STRONG IRREGULARITY OF BOUNDED BILINEAR MAPPINGS

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ABSTRACT. In this paper first we give a lower bound for the topological centres of a bounded bilinear map, and then we characterize the topological centres of certain bilinear mappings, say Banach module actions.

1. PRELIMINARIES

Suppose that $f : X \times Y \to Z$ is a bounded bilinear mapping on the normed spaces $X$, $Y$ and $Z$ and let $X^*$ and $X^{**}$ be the first and second dual of $X$, respectively. The adjoint of $f$ is the bounded bilinear mapping $f^* : Z^* \times X \to Y^*$ defined by

$$
\langle f^*(z^*, x), y \rangle = \langle z^*, f(x, y) \rangle \quad (x \in X, y \in Y, z^* \in Z^*).
$$

Continuing this method, the higher rank adjoints of $f$ can be verified by setting $f^{**} = (f^*)^*$ and so on.

The mapping $f^*$ will be considered as the bounded bilinear mapping from $Y \times X$ into $Z$ defined by $f^*(y, x) = f(x, y)$.

The first and second topological centers of $f$ are defined as follows, respectively:

$$
Z(f) : = \{ z^{**} \in X^{**}; y^{**} \to f^{***}(x^{**}, y^{**}) : Y^{**} \to Z^{**} \text{ is } \omega^* \text{ – continuous} \}
$$

$$
Z'(f) : = \{ y^{**} \in Y^{**}; z^{**} \to f^{****}(z^{**}, y^{**}) : X^{**} \to Z^{**} \text{ is } \omega^* \text{ – continuous} \}.
$$

When $f$ is the product $\pi$ of a Banach algebra $A$, we usually show its topological centers by $Z(A^{**})$ and $Z'(A^{**})$. It can be shown that $\pi^{***}$ and $\pi^{****}$ are really the first and second Arens products of $A^{**}$ which will be denoted by $\Box$ and $\odot$, respectively.

The mapping $f$ is called (Arens) regular when $f^{***} = f^{****}$. The Banach algebra $A$ is said to be Arens regular if its product mapping is regular.

The bilinear mapping $f$ is said to be strongly left (resp. right) irregular if $Z(f) = X$ (resp. $Z'(f) = Y$). The subject of Arens regularity of bilinear mappings are investigated in [1, 2, 4, 5, 6, 8].

Key words and phrases. Arens product, bounded bilinear map, Banach module action, topological centre, second dual.
2. Main results

A bounded bilinear mapping \( f : \mathcal{X} \times \mathcal{A} \to \mathcal{X} \) is said to be approximately unital if there exist a bounded net \( \{ e_\alpha \} \) in \( \mathcal{A} \) such that \( \lim_{\alpha} g(x, e_\alpha) = x \), for all \( x \in \mathcal{X} \). We commence with the following result which describes the topological centres of such a mapping.

**Theorem 2.1.** For every approximately unital bounded bilinear mapping \( f : \mathcal{X} \times \mathcal{A} \to \mathcal{X} \) on normed spaces \( \mathcal{A} \) and \( \mathcal{X} \), \( Z(f) = \mathcal{X}^* \) and \( Z(f^*) = \mathcal{A}_X \); in which, \( \mathcal{A}_X := \{ x^{**} \in \mathcal{X}^{**} : J_{\mathcal{X}}(x^{**}) = (J_X)^*(x^{**}) \} \).

As an immediate consequence we have:

**Corollary 2.2.** Let \( g : \mathcal{X} \times \mathcal{A} \to \mathcal{X} \) be an approximately unital bounded bilinear mapping on normed spaces \( \mathcal{A} \) and \( \mathcal{X} \), then \( g^* \) is regular if and only if \( \mathcal{X} \) is reflexive.

The next result studies the strong irregularity of \( \pi_1^{**} \) and \( \pi_2^* \), in which \( \pi_1 \) and \( \pi_2 \) are Banach module actions.

**Theorem 2.3.** Let \( (\pi_1, \mathcal{X}) \) and \( (\pi_2, \mathcal{X}) \) be approximately unital left and right Banach \( \mathcal{A} \)-modules, respectively. Then

\[
Z(\pi_1^{**}) = \mathcal{X}^* = Z(\pi_2^*), \quad \text{and} \quad Z(\pi_1^*) = \mathcal{A}_X = Z(\pi_2^*);
\]

in particular, \( \pi_1^{**} \) and \( \pi_2^* \) are left strongly irregular.

As an straightforward application of the latter theorem we have the next one which is a generalization of a result of [6].

**Corollary 2.4.** (See [6, Corollary 2.4]). For the multiplication \( \pi \) of a Banach algebra \( \mathcal{A} \) having a right (respectively, left) bounded approximate identity, \( \pi^* \) (respectively, \( \pi^{**} \)) is left strongly regular; that is, \( Z(\pi^*) = \mathcal{A}^* \) (respectively, \( Z(\pi^{**}) = \mathcal{A}^{**} \)).

As another application of Theorem 2.3 we deduce the next result of [8], (which in turn is a generalization of [5, Proposition 4.5])

**Corollary 2.5.** ([8, Proposition 3.6]). Let \( (\pi_1, \mathcal{X}) \) and \( (\pi_2, \mathcal{X}) \) be approximately unital left and right Banach \( \mathcal{A} \)-modules, respectively. Then the following assertions are equivalent:

(i) \( \pi_1^{**} \) is regular;

(ii) \( \pi_2^* \) is regular;

(iii) \( \mathcal{X} \) is reflexive.

**References**


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