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Equivalence Classes of Operators on $\mathcal{B}(\mathcal{H})$

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Abstract

Let $\mathcal{L}(\mathcal{B}(\mathcal{H}))$ be the algebra of all linear operators on $\mathcal{B}(\mathcal{H})$ and $\mathcal{P}$ be a property on $\mathcal{B}(\mathcal{H})$. For $\phi_1, \phi_2 \in \mathcal{L}(\mathcal{B}(\mathcal{H}))$ we say that $\phi_1 \sim_{\mathcal{P}} \phi_2$ whenever $\phi_1(T)$ has property $\mathcal{P}$ if and only if $\phi_2(T)$ has this property. In particular, if $\mathcal{I}$ is the identity map on $\mathcal{B}(\mathcal{H})$ then $\phi \sim_{\mathcal{P}} \mathcal{I}$ means that ϕ preserves property $\mathcal{P}$ in both directions. Each property $\mathcal{P}$ produces an equivalence class on $\mathcal{L}(\mathcal{B}(\mathcal{H}))$. We study the relation between equivalence classes with respect to different properties.

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1 Introduction

Let $\mathcal{H}$ be an infinite-dimensional separable complex Hilbert space and $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. We denote by $\mathcal{F}(\mathcal{H})$ and $\mathcal{K}(\mathcal{H})$ the ideals of all finite rank and compact operators in $\mathcal{B}(\mathcal{H})$, respectively. The Calkin algebra of $\mathcal{H}$ is the quotient algebra $\mathcal{C}(\mathcal{H}) = \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be a Fredholm operator if both its kernel and cokernel are finite-dimensional. An operator which has a closed range and either its kernel or cokernel is finite dimensional is called a semi-Fredholm operator. We denote by $\mathcal{SF}(\mathcal{H})$, $\mathcal{FR}(\mathcal{H})$ the sets of semi-Fredholm and Fredholm operators, respectively. It is easy to see that $U \in \mathcal{B}(\mathcal{H})$ is semi-Fredholm if and only if $U + K(\mathcal{H})$ is right or left invertible in the Calkin algebra $\mathcal{C}(\mathcal{H})$. Let $A \in \mathcal{B}(\mathcal{H})$, if there exists $B \in \mathcal{B}(\mathcal{H})$ such that $ABA = A$ then A is called generalized invertible and B is said to be a generalized inverse of A. Note that $A \in \mathcal{B}(\mathcal{H})$ is generalized invertible if and only if $Im(A)$ is closed, where $Im(A)$ denotes the range of A [4].

Throughout this paper we use the following notations for some specific properties:

(i) "f" is the property of "being finite-rank";
(ii) "k" is the property of "being compact";
(iii) "fr" is the property of "being Fredholm";
(iv) "sf" is the property of "being semi-Fredholm";
(v) "g" is the property of "being generalized invertible";

Let $\mathcal{I}$ denote the identity operator of $\mathcal{L}(\mathcal{B}(\mathcal{H}))$ and $\phi \in \mathcal{L}(\mathcal{B}(\mathcal{H}))$. Then $\phi \sim_{\mathcal{P}} \mathcal{I}$ means that ϕ preserves the property $\mathcal{P}$ in both directions, that is $\phi(T)$ has property $\mathcal{P}$ if and only if T has this property. Mbekhta and Šemrl in [3] study those ϕ which satisfy $\phi \sim_g \mathcal{I}$ and $\phi \sim_{sf} \mathcal{I}$. In general, if ψ is a linear operator on $\mathcal{B}(\mathcal{H})$ which preserves property $\mathcal{P}$ in both directions then $\psi\phi \sim_{\mathcal{P}} \phi$ for all $\phi \in \mathcal{L}(\mathcal{B}(\mathcal{H}))$. Also if υ is a linear operator on $\mathcal{B}(\mathcal{H})$ which does not preserve property $\mathcal{P}$ in both directions then for each surjective linear operator ϕ, $\upsilon\phi \nsim_{\mathcal{P}} \phi$.

2 Main Results

In the following lemma $(i) \Leftrightarrow (ii)$ is [2, lemma 2.2] and $(i) \Leftrightarrow (iii)$ is [3, Lemma 2.2].

Lemma 2.1. *Let $K \in \mathcal{B}(\mathcal{H})$. Then the following are equivalent*
(i) *K is compact,*
(ii) *for every $B \in \mathcal{FR}(\mathcal{H})$ we have $B + K \in \mathcal{FR}(\mathcal{H})$,*
(iii) *for every $B \in \mathcal{SF}(\mathcal{H})$ we have $B + K \in \mathcal{SF}(\mathcal{H})$.*

Take $C = \{T \in \mathcal{B}(\mathcal{H}) \mid$ for every operator $A \in \mathcal{B}(\mathcal{H})$ with $Im(A)$ not closed, there exists $\lambda \in \mathbb{C}$ such that $A + \lambda T \neq 0$ and $Im(A + \lambda T)$ is closed $\}$. It is proved in [1, Lemma 3.1] that $C = \mathcal{SF}(\mathcal{H})$.

Theorem 2.2. *If $\phi_1, \phi_2 : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ are surjective linear maps then*
(i) $\phi_1 \sim_g \phi_2 \quad \Rightarrow \quad \phi_1 \sim_{sf} \phi_2$;
(ii) $\phi_1 \sim_g \phi_2 \quad \Rightarrow \quad \phi_1 \sim_f \phi_2$;
(iii) $\phi_1 \sim_{fr} \phi_2 \quad \Rightarrow \quad \phi_1 \sim_k \phi_2$;
(iv) $\phi_1 \sim_{sf} \phi_2 \quad \Rightarrow \quad \phi_1 \sim_k \phi_2$;

As we see in Theorem 2.2, for surjective linear maps ϕ_1, ϕ_2 on $\mathcal{B}(\mathcal{H})$, $\phi_1 \sim_g \phi_2 \Rightarrow \phi_1 \sim_{sf} \phi_2 \Rightarrow \phi_1 \sim_k \phi_2$, $\phi_1 \sim_{fr} \phi_2 \Rightarrow \phi_1 \sim_k \phi_2$, $\phi_1 \sim_g \phi_2 \Rightarrow \phi_1 \sim_f \phi_2$. In what follows we give some examples to show that the reverse implications do not hold in general.

Example 2.3. *Let (α_n) be a sequence of non-zero complex numbers such that $\alpha_n \to 0$ as $n \to \infty$ and let $\{e_n : n \in \mathcal{N}\}$ be an orthonormal basis for $\mathcal{H}$. Suppose that $T : \mathcal{H} \to \mathcal{H}$ is defined by*

$$T(x) = \sum_{j=1}^{\infty} \alpha_j e_j \otimes e_j(x), \qquad (x \in \mathcal{H}).$$

It is clear that T is an injective bounded linear operator. Also

$$Im(T) = \{y \in H : \sum_{j=1}^{\infty} \frac{|\langle y, e_j\rangle|^2}{|\alpha_j|^2} < \infty\},$$

since for $y \in \mathcal{H}$, $y = T(x)$ for some $x \in \mathcal{H}$ if and only if

$$\sum_{j=1}^{\infty} e_j \otimes e_j(y) = \sum_{j=1}^{\infty} \alpha_j e_j \otimes e_j(x).$$

This is equivalent to $\langle x, e_j\rangle = \langle y, e_j\rangle/\alpha_j \quad (j \in \mathbb{N})$. Also for $y \in \mathcal{H}$, we have $y = lim_{n\to\infty} T(x_n)$ where $x_n = \sum_{j=1}^{n} \frac{\langle y, e_j\rangle}{\alpha_j} e_j$. Thus $Im(T)$ is dense in $\mathcal{H}$. Suppose that $\sum_{n=1}^{\infty} |\alpha_n|^2$ converges then $y = \sum_{n=1}^{\infty} \alpha_n e_n \in \mathcal{H}\backslash Im(T)$. This shows that $Im(T)$ is a proper subspace of H.

Now let $\phi = L_T$ where $L_T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is defined by $L_T(S) = TS \quad (S \in \mathcal{B}(\mathcal{H}))$. It is clear that $\phi \sim_k \mathcal{I}$ and $\phi \sim_f \mathcal{I}$ but $\phi \nsim_{fr} \mathcal{I}$, also $\phi \nsim_{sf} \mathcal{I}$ and hence $\phi \nsim_g \mathcal{I}$, because $\mathcal{I}(I)$ is a Fredholm operator but $\phi(I)$ does not have a closed range and hence it is not semi-Fredholm. Here I denotes the identity operator on $\mathcal{H}$.

Example 2.4. *We show that in general $\phi_1 \sim_{sf} \phi_2$ or $\phi_1 \sim_{fr} \phi_2$ does not imply that $\phi_1 \sim_g \phi_2$.*

Theorem 2.5. *Let $\mathcal{P}$ be a property on $\mathcal{B}(\mathcal{H})$. If $\phi_1, \phi_2 : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ are bijective linear maps such that $\phi_1 \sim_{\mathcal{P}} \phi_2$ then $\phi_1 {\phi_2}^{-1}$ preserves property $\mathcal{P}$ in both directions.*

Now a question arises in mind: *what is the relation between ϕ_1 and ϕ_2 when* $\phi_1 \sim_{\mathcal{P}} \phi_2$? The next remark illustrates some situations.

Remark (i) Let $\tau : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be the linear map which takes T to its transpose. If $\phi \in \mathcal{B}(\mathcal{H})$ then it is easy to see that $\tau\phi \sim_g \phi$ and hence $\tau\phi \sim_{sf} \phi$, $\tau\phi \sim_k \phi$, $\tau\phi \sim_f \phi$. Also $\tau\phi \sim_{fr} \phi$.
(ii) Let A, B be Fredholm operators and $\lambda : \mathcal{B}(\mathcal{H}) \to \mathcal{K}(\mathcal{H})$ a linear map. Suppose that R_B denotes the operator $T \mapsto TB$ on $\mathcal{B}(\mathcal{H})$. If $\phi_1, \phi_2 \in \mathcal{L}(\mathcal{B}(\mathcal{H}))$ are related as $\phi_2 = L_A R_B \phi_1 + \lambda$ or $\phi_2 = L_A R_B \tau\phi_1) + \lambda$, then it is easy to see that $\phi_1 \sim_{sf} \phi_2$, $\phi_1 \sim_{fr} \phi_2$.
(iii) Let A, B be Fredholm operators and $\lambda : \mathcal{B}(\mathcal{H}) \to \mathcal{F}(\mathcal{H})$ a linear map. If $\phi_1, \phi_2 \in \mathcal{L}(\mathcal{B}(\mathcal{H}))$ are such that $\phi_2 = L_A R_B \phi_1 + \lambda$ or $\phi_2 = L_A R_B \tau\phi_1 + \lambda$ then

$\phi_1 \sim_g \phi_2$.

Question 1. Let $\phi_1 \sim_g \phi_2$. Are there $A,\ B \in \mathcal{FR}(\mathcal{H})$, and a linear map $\lambda : \mathcal{B}(\mathcal{H}) \to \mathcal{F}(\mathcal{H})$ such that $\phi_2 = L_A R_B \phi_1 + \lambda$ or $\phi_2 = L_A R_B \tau \phi_1 + \lambda$?

Question 2. Let $\phi_1 \sim_{sf} \phi_2$ or $\phi_1 \sim_{fr} \phi_2$. Are there $A,\ B \in \mathcal{FR}(\mathcal{H})$, and a linear map $\lambda : \mathcal{B}(\mathcal{H}) \to \mathcal{K}(\mathcal{H})$ such that $\phi_2 = L_A R_B \phi_1 + \lambda$ or $\phi_2 = L_A R_B \tau \phi_1 + \lambda$?

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