X_i \le x)}{nC_n(X_i)},$$

which is the usual so-called Nelson–Aalen estimator of Λ. Moreover, $\widehat{\Lambda}_n$ is the cumulative hazard function of the PL-estimator $\widehat{F}_n$ defined in (1.3).

In the independent framework, Burke et al. (1981, 1988) used the Komlós, Major and Tusnády theorem to obtain strong Gaussian approximations of the estimation processes in the random censorship model. For a left-truncated data, Tse (2000) has established strong Gaussian approximation of the PL-process $\sqrt{n}[\widehat{F}_n(t) - F(t)]$ by a two parameter Kiefer type process at the almost sure rate of $O(n^{-1/8}(\log n)^{3/2})$. In left-truncation and right censorship (LTRC) model, Zhou and Yip (1999) initiated and Tse (2003) established strong Gaussian approximation of the PL-process by a two parameter Gaussian process at the almost sure rate of $O(n^{-1/8}(\log n)^{3/2})$, a rate that reflects the two-dimensional nature of the LTRC model. Sun and Zhou (2001) obtained strong representations for both the PL and the Nelson–Aalen estimators in the case of truncated dependent data.

For $0 < p < 1$, the pth quantile of $F(t)$ is defined by

$$Q(p) = \inf\{x \in R; F(x) \ge p\}, \tag{1.7}$$

and the sample estimator of $Q(p)$ is defined by

$$Q_n(p) = \inf\{x \in R; \widehat{F}_n(x) \ge p\}, \tag{1.8}$$

when $\widehat{F}_n$ is the PL-estimator define in (1.3).

The role of the quantile function in statistical data modeling was emphasized by Parzen (1979). In econometrics, Gastwirth (1971) used the quantile function to give a succinct definition of the Lorenz curve, which measure inequality in distribution of resources and in size distribution.

In the independent framework with no truncation, the properties of estimator Q_n (where $\widehat{F}_n$ is replaced by the empirical d.f. F_n) have been extensively studied (see e.g. Csörgő, 1983; Shorack and Wellner, 1986). Gűrler et al. (1993) obtained weak and strong quantile representations for randomly truncated data. Based on LTRC model, Tse (2005), established strong Gaussian approximation of the normed PL-quantile process $\rho_n(p) := \sqrt{n}f(Q(p))[Q(p)-Q_n(p)]$ by a two parameter Gaussian process at the almost sure rate of $O(n^{-1/8}(\log n)^{3/2})$.

Under ϕ-mixing condition (for the definition see Doukhan, 1996), the Bahadur representation Q_n (where $\widehat{F}_n$ is replaced by the empirical d.f. F_n) was obtained by Sen (1972) and the extension to the strong mixing (see definition below) case was obtained by Yoshihara (1995). Under strong mixing condition, the strong approximation of the normed quantile process $\rho_n(p)$ by a two parameter Gaussian process at the rate $O((\log n)^{-\lambda})$ for some $\lambda > 0$, was obtained by Fotopoulos et al. (1994) and was later improved by Yu (1996).

The main aim of this paper is to derive strong Gaussian approximations of the PL-process and normed PL-quantile process, for the case of truncated data which the underling lifetime are assumed to be strong mixing whose definition is given below. As a result, we obtain the Law of the iterated logarithm for PL-process.

We consider the strong mixing dependence, which amounts to a form of asymptotic independence between the past and the future as shown by its definition.

Definition 1. Let $\{X_i, i \geq 1\}$ denote a sequence of random variables. Given a positive integer m, set

$$\alpha(m) = \sup_{k\geq 1}\{|P(A\cap B) - P(A)P(B)|; A \in \mathcal{F}_1^k, B \in \mathcal{F}_{k+m}^{\infty}\}, \tag{1.9}$$

where $\mathcal{F}_i^k$ denote the σ-field of events generated by $\{X_j; i \leq j \leq k\}$. The sequence is said to be strong mixing (α-mixing) if the mixing coefficient $\alpha(m) \to 0$ as $m \to \infty$.

Among various mixing conditions used in the literature, α-mixing, is reasonably weak and has many practical applications. There exists many processes and time series fulfilling the strong mixing condition. As a simple example we can consider the Gaussian AR(1) process for which

$$Z_t = \rho Z_{t-1} + \varepsilon_t,$$

where $|\rho| < 1$ and ε_t's are independently identically distributed random variables with standard normal distribution. It can be shown (see Ibragimov and Linnik, 1971, pp. 312–313) that $\{Z_t\}$ satisfies strong mixing condition. The stationary autoregressive moving average (ARMA) processes, which are widely applied in time series analysis, are α-mixing with exponential mixing coefficient, i.e., $\alpha(n) = \mathrm{e}^{-\nu n}$ for some $\nu > 0$. The threshold models, the EXPAR models (see Ozaki, 1979), the simple ARCH models (see Engle, 1982; Masry and Tjostheim, 1995, 1997) and their extensions (see Diebolt and Guégan, 1993) and the bilinear Markovian models are geometrically strongly mixing under some general ergodicity conditions. Auestad and Tjostheim (1990) provided excellent discussions on the role of α-mixing for model identification in nonlinear time series analysis.

Our main assumption is the following.

A. $\{X_i\}_{i\geq 1}$ is a sequence of stationary α-mixing rv's with mixing coefficient $\alpha(n) = O(\mathrm{e}^{-(\log n)^{1+\nu}})$ for some $\nu > 0$.

The layout of the paper is as follows: Section 2, contains main results. The proofs of the main results are relegated to Section 3.

2. Main results

In this section, we construct a two parameter mean zero Gaussian process that strongly uniformly approximate the empirical process $\alpha_n(t) := \sqrt{n}[\widehat{\Lambda}_n(t) - \Lambda(t)]$. Utilizing the relationship between $\alpha_n(t)$ and $\beta_n(t)$, where $\beta_n(t) := \sqrt{n}[\widehat{F}_n(t) - F(t)]$ we then establish similar result for $\beta_n(t)$. The counterpart of these results for the censored dependent model was established by Fakoor and Nakhaei Rad (2009).

Theorem 1. *Let $a_G < a_F$ and $b < b_F$. Suppose that Assumption* **A** *is satisfied. On a rich probability space, there exists a two parameter mean zero Gaussian process $B(u, v)$ for $u, v \geq 0$, such that,*

$$\sup_{0\leq t\leq b}|\alpha_n(t) - B(t, n)| = O((\log n)^{-\lambda}) \quad a.s., \tag{2.1}$$

$$\sup_{0\leq t\leq b}|\beta_n(t) - (1 - F(t))B(t, n)| = O((\log n)^{-\lambda}) \quad a.s., \tag{2.2}$$

for some $\lambda > 0$.

Remark 1. In the α-mixing case, we cannot achieve the same rate as in the iid case i.e. $O(n^{-1/8}(\log n)^{3/2})$ (see Tse, 2000). The main reason is that our approach utilizes the strong approximation introduced by Dhompongsa (1984) as a Kiefer process with a negligible reminder term of order $O(n^{-1/2}(\log n)^{-\lambda})$. This is not as sharp as in iid case.

Corollary 1. *Under Assumption* **A**, *we have,*

$$\sup_{0\le t\le b} |\widehat{\Lambda}_n(t) - \Lambda(t)| = O\left(\frac{\log\log n}{n}\right)^{1/2} \quad a.s., \tag{2.3}$$

$$\sup_{0\le t\le b} |\widehat{F}_n(t) - F(t)| = O\left(\frac{\log\log n}{n}\right)^{1/2} \quad a.s. \tag{2.4}$$

In the next theorem, we construct a two parameter mean zero Gaussian process that strongly uniformly approximate the normed PL-quantile process $\rho_n(p)$.

Theorem 2. *Let $0 < p_0 \le p_1 < 1$. Under Assumption* **A**, *assume that F is Lipschitz continuous and that F is twice continuously differentiable on $[Q(p_0)-\delta, Q(p_1)+\delta]$ for some $\delta > 0$, such that f is bounded away from zero, then there exists a two parameter mean zero Gaussian process $B(x, u)$ for $x, u \ge 0$, such that,*

$$\sup_{p_0\le p\le p_1} |\rho_n(p) - (1-p)B(Q(p), n)| = O((\log n)^{-\lambda}) \quad a.s.,$$

for some $\lambda > 0$.

3. Proofs

Proof of Theorem 1. We start with the usual decomposition of $\alpha_n(t)$.

$$\alpha_n(t) = \sqrt{n}[\widehat{\Lambda}_n(t) - \Lambda(t)] = \frac{\sqrt{n}[F_n^*(t) - F^*(t)]}{C(t)} + \int_0^t \frac{F_n^*(u) - F^*(u)}{C^2(u)} \mathrm{d}C(u) + R_{n1}(t).$$

where

$$n^{-1/2}R_{n1}(t) = \int_0^t \frac{C(u) - C_n(u)}{C_n(u)C(u)} \mathrm{d}F_n^*(u).$$

Furthermore, it follows from Theorem 3 of Dhompongsa (1984) that there exists a Kiefer process $\{K(s, t), s, t \ge 0\}$, with covariance function

$$E[K(s, t)K(s', t')] = \Gamma(s, s') \min(t, t'),$$

where

$$\Gamma(s, s') = Cov(g_1(s), g_1(s')) + \sum_{j=2}^{\infty}[Cov(g_1(s), g_j(s')) + Cov(g_1(s'), g_j(s))],$$

$g_j(s) = I(X_j \le s) - F^*(s)$, such that for some $\lambda > 0$ depending only on ν,

$$\sup_{x\ge 0} |F_n^*(x) - F^*(x) - n^{-1}K(x, n)| = O(n^{-\frac{1}{2}}(\log n)^{-\lambda}) \quad a.s. \tag{3.1}$$

Define, for $0 \le t \le b$ the sequence of Gaussian processes

$$B(t, n) := \frac{K(t, n)/\sqrt{n}}{C(t)} + \int_0^t \frac{K(u, n)/\sqrt{n}}{C(u)^2} \mathrm{d}C(u), \tag{3.2}$$

where $K(s, t)$ is the Kiefer process in (3.1). Let

$$\beta(t, n) = \sqrt{n}[F_n^*(t) - F^*(t)] - K(t, n)/\sqrt{n}$$

Theorem 1 is about the order

$$\sup_{0\le t\le b} |\alpha_n(t) - B(t, n)| = \sup_{0\le t\le b} |R_{n1}(t) + R_{n2}(t)|, \tag{3.3}$$

where

$$R_{n2}(t) = \frac{\beta(t, n)}{C(t)} + \int_0^t \frac{\beta(u, n)}{C(u)^2} \mathrm{d}C(u).$$

To deal with $R_{n1}(t)$, it follows from Theorem 3.2 of Cai and Roussas (1992) that

$$\sup_{t\geq 0}|F_n^*(t)-F^*(t)|=O\left(\sqrt{\frac{\log\log n}{n}}\right)\quad a.s., \tag{3.4}$$

and

$$\sup_{t\geq 0}|G_n^*(t)-G^*(t)|=O\left(\sqrt{\frac{\log\log n}{n}}\right)\quad a.s. \tag{3.5}$$

By (1.6), (3.4) and (3.5),

$$\sup_{t\geq 0}|C_n(t)-C(t)|=O\left(\sqrt{\frac{\log\log n}{n}}\right)\quad a.s. \tag{3.6}$$

Therefore, by (3.6), we have

$$\sup_{0\leq t\leq b}|R_{n1}(t)|=O\left(\sqrt{\frac{\log\log n}{n}}\right)\quad a.s. \tag{3.7}$$

Note that $\inf_{0\leq x\leq b}C(x)>0$.

Next, by applying (3.1), we have

$$\sup_{0\leq t\leq b}|R_{n2}(t)|=O((\log n)^{-\lambda})\quad a.s. \tag{3.8}$$

Combining (3.3), (3.7) and (3.8) we obtain (2.1). It can be shown that

$$\widehat{F}_n(t)-F(t)=(1-F(t))[\widehat{\Lambda}_n(t)-\Lambda(t)]+O\left(\frac{\log\log n}{n}\right)\quad a.s. \tag{3.9}$$

Therefore (2.2) is proved via (3.9) and (2.1). □

Proof of Corollary 1. . By (3.2) and the law of the iterated logarithm for Kiefer processes (see, Theorem A. in Berkes and Philipp, 1977), we have,

$$\sup_{0\leq t\leq b}|B(t,n)|\leq C\sup_{0\leq t\leq b}|K(t,n)|/\sqrt{n}=O(\log\log n)^{1/2}\quad a.s.,$$

where C is a positive constant. From (2.1) and (2.2) and the above inequality we obtain the results. □

The proof of Theorem 2, is mainly based on the following Lemmas of Lemdani et al. (2005). Lemma 1 shows that $\widehat{F}_n$ composed with Q_n is an approximate identity up to order $O(n^{-\frac{1}{2}}(\log n)^{-\lambda})$. Lemmas 2 and 3 give global and local bounds for the deviation between Q_n and Q.

Lemma 1. *Let $0<p_0\leq p_1<1$. Under Assumption* **A**, *assuming that F continuous, then*

$$\sup_{p_0\leq p\leq p_1}|\widehat{F}_n(Q_n(p))-p|=O\left(n^{-\frac{1}{2}}(\log n)^{-\lambda}\right)\quad a.s.$$

Lemma 2. *Let $0<p_0\leq p_1<1$. Under Assumption* **A**, *assuming that $F'=f$ is bounded away from zero on $[Q(p_0)-\delta,Q(p_1)+\delta]$ for some $\delta>0$, we have*

$$\sup_{p_0\leq p\leq p_1}|Q_n(p)-Q(p)|=O\left(\sqrt{\frac{\log\log n}{n}}\right)\quad a.s.$$

Lemma 3. *Let $K_2>0$ and $0\leq b<b_F$. Under Assumption* **A**, *assuming that F Lipschitz continuous on $[0,b]$. Then*

$$\sup_{t,s\in J_n}|\beta_n(t)-\beta_n(s)|=O((\log n)^{-\lambda})\quad a.s.$$

for some $\lambda>0$, where $J_n=\{t,s:|t-s|<K_2\eta_n,0\leq s,t\leq b\}$ and $\eta_n=O\left(\sqrt{\frac{\log\log n}{n}}\right)$.

Proof of Theorem 2. Let $s=Q_n(p)$ and $t=Q(p)$, $p_0\leq p\leq p_1$, Lemma 2 yields $\sqrt{n}|s-t|=O(\log\log n)$ almost surely. Applying Lemma 3 gives,

$$\widehat{F}_n(Q_n(p)) - \widehat{F}_n(Q(p)) = F(Q_n(p)) - F(Q(p)) + O(n^{-\frac{1}{2}}(\log n)^{-\lambda}) \quad a.s. \tag{3.10}$$

By Lemma 1, $\widehat{F}_n(Q_n(p))$ can be replaced by p up to $O(n^{-\frac{1}{2}}(\log n)^{-\lambda})$. For the right hand side, a Taylor expansion of the first term about $Q(p)$ up to second order term gives,

$$f(Q(p))[Q_n(p) - Q(p)] + O([Q_n(p) - Q(p)]^2) + O(n^{-\frac{1}{2}}(\log n)^{-\lambda}) \quad a.s., \quad \textit{for } p_0 \le p \le p_1.$$

Invoking Lemma 2 and rearranging terms in (3.10), we have,

$$\sqrt{n}f(Q(p))[Q_n(p) - Q(p)] = \sqrt{n}[p - \widehat{F}_n(Q(p))] + O((\log n)^{-\lambda}) \quad a.s., \quad \textit{for } p_0 \le p \le p_1.$$

Since F is continuous, $F(Q(p)) = p$. Recalling the definitions of the PL-process β_n and the normed PL-quantile process ρ_n, we have,

$$\rho_n(p) = \beta_n(Q(p)) + O((\log n)^{-\lambda}) \quad a.s., \tag{3.11}$$

for $p_0 \le p \le p_1$. By using Theorem 1 and (3.11), the theorem is proved. □

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