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WEAKLY STOCHASTIC RUNGE-KUTTA METHOD WITH ORDER 2

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ABSTRACT. Many deterministic systems are described by Ordinary differential equations and can often be improved by including stochastic effects, but numerical methods for solving stochastic differential equations(SDEs) are required, and work in this area is far less advanced than for deterministic differential equations. In this paper,first we follow [7] to describe Runge-Kutta methods with order 2 from Taylor approximations in the weak sense and present two well known Runge-Kutta methods, RK2-TO and RK2-PL. Then we obtain a new 3-stage explicit Runge-Kutta with order 2 in weak sense and compare the numerical results among these three methods.

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1. Introduction

Many physical systems are modeled by SDEs, where random effects are being modeled by a Wiener process (see, for example, [6], [4], [3]) that is nowhere differentiable [2]. Because such differential equations cannot usually be solved analytically, so numerical methods are required and should be designed to perform with a certain order of accuracy.

Consider d-dimensional Wiener process $\{W_t = (W_t^1, ..., W_t^d)\}$ and d-dimensional stochastic differential equation

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t \ t_0 \leq t \leq T \tag{1}$$

where $a = (a^1, ..., a^m)$ is m-dimensional drift vector and $b = (b^{ij})$ is $m \times d$ diffusion matrix that we express in form $b^j = (b^{1j}, ..., b^{dj}), j = 1, 2, ..., m$.

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General form of s-stage explicit Runge-Kutta for approximating stochastic differential equation (1)be stated as follows[1]:

$$\bar{X}_{n+1} = \bar{X}_n + \triangle \sum_{j=1}^{s} \alpha_j a(t_n + \mu_j \triangle, \eta_j) + \sum_{k=1}^{m} \triangle \hat{W}_n^k \sum_{j=1}^{s} \beta_j^k b^k(t_n + \mu_j \triangle, \eta_j) + R, \quad (2)$$

where $\mu_1 = 0$, $\eta_1 = X_n$ and

$$\eta_j = \bar{X}_n + \triangle \sum_{i=1}^{j-1} \lambda_{ji} a(t_n + \mu_i \triangle, \eta_i) + \sum_{k=1}^{m} \triangle \hat{W}_n^k \sum_{i=1}^{j-1} \gamma_{ji}^k b^k(t_n + \mu_i \triangle, \eta_i) \ , \ j = 1, ..., s$$

and R is the residual term. Numerical constants $\alpha_j, \beta_j^k, \mu_j, \lambda_{ij}, \gamma_{ij}^k$ and R should be chosen. Approximation (2) is β-equivalence with simplify form of Taylor expansion with order β.
Generalized Butcher array of coefficient in (2) will be:

$$\begin{array}{c|cccc|cccc|c|cccc}
\mu_2 & \lambda_{21} & & & & \gamma_{21}^1 & & & & & \gamma_{21}^m & & & \\
\vdots & \vdots & \ddots & & & \vdots & \ddots & & & \vdots & \vdots & \ddots & & \\
\mu_s & \lambda_{s1} & \dots & \lambda_{s,s-1} & & \gamma_{s1}^1 & \dots & \gamma_{s,s-1}^1 & & & \gamma_{s1}^m & \dots & \gamma_{s,s1}^m & \\
\hline
R & \alpha_1 & \dots & \alpha_{s-1} & \alpha_s & \beta_1^1 & \dots & \beta_{s-1}^1 & \beta_s^1 & \dots & \beta_1^m & \dots & \beta_{s-1}^m & \beta_s^m
\end{array}$$

where the first matrix is deterministic coefficients and the rest matrices correspond to stochastic parts depends on the Wiener Process components.

2. Weak approximation and Itô-Taylor expansion

Suppose functions $a = a(t, x), b^j = b^j(t, x)$ in (1) are defined on $[t_0, T] \times \mathbb{R}$ and satisfy in both Lipschitz and linear growth bound conditions in x_0. These assumptions, ensure the existence of a unique solution of the SDE (1) with the initial condition $X_{t_0} = X_0$ if X_0 is μ_{t_0}-measurable.
Let $X_{t,x}$ denote the solution of (1) starting at time $t \in [t_0, T]$ and $x \in \mathbb{R}$. Let φ_p be the space of all functions $f(t, x)$ defined on $[t_0, T] \times \mathbb{R}$ which have polynomial growth (with respect to x), and suppose φ_p^β define the subspace of functions $f \in \varphi_p$, for which all partial derivatives up to order $\beta = 1, 2, ...$ belong to φ_p.
Consider the following one-step approximation for d-dimensional equation (1)

$$\bar{X}_{t,x}(t + h) = \bar{X}_{t,x}(t) + A(t, x, h, \xi), \quad (3)$$

where A is some $\mathbb{R}$-valued function and ξ is a random vector.
Suppose $t_0 < t_1 < \cdots < t_N = T$ is an equidistant partition of $[t_0, T]$ with step size $\Delta = (T - t_0)/N$. The discrete approximation of one-step approximation (3) is

$$\begin{aligned} \bar{X}_0 &= X_0 \\ \bar{X}_{n+1} &= \bar{X}_n + A(t_n.\bar{X}_n, \Delta, \xi_n).\ n = 0, ..., N-1 \end{aligned} \quad (4)$$

Discrete approximation $\bar{X} = \{\bar{X}_0, \bar{X}_1, ..., \bar{X}_N\}$ converge weakly to X with order β if for each $g \in \varphi_p^{2\beta+2}$, there exist $k_g \geq 0$ such that

$$|E[g(\bar{X}_N) - g(X_T)]| \leq k_g \triangle^\beta, \tag{5}$$

where β is the order of the scheme[4].

Definition 2.1. Vector $\alpha = (j_1, j_2, ..., j_\ell)$ where $j_i \in \{0, 1, 2, ..., m\}$ and $i \in \{1, 2, ..., \ell\}$, $m = 1, 2, 3, ...$ is called *multi-index vector with length* $\ell = \ell(\alpha) = \{1, 2, ..., \ell\}$.

Definition 2.2. A set $\mathcal{A} \subset \mathcal{M}$ is a *hierarchical set* if

a) $\mathcal{A}$ is non-empty ($\mathcal{A} \neq \phi$)
b) Length of multi-index $\mathcal{A}$ is uniformly bounded ($\sup_{\alpha \in \mathcal{A}} \ell(\alpha) < \infty$)
c) For any $\alpha \in \mathcal{A} \setminus \{\nu\}$, then $-\alpha \in \mathcal{A}$ where ν is a zero-length multi-index. ($-\alpha$ is a vector when the first component of α be omitted).

Definition 2.3. For hierarchical set $\mathcal{A}$, the remain set $\mathcal{B}(\mathcal{A})$ is defined:

$$\mathcal{B}(\mathcal{A}) = \{\alpha \in \mathcal{M} \setminus \mathcal{A} \ : \ -\alpha \in \mathcal{A}\}. \tag{6}$$

Consider the following operators

$$\begin{aligned} L^{(0)} &= \frac{\partial}{\partial t} + \sum_{i=1}^{d} a^i \frac{\partial}{\partial x^i} + \frac{1}{2} \sum_{i,j=1}^{d} c^{ij} \frac{\partial^2}{\partial x^i \partial x^j} \\ L^{(k)} &= \sum_{i=1}^{d} b^{ik} \frac{\partial}{\partial x^i} \ k = 1, ..., m \end{aligned} \tag{7}$$

where $c^{ij} = \sum_{k=1}^{m} b^{ik} b^{jk}$, $i, j = 1, ..., d$.

Suppose $\Gamma_\beta (\beta \in N)$ be the set of all multi-index $\alpha = (j_1, ..., j_l)$, $j_k \in \{0, 1, ..., m\}$ with length $l \in \{1, ..., \beta\}$. Let $f : [t_0, T] \times R^d \longrightarrow R^d$ be a function, if reminder of Itô-Taylor expansion of $f(t, X_t)$ an hierarchical set $\Gamma_\beta \cup \{\nu\}$ be omitted, then we obtain Truncated Itô-Taylor expansion with order β [8]:

$$f(t, X_t) \simeq f(t_0, X_{t_0}) + \sum_{\alpha \in \Gamma_\beta} (L^\alpha f)(t_0, X_{t_0}) I_\alpha \tag{8}$$

where if $\alpha = (j_1, ..., j_l)$ then $L^\alpha = L^{j_1} o ... o L^{j_l}$, $I_\alpha = \int_t^{t+\triangle} \int_t^{s_\ell} ... \int_t^{s_2} dW_{s_1}^{j_1} ... dW_{s_\ell}^{j_\ell}$ and $dW^{(0)} = dt$.

If $f(t, x) = x$, then the one-step approximation will be:

$$\bar{X}_{t,x}(t + \triangle) = x + \sum_{\alpha \in \Gamma_\beta} (L^\alpha f)(t, X_t) I_\alpha \tag{9}$$

such that

$$\bar{X}_{n+1} = \bar{X}_n + \sum_{\alpha \in \Gamma_\beta} (L^\alpha f)(t_n, \bar{X}_n) I_{\alpha,n}. \tag{10}$$

The scheme which construct from Truncated Itô-Taylor expansion with order β be called weakly Taylor form with order β, so the simplify Taylor form with order 2 will be [7]:

$$\begin{aligned}\bar{X}_{n+1} &= \bar{X}_n + b \triangle \hat{W}_n + a \triangle + \frac{1}{2} b b_{01}((\triangle \hat{W}_n)^2 - \triangle) \\ &+ \frac{1}{2}(b_{10} + a b_{01} + \frac{1}{2} b^2 b_{02} + b a_{01}) \triangle \triangle \hat{W}_n \\ &+ \frac{1}{2}(a_{10} + a a_{01} + \frac{1}{2} b^2 a_{02}) \triangle^2,\end{aligned} \tag{11}$$

where for $g = g(t, x)$, $t, x \in R$, we have

$$g = g_{00} = g(t_n, \bar{X}_n) \ , \ g_{ij} = \frac{\partial^{i+j} g}{\partial t^i \partial x^j}(t_n, \bar{X}_n) \ , \ \textit{and} \ \triangle \hat{W}_n \sim N(0, \triangle)$$

Definition 2.4. *Two diffusion process* $\{Y_t\}$,$\{Z_t\}$ *is n-equivalent in weak sense* if their weakly Itô-Taylor expansion with order n at any point are equal and be denoted $Y_t \overset{(n)}{\simeq} Z_t$.

As an example, for $d = m = 1$, we have

$$\begin{aligned}(\triangle \hat{W})^3 &\overset{(2)}{\simeq} 3 \triangle (\triangle \hat{W}) \\ \triangle (\triangle \hat{W})^2 &\overset{(2)}{\simeq} \triangle^2 \\ (\triangle \hat{W})^2 &\overset{(2)}{\simeq} 3 \triangle^2\end{aligned} \tag{12}$$

Obviously variables $\triangle^i (\triangle \hat{W})^j$ in mean-square sense with order $3, 7/2, 4, ...$ equivalence with zero, i.e.,

$$\triangle^i (\triangle \hat{W})^j \overset{(2)}{\simeq} 0 \ , \ i + j/2 \geq 5/2 \tag{13}$$

3. Runge-Kutta methods with order 2

Similar to the deterministic case, for the truncated expansion of (2), we have to obtain truncated expansion with order β of $f(t + \triangle, X_t + \triangle X)$ with respect to terms $\triangle$ and $\triangle X = X_{t+\triangle} - X_t$.
Tocino and Ardanuy calculated this expansion for $\beta = 2$ in multi-dimensional and for $\beta = 3$ in the scalar case [9].

A 2-equivalence between process and truncated expansion with order 2 is

$$f(t+\triangle, X_t+\triangle X) \overset{(2)}{\simeq} f + \frac{\partial f}{\partial t}\triangle + \sum_{i=1}^{d}\frac{\partial f}{\partial x^i}\triangle X^i + \Bigg\{\frac{\partial^2 f}{\partial t^2} + \sum_{i,j=1}^{d} c^{ij}\frac{\partial^3 f}{\partial t\partial x^i\partial x^j}$$
$$+\frac{1}{2}\sum_{i,j,k=1}^{d}\left(\sum_{l=1}^{d} c^{kl}\frac{\partial c^{ij}}{\partial x^l}\right)\frac{\partial^3 f}{\partial x^i\partial x^j\partial x^k} + \frac{1}{4}\sum_{i,j,k,l=1}^{d} c^{ij}c^{kl}\frac{\partial^4 f}{\partial x^i\partial x^j\partial x^k\partial x^l}\Bigg\}\frac{\triangle^2}{2} \quad (14)$$
$$+\sum_{i=1}^{d}\left(\frac{\partial^2 f}{\partial t\partial x^i} + \frac{1}{2}\sum_{j,k=1}^{d} c^{jk}\frac{\partial^3 f}{\partial x^i\partial x^j\partial x^k}\right)\triangle\triangle X^i + \sum_{i,j=1}^{d}\frac{\partial^2 f}{\partial x^i\partial x^j}\frac{\triangle X^i\triangle X^j}{2}$$

where all functions are evaluated at (t, X_t). In scalar case $d = m = 1$, by using the notation $g^{ij} = \dfrac{\partial^{i+j} g}{\partial t^i\partial x^j}(t_n.\bar{X}_n)$, the formula (14) is reduced as follows:

$$\begin{aligned} f(t+\triangle, X_t+\triangle X) &\overset{(2)}{\simeq} f_{00} + f_{10}\triangle + f_{01}\triangle X \\ &+ \left(f_{20} + b^2 f_{12} + b^3 b_{01} f_{03} + \frac{b^4}{4} f_{04}\right)\frac{\triangle^2}{2} \qquad (15)\\ &+ \left(f_{11} + \frac{b^2}{2} f_{03}\right)\triangle\triangle X + f_{02}\frac{(\triangle X)^2}{2} \end{aligned}$$

3.1. Two-stage Runge-Kutta method. For $s = 2$ in scalar case, Runge-Kutta method (2) reduces to the following form:

$$\begin{aligned} \bar{X}_{n+1} &= \bar{X}_n + \{\alpha_1 a + \alpha_2 a(t_n+\mu\triangle, \eta)\}\triangle \\ &+\{\beta_1 b + \beta_2 b(t_n+\mu\triangle, \eta)\}\triangle\hat{W} + R, \qquad (16) \end{aligned}$$

where $\eta = \bar{X}_n + \lambda a\triangle + \gamma b\triangle\hat{W}$.
Suppose coefficient a, b belong to φ_p^6. By using (15) and the equivalence formulae (12),(13), a 2-equivalence approximation for (16) be obtained and compared with another approximation from simplify form of the Taylor method. Let $\dfrac{\partial b}{\partial x} = k$ is constant, in this case $b_{11} = b_{02} = b_{03} = 0$, then the system of equations has an unique solution, and the following method be constructed:

$$\begin{aligned} \bar{X}_{n+1} &= \bar{X}_n + \frac{1}{2}b\triangle\hat{W}_n + \frac{1}{2}b(t_n+\triangle, \bar{X}_n + a\triangle + b\triangle\hat{W}_n)\triangle\hat{W}_n \\ &+ \frac{1}{2}a\triangle + \frac{1}{2}a(t_n+\triangle, \bar{X}_n + a\triangle + b\triangle\hat{W}_n)\triangle - \frac{1}{2}b\frac{\partial b}{\partial x}\triangle. \qquad (17) \end{aligned}$$

The Butcher array of the method is:

1	1		1	
$-1/2bb_{01}\triangle$	1/2	1/2	1/2	1/2

3.2. RK2-TO method. Tocino and Ardanuy [7] obtained the following RK method with order 2 and $s > 2$:

$$\begin{aligned} \hat{X}_{n+1} &= \hat{X}_n + \{\alpha_1 b + \alpha_2 a(t_n + \mu\triangle, \eta)\} \triangle \\ &\quad + \{\beta_1 b + \beta_2 b(t_n + \mu\triangle, \bar{\eta}) + \beta_3 b(t_n + \mu\triangle, \bar{\bar{\eta}})\} \triangle \hat{W} + R, \end{aligned} \tag{18}$$

where

$$\begin{aligned} \eta &= \bar{X}_n + \lambda a \triangle + \gamma b \triangle \hat{W}, \qquad \bar{\eta} = \bar{X}_n + \bar{\lambda} a \triangle + \bar{\gamma} b \triangle \hat{W}, \\ \bar{\bar{\eta}} &= \bar{X}_n + \bar{\bar{\lambda}} a \triangle + \bar{\bar{\gamma}} b \triangle \hat{W}. \end{aligned}$$

By using (15) and (12),(13) a 2-equivalence in weak sense for approximation (18) can be obtained and by comparing it with weakly Taylor expansion with order 2, a system of equations can be derived.
An one parameter solution of the system is:

$$\alpha_1 = \alpha_2 = \frac{1}{2}\ ,\ \gamma = \lambda = \mu = 1\ ,\ \beta_1 = \frac{1}{2}\ ,\ \beta_2 = \frac{1}{2 + 6\bar{\gamma}^2}\ ,\ \beta_3 = \frac{3\bar{\gamma}^2}{2 + 6\bar{\gamma}^2}$$
$$\bar{\lambda} = \bar{\bar{\lambda}} = 1\ ,\ \bar{\bar{\gamma}} = \frac{-1}{3\bar{\gamma}}\ ,\ \bar{\gamma} \neq 0. \tag{19}$$

Take $R = \frac{1}{2} b b_{01}((\triangle\hat{W})^2 - \triangle)$, and for any $\bar{\gamma} \neq 0$, from (19) the following scheme be obtained:

$$\begin{aligned} \bar{X}_{n+1} &= \bar{X}_n + \frac{1}{2} b \triangle \hat{W}_n + \frac{1}{2 + 6\bar{\gamma}^2} b(t_n + \triangle, \bar{X}_n + a \triangle + \bar{\gamma} b \triangle \hat{W}_n) \triangle \hat{W}_n \\ &\quad + \frac{3\bar{\gamma}^2}{2 + 6\bar{\gamma}^2} b(t_n + \triangle, \bar{X}_n + a \triangle - \frac{1}{3\bar{\gamma}} b \triangle \hat{W}_n) \triangle \hat{W}_n \\ &\quad + \frac{1}{2} a \triangle + \frac{1}{2} a(t_n + \triangle, \bar{X}_n + a \triangle + b \triangle \hat{W}_n) \triangle + \frac{1}{2} b \frac{\partial b}{\partial x}((\triangle \hat{W}_n)^2 - \triangle), \end{aligned} \tag{20}$$

where it is 2-equivalence with Taylor method with order 2.

3.3. RK2-PL method. In autonomous case, $d = 1, 2, ...$ with constant diffusion coefficient, $m = 1$, Kloden and Platen [4], offered the following explicit weakly form with order 2:

$$\begin{aligned} Y_{n+1} = Y_n &+ \frac{1}{2}(a(\bar{\Upsilon}) + a) \triangle \\ &+ \frac{1}{4}\Big(b(\bar{\Upsilon}^+) + b(\bar{\Upsilon}^-) + 2b\Big) \triangle W \\ &+ \frac{1}{4}\Big(b(\bar{\Upsilon}^+) - b(\bar{\Upsilon}^-)\Big)\{(\triangle\hat{W})^2 - \triangle\}\triangle^{-\frac{1}{2}} \end{aligned} \tag{21}$$

with

$$\bar{\Upsilon} = Y_n + a \triangle + b \triangle \hat{W}, \qquad \bar{\Upsilon}^{\pm} = Y_n + a \triangle \pm b\sqrt{\triangle}$$

3.4. Three-stage Runge-Kutta method. By using the general form of Runge-Kutta method, and for more accurate approximation, we consider the following form:

$$\begin{aligned}\bar{X}_{n+1} = \bar{X}_n \quad &+ \quad \{\alpha_1 a + \alpha_2 a(t_n + \mu_2\triangle, \eta_2) + \alpha_3 a(t_n + \mu_3\triangle, \eta_3)\} \triangle \\ &+ \quad \{\beta_1 b + \beta_2 b(t_n + \mu_2\triangle, \eta_2) + \beta_3 b(t_n + \mu_3\triangle, \eta_3)\} \triangle W,\end{aligned} \tag{22}$$

where

$$\begin{aligned}\eta_2 &= \bar{X}_n + \lambda_{21} a \triangle + \gamma_{21} b \triangle W, \\ \eta_3 &= \bar{X}_n + \lambda_{31} a \triangle \\ &\quad + \lambda_{32} a(t_n + \mu_2\triangle, \eta_2) \triangle + \gamma_{31} b \triangle W + \gamma_{32} b(t_n + \mu_2\triangle, \eta_2) \triangle W\end{aligned}$$

and coefficient a and b belong to φ_p^6. By using (15) and equivalence forms (12) and (13), we have

$$\begin{aligned}a(t_n + \mu_2\triangle, \bar{X}_n \quad &+ \quad \lambda_{21} a \triangle + \gamma_{21} b \triangle W)\triangle \overset{(2)}{\simeq} a_{01} b \gamma_{21} \triangle \triangle W + a \triangle \\ &+ \quad a_{10}\mu_2 \triangle^2 + a a_{01} \lambda_{21} \triangle^2 + \frac{1}{2} a_{02} b^2 \gamma_{21}^2 \triangle^2,\end{aligned} \tag{23}$$

and similarly

$$\begin{aligned}b(t_n + \mu_2\triangle, \bar{X}_n \quad &+ \quad \lambda_{21} a \triangle + \gamma_{21} b \triangle W) \triangle W \overset{(2)}{\simeq} b \triangle W + b b_{01} \gamma_{21} (\triangle W)^2 \\ &+ \quad b_{10}\mu_2 \triangle \triangle W + a b_{01} \lambda_{21} \triangle \triangle W + \frac{3}{2} b^2 b_{02} \gamma_{21}^2 \triangle \triangle W \\ &+ \quad b(b_{11} + \frac{1}{2} b^2 b_{03}) \mu_2 \gamma_{21} \triangle^2 + a b b_{02} \lambda_{21} \gamma_{21} \triangle^2 .\end{aligned} \tag{24}$$

Now, we obtain truncated expansion with order two of $a(t_n + \mu_3\triangle, \eta_3)$. For simplicity, we take

$$\begin{aligned}M \quad &= \quad \eta_3 - \bar{X}_n = \lambda_{31} a \triangle + \lambda_{32} a(t_n + \mu_2\triangle, \eta_2) + \gamma_{31} b \triangle W \\ &\quad + \gamma_{32} b(t_n + \mu_2\triangle, \eta_2) \triangle W\end{aligned}$$

with (14), we have

$$\begin{aligned}a(t_n + \mu_3\triangle, \bar{X}_n + M) \overset{(2)}{\simeq} a \quad &+ \quad a_{10}\mu_3 \triangle + a_{01} M \\ &+ \quad \left(a_{02} + b^2 a_{12} + b^3 b_{01} a_{03} + \frac{b^4}{4} a_{04}\right) \frac{\mu_2^2 \triangle^2}{2} \\ &+ \quad \left(a_{11} + \frac{b^2}{2} a_{03}\right) \mu_2 \triangle M + a_{02} \frac{M^2}{2}.\end{aligned}$$

Therefore

$$\begin{aligned}a(t_n + \mu_3\triangle, \eta_3)\triangle \overset{(2)}{\simeq} a\triangle \quad &+ \quad a_{10}\mu_3 \triangle^2 + a_{01} \triangle .M \\ &+ \quad (a_{11} + \frac{b^2}{2} a_{03}) \mu_3 \triangle^2 .M + a_{02} \frac{M^2.\triangle}{2}.\end{aligned} \tag{25}$$

This 2-equivalence and (12) , (13) give the following equivalence forms:

$$\begin{aligned} &\triangle M \overset{(2)}{\simeq} (\lambda_{31}+\lambda_{32})a \triangle^2 + (\gamma_{31}+\gamma_{32})b \triangle \triangle W + bb_{01}\gamma_{01}\gamma_{32} \triangle^2 \\ &\triangle^2 M \overset{(2)}{\simeq} 0 \\ &\triangle M^2 \overset{(2)}{\simeq} (\gamma_{31}+\gamma_{32})^2 b^2 \triangle^2 . \end{aligned} \tag{26}$$

By substituting the above equivalence forms in (25)

$$\begin{aligned} a(t_n+\mu_3\triangle,\eta_3)\triangle \overset{(2)}{\simeq} a\triangle &+ a_{10}\mu_3 \triangle^2 (\lambda_{31}+\lambda_{32})aa_{01} \triangle^2 \\ &+ (\gamma_{31}+\gamma_{32})a_{01}b \triangle \triangle W + a_{01}bb_{01}\gamma_{21}\gamma_{32} \triangle^2 \\ &+ \frac{1}{2}(\gamma_{31}+\gamma_{32})^2 a_{02}b^2 \triangle^2 \end{aligned} \tag{27}$$

Similarly, we obtain the Taylor expansion with order 2 of $b(t_n+\mu_3\triangle,\eta_3) \triangle W$, and by using (15) and equivalence (12),(13), we have:

$$\begin{aligned} b(t_n+\mu_3\triangle,\bar{X}_n+M) \triangle W &\overset{(2)}{\simeq} b \triangle W + b_{10}\mu_3 \triangle \triangle W + b_{01}M. \triangle W \\ &+ \left(b_{11}+\frac{b^2}{2}b_{03}\right)\mu_3 \triangle \triangle WM + b_{02}\frac{M^2 \triangle W}{2}, \end{aligned} \tag{28}$$

$$\begin{aligned} M \triangle W &\overset{(2)}{\simeq} (\lambda_{31}+\lambda_{32})a \triangle \triangle w + (\gamma_{31}+\gamma_{32})b(\triangle W)^2 + a_{01}b\lambda_{32}\gamma_{21} \triangle^2 \\ &+ 3bb_{01}\gamma_{21}\gamma_{32} \triangle \triangle W + b_{10}\mu_2\gamma_{32} \triangle^2 \\ &+ ab_{01}\lambda_{21}\gamma_{32} \triangle^2 + \frac{3}{2}b^2 b_{02}\gamma_{21}^2\gamma_{32}\triangle^2, \\ M \triangle \triangle W &\overset{(2)}{\simeq} (\gamma_{31}+\gamma_{32})b\triangle^2, \\ M^2 \triangle W &\overset{(2)}{\simeq} 2(\lambda_{31}+\lambda_{32})(\gamma_{31}+\gamma_{32})ab \triangle^2 \\ &+ 3(\gamma_{31}+\gamma_{32})^2 b^2 \triangle \triangle W + 3b^2 b_{01}\gamma_{21}\gamma_{32}(\gamma_{31}+\gamma_{32}) \triangle^2 . \end{aligned} \tag{29}$$

Substitute the above equivalence in (28). Then

$$\begin{aligned} b(t_n &+ \mu_3\triangle,\eta_3) \triangle W \overset{(2)}{\simeq} b \triangle W + b_{10}\mu_3 \triangle \triangle W + (\lambda 31+\lambda 32)ab_{01} \triangle \triangle W \\ &+ (\gamma 31+\gamma_{32})bb_{01}(\triangle W)^2 + \lambda_{32}\gamma_{21}a_{01}bb_{01} \triangle^2 + 3\gamma_{21}\gamma_{32}bb_{01}^2 \triangle \triangle W \\ &+ \mu_2\gamma_{32}b_{01}b_{10} \triangle^2 + \lambda_{21}\gamma_{32}ab_{01}^2 \triangle^2 + \frac{3}{2}\gamma_{21}^2\gamma_{32}b^2 b_{01}b_{02} \triangle^2 \\ &+ \mu_3(\gamma_{31}+\gamma_{32})(b_{11}+\frac{b^2}{2}b_{03})b \triangle^2 + (\lambda_{31}+\lambda_{32})(\gamma_{31}+\gamma_{32})abb_{02} \triangle^2 \\ &+ \frac{3}{2}(\gamma_{31}+\gamma_{32})^2 b^2 b_{02} \triangle \triangle W + \frac{3}{2}\gamma_{21}\gamma_{32}(\gamma_{31}+\gamma_{32})b^2 b_{01}b_{02}\triangle^2 \end{aligned} \tag{30}$$

Now, by substituting (23),(24),(27) and (30) in (22), we can conclude, that the following method is 2-equivalence:

$$
\begin{aligned}
\bar{X}_{n+1} &= \bar{X}_n + (\alpha_1+\alpha_2+\alpha_3)a\triangle + (\beta_1+\beta_2+\beta_3)b\triangle W \\
&+(\alpha_2\gamma_{21}+\alpha_3(\gamma_{31}+\gamma_{32}))a_{01}b\triangle\triangle W + (\alpha_2\mu_2+\alpha_3\mu_3)a_{10}\triangle^2 \\
&+(\alpha_2\lambda_{21}+\alpha_3(\lambda_{31}+\lambda_{32}))aa_{01}\triangle^2 + \frac{1}{2}(\alpha_2\gamma_{21}^2+\alpha_3(\gamma_{31}+\gamma_{32})^2)a_{02}b^2\triangle^2 \\
&+\gamma_{21}(\alpha_3\gamma_{32}+\beta_3\lambda_{32})a_{01}bb_{01}\triangle^2 + (\beta_2\gamma_{21}+\beta_3(\gamma_{31}+\gamma_{32}))bb_{01}(\triangle W)^2 \\
&+(\beta_2\mu_2+\beta_3\mu_3)b_{10}\triangle\triangle W + (\beta_2\lambda_{21}+\beta_3(\lambda_{31}+\lambda_{32}))ab_{01}\triangle\triangle W \\
&+\frac{3}{2}(\beta_2\gamma_{21}^2+\beta_3(\gamma_{31}+\gamma_{32})^2)b^2b_{02}\triangle\triangle W \\
&+(\beta_2\mu_2\gamma_{21}+\beta_3\mu_3(\gamma_{31}+\gamma_{32}))b(b_{11}+\frac{b^2}{2}b_{03})\triangle^2 \\
&+(\beta_2\lambda_{21}\gamma_{21}+\beta_3(\lambda_{31}+\lambda_{32})(\gamma_{31}+\gamma_{32}))abb_{02}\triangle^2 + 3\beta_3\gamma_{21}\gamma_{32}bb_{01}^2\triangle\triangle W \\
&+\beta_3\mu_2\gamma_{32}b_{01}b_{10}\triangle^2 + \beta_3\lambda_{21}\gamma_{32}ab_{01}^2\triangle^2 + \frac{3}{2}\beta_3\gamma_{21}^2\gamma_{32}b^2b_{01}b_{02}\triangle^2 \\
&+\frac{3}{2}\beta_3\gamma_{21}\gamma_{32}(\gamma_{31}+\gamma_{32})b^2b_{01}b_{02}\triangle^2 + R
\end{aligned}
$$

Where the coefficients of the method satisfy with following system of equations, then it coincide with truncated Taylor expansion with order 2.

$$
\begin{aligned}
\alpha_1+\alpha_2+\alpha_3 = 1 \quad &, \quad \beta_1+\beta_2+\beta_3 = 1 \\
\alpha_2\mu_2+\alpha_3\mu_3 = \frac{1}{2} \quad &, \quad \alpha_2\gamma_{21}+\alpha_3(\gamma_{31}+\gamma_{32}) = \frac{1}{2} \\
\alpha_2\lambda_{21}+\alpha_3(\lambda_{31}+\lambda_{32}) = \frac{1}{2} \quad &, \quad \alpha_2\gamma_{21}^2+\alpha_3(\gamma_{31}+\gamma_{32})^2 = \frac{1}{2} \\
\beta_3\gamma_{21}\gamma_{32} = 0 \quad &, \quad \gamma_{21}(\alpha_3\gamma_{32}+\beta_3\lambda_{32}) = 0 \\
\beta_3\mu_2\gamma_{32} = 0 \quad &, \quad \beta_2\mu_2+\beta_3\mu_3 = \frac{1}{2} \\
\beta_3\lambda_{21}\gamma_{32} = 0 \quad &, \quad \beta_2\lambda_{21}+\beta_3(\lambda_{31}+\lambda_{32}) = \frac{1}{2} \\
\beta_3\gamma_{21}^2\gamma_{32} = 0 \quad &, \quad \beta_2\gamma_{21}^2+\beta_3(\gamma_{31}+\gamma_{32})^2 = \frac{1}{6} \\
\beta_3\gamma_{21}\gamma_{32}(\gamma_{31}+\gamma_{32}) = 0 \quad &, \quad \beta_2\mu_2\gamma_{21}+\beta_3\mu_3(\gamma_{31}+\gamma_{32}) = 0 \\
\beta_2\lambda_{21}\gamma_{21}+\beta_3(\lambda_{31}+\lambda_{32})(\gamma_{31}+\gamma_{32}) = 0 &
\end{aligned}
\tag{31}
$$

where $R = \frac{1}{2}bb_{01}((\triangle W)^2 - \triangle)$.
The above system in Maple environment be solved and observed that the system has two classes of two parameters solution and three groups of one parameters solution that we discuss on these answers.

Case 1: Two parameter solution has the following Butcher′s array:

$$
\begin{array}{c|ccc|ccc}
\mu_2 & \lambda_{21} & & & \gamma_{21} & & \\
\mu_3 & \lambda_{31} & 0 & & \gamma_{31} & 0 & \\
\hline
 & \alpha_1 & \alpha_2 & \alpha_3 & \beta_1 & \beta_2 & \beta_3
\end{array}
$$

where:

$$
\begin{aligned}
\lambda_{21} &= \mu_2 \\
\mu_3 &= 3\frac{\gamma_{21}\mu_2(\mu_2-\gamma_{21}^2)^2}{(\mu_2-3\gamma_{21}^2)^2(\mu_2-\gamma_{21})(\gamma_{21}-1)} \\
\lambda_{31} &= 3\frac{\gamma_{21}\mu_2(\mu_2-\gamma_{21}^2)^2}{(\mu_2-3\gamma_{21}^2)^2(\mu_2-\gamma_{21})(\gamma_{21}-1)} \\
\gamma_{31} &= \frac{\mu_2(\mu_2-\gamma_{21}^2)}{(\mu_2-\gamma_{21})(\mu_2-3\gamma_{21}^2)} \\
\alpha_1 &= \frac{1}{2}\frac{\mu_2^2+\gamma_{21}^2\mu_2-\mu_2\gamma_{21}-\mu_2-3\gamma_{21}^3+3\gamma_{21}^2}{\mu_2(\mu_2-\gamma_{21}^2)} \\
\alpha_2 &= -\frac{1}{2}\frac{2\mu_2\gamma_{21}+\mu_2-3\gamma_{21}^2}{\mu_2^2\gamma_{21}-3\gamma_{21}^3\mu_2+3\gamma_{21}^4-\mu_2^2} \\
\alpha_3 &= \frac{1}{2}\frac{(\gamma_{21}-1)(\mu_2-\gamma_{21})^2(\mu_2-3\gamma_{21}^2)^2}{\mu_2(\mu_2-\gamma_{21}^2)(\mu_2^2\gamma_{21}-3\gamma_{21}^3\mu_2+3\gamma_{21}^4-\mu_2^2)} \\
\beta_1 &= \frac{1}{6}\frac{5\mu_2^3-6\gamma_{21}^2\mu_2^2-\mu_2^2\gamma_{21}-3\mu_2^2-3\mu_2\gamma_{21}^4+8\gamma_{21}^2\mu_2+9\gamma_{21}^5-9\gamma_{21}^4}{(\mu_2-\gamma_{21}^2)^2\mu_2} \\
\beta_2 &= -\frac{1}{2}\frac{\mu_2-\gamma_{21}^2}{\mu_2^2\gamma_{21}-3\gamma_{21}^3\mu_2+3\gamma_{21}^4-\mu_2^2} \\
\beta_3 &= \frac{1}{6}\frac{(\mu_2\gamma_{21}-3\gamma_{21}^3+3\gamma_{21}^2-\mu_2)(\mu_2-\gamma_{21})^2(\mu_2-3\gamma_{21}^2)^2}{\mu_2(\mu_2^2\gamma_{21}-3\gamma_{21}^3\mu_2+3\gamma_{21}^4-\mu_2^2)(\mu_2-\gamma_{21}^2)^2}
\end{aligned}
$$

Case 2: For $\gamma_{31} \neq \{0, 1, -1/2\}$, one-parameter solution has this form:

$$
\begin{array}{c|ccc|ccc}
1 & 1 & & & 1 & & \\
\frac{3\gamma_{31}^2}{1+2\gamma_{31}} & \frac{3\gamma_{31}^2}{1+2\gamma_{31}} & 0 & & \gamma_{31} & 0 & \\
\hline
 & 1/2 & 1/2 & 0 & \frac{3\gamma_{31}^2-3\gamma_{31}-1}{6\gamma_{31}^2} & \frac{\gamma_{31}}{2(\gamma_{31}-1)} & -\frac{1+2\gamma_{31}}{6\gamma_{31}^2(-1+\gamma_{31}}
\end{array}
$$

In the particular case $\gamma_{31} = 1/3$, has the following Butcher array:

$$
\begin{array}{c|ccc|ccc}
1 & 1 & & & 1 & & \\
1/5 & 1/5 & 0 & & 1/3 & 0 & \\
\hline
 & 1/2 & 1/2 & 0 & -5/2 & -1/4 & 15/4
\end{array}
$$

Case 3: For $\beta_2 \neq 0$, one-parameter solution is in form:

$$
\begin{array}{c|ccc|ccc}
\frac{1}{2\beta_2} & \frac{1}{2\beta_2} & & & 0 & & \\
0 & 0 & 0 & & 1 & 0 & \\
\hline
 & -\beta_2+\frac{1}{2} & \beta_2 & 1/2 & -\beta_2+\frac{5}{6} & \beta_2 & 1/6
\end{array}
$$

And for $\beta_2 = 1$, the Butcher$'$s array is:

$$\begin{array}{c|ccc|ccc} 1/2 & 1/2 & & & 0 & & \\ 0 & 0 & 0 & & 1 & 0 & \\ \hline & -1/2 & 1 & 1/2 & -1/6 & 1 & 1/6 \end{array}$$

Case 4: For $\beta_3 \neq 0$, one-parameter solution is:

$$\begin{array}{c|ccc|ccc} 0 & 0 & & & 1 & & \\ \frac{1}{2\beta_3} & \frac{1}{2\beta_3} & 0 & & 0 & 0 & \\ \hline & \frac{1}{2}-\beta_3 & 1/2 & \beta_3 & \frac{5}{6}-\beta_3 & 1/6 & \beta_3 \end{array}$$

And for $\beta_3 = 1$, the method determine with the following Butcher$'$s array:

$$\begin{array}{c|ccc|ccc} 0 & 0 & & & 1 & & \\ 1/2 & 1/2 & 0 & & 0 & 0 & \\ \hline & -1/2 & 1/2 & 1 & -1/6 & 1/6 & 1 \end{array}$$

Case 5: For $\mu_2 \neq 0$, two parameters solution are:

$$\begin{array}{c|ccc|ccc} \mu_2 & \mu_2 & & & 0 & & \\ 0 & -\lambda_{32} & \lambda_{32} & & 1 & 0 & \\ \hline & \frac{\mu_2-1}{2\mu_2} & \frac{1}{2\mu_2} & 1/2 & \frac{5\mu_2-3}{6\mu_2} & \frac{1}{2\mu_2} & 1/6 \end{array}$$

and in particular case $\mu_2 = 1/2, \lambda_{32} = 1$, Butcher$'$s array is:

$$\begin{array}{c|ccc|ccc} 1/2 & 1/2 & & & 0 & & \\ 0 & -1 & 1 & & 1 & 0 & \\ \hline & -1/2 & 1 & 1/2 & -1/6 & 1 & 1/6 \end{array}$$

All solution of 3-stage Runge-Kutta method with order 2, except case 5 that $\mu_3 = 0$, we have $\gamma_{32} = \lambda_{32} = 0$.

4. Numerical results

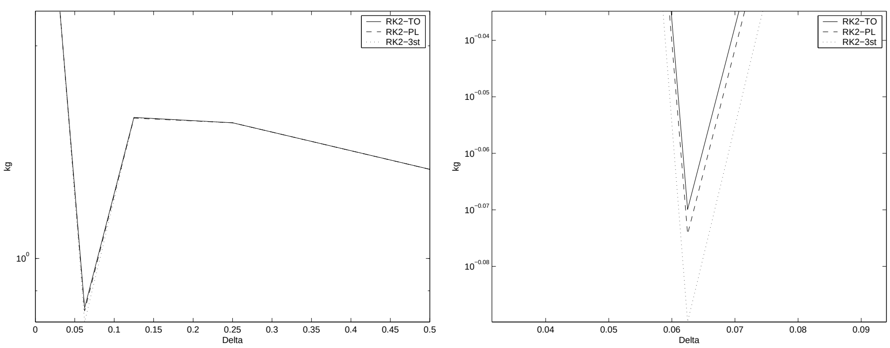

FIGURE 1. k_g with respect to $\triangle$ for approximating $E[X]$

k_g	$St.dev$	$Error$	$\triangle$	
21.7469	6.13786	5.43673	2^{-1}	
35.9861	7.71982	2.24913	2^{-2}	
38.3568	8.74094	0.599326	2^{-3}	$RK2-TO$
7.10071	9.38082	0.0277606	2^{-4}	
171.812	9.92599	0.167785	2^{-5}	
21.7269	6.13479	5.43172	2^{-1}	
36.0028	7.72699	2.25018	2^{-2}	
38.0028	8.74195	0.599633	2^{-3}	$RK2-PL$
6.96173	9.38113	0.0271942	2^{-4}	
172.376	9.92629	0.168336	2^{-5}	
21.7424	6.138	5.4355	2^{-1}	
35.9489	7.7198	2.24681	2^{-2}	
38.1766	8.74103	0.596509	2^{-3}	$RK2-3st$
6.50332	9.38097	0.0254036	2^{-4}	
173.413	9.92609	0.169348	2^{-5}	

TABLE 1. Numerical results for three method $RK2-PL, RK2-TO$ and $RK2-3st$ for approximating $E[X_t]$

In this section, the new 3-stage Runge-Kutta method compare with other well-known methods. We state two examples. Runge-Kutta method (20) and 3-stage Runge-Kutta in case 2 for $\gamma_{31} = 1/3$ are denoted by RK2-TO and RK2-3st respectively. In the following examples, we consider one-dimensional ($d = m = 1$) nonlinear differential equation. Our aim is to estimate $E[g(X_t)]$ for $g(x) = x$ and $g(x) = x^2$, where X_t is the solution of differential equation. Here, $N = 5000$ simulation paths with step size $\triangle = 2^{-1}, ..., 2^{-5}$ be applied for approximation of the mathematical expectation.
The mean, the standard deviation of the errors and an estimation of the k_g in (5) for each methods are presented in Table 1 and Table 2.
The CPU time be plotted for each test examples. All results have been calculated with the same conditions for all methods in Mathematica and Matlab.

Example 1. Consider nonlinear stochastic differential equation

$$dX_t = \left(\frac{1}{3}X_t^{\frac{1}{3}} + 6X_t^{\frac{2}{3}}\right)dt + X_t^{\frac{2}{3}}dW_tX_0 = 1$$

with exact solution $X_t = \left(2t + 1 + \frac{W_t}{3}\right)^3$. We want to estimate $E[X_1] = 28$. Result be presented in the following table.

In the following figure, the value of k_g with respect to $\triangle$ for the same order methods RK2-TO, RK2-PL and RK2-3st are compared. A truncated of the figure with larger scale shows the differences of k_g among these methods.

Figure 2 shows CPU time of the mention methods in table 1.

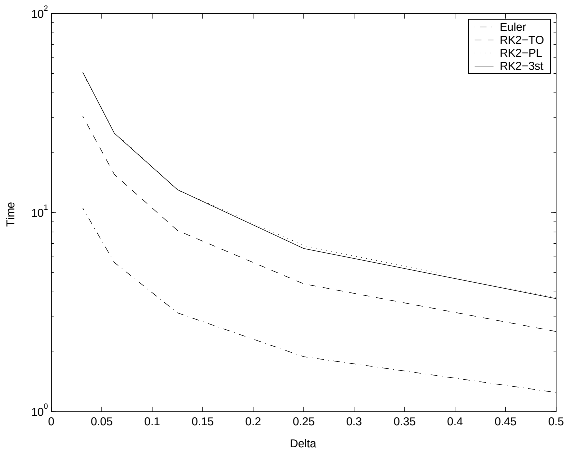

FIGURE 2. CPU time for approximating $E[X]$

Numerical results in Table 1 and group of k_g in Figure 1 shows that new Runge-Kutta works better than others. The mean value of error for $\triangle < 2^{-4} = 0.0625$ increase and so k_g be large.

Example 2. Consider nonlinear stochastic differential equation in Example 1, with exact solution $X_t = \left(2t + 1 + \frac{W_t}{3}\right)^3$, and we want to estimate $E[X_t^2] = 869 + \frac{5}{3^5}$. Result be presented in the following table.

In the following figure, the value of k_g with respect to $\triangle$ for the same order methods RK2-TO, RK2-PL and RK2-3st are compared. A truncated of this figure with larger scale shows the differences of k_g among these methods.

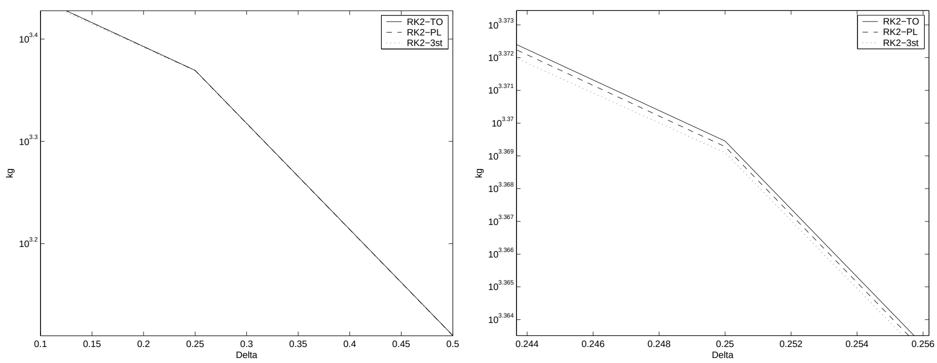

FIGURE 3. k_g with respect to $\triangle$ for approximating $E[X]$

k_g	$St.dev$	$Error$	$\triangle$	
1289.01	300.921	322.254	2^{-1}	
2341.28	442.732	146.33	2^{-2}	
2677.43	540.098	41.8349	2^{-3}	$RK2-TO$
360.387	617.493	1.40776	2^{-4}	
23459	695.458	2.9092	2^{-5}	
1288.26	296.48	322.065	2^{-1}	
2340.36	440.605	146.273	2^{-2}	
2677.38	539.145	41.8341	2^{-3}	$RK2-PL$
370.004	617.129	1.44533	2^{-4}	
23296.7	695.353	2.946	2^{-5}	
1288.8	300.989	322.210	2^{-1}	
2339.36	442.819	146.21	2^{-2}	
2667.45	540.182	41.679	2^{-3}	$RK2-3st$
394.877	617.565	1.54249	2^{-4}	
23551.1	695.503	2.9991	2^{-5}	

Table 2. Numerical results for three methods $RK2-PL, RK2-TO$ and $RK2-3st$ for approximating $E[X_t^2]$

Figure 4 shows CPU time of the mention methods in table 2

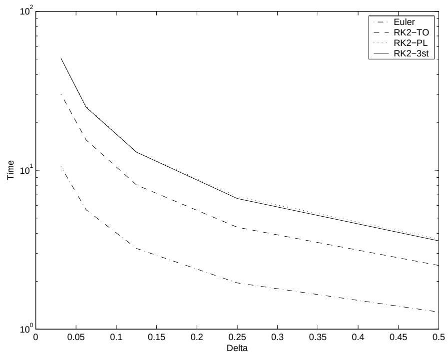

Figure 4. CPU time for approximating $E[X]$

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