A Modified Moving Element-Free Petrov-Galerkin Viscous Method

A. R. Soheili[1], M. Ghorbany[2]

Abstract: *Moving meshless methods are a new generation of numerical methods for time-dependent PDEs, specially tailored to handle shocks or regions with large gradients. These methods link the moving finite element method (MFE) with meshless methods, where instead of hat functions, bases derived from any meshless methods can be used. An initial distribution of mesh points is found using a suitable monitor function based on the equidistribution principle. A penalty is appended to the energy functional to prevent high mesh velocity and collisions of nodes. Here, we employ a decreasing function of the distance between particles as the viscosity function. We develop a modified version of an algorithm developed earlier, the Moving Element-Free Petrov-Galerkin Viscous Method (MEFPGVM), which combines moving finite element (MFE) and element-free Galerkin (EFG) methods, along with initial mesh equidistribution. Numerical solutions for the 1–D Burgers' and Korteweg-de Vries equations demonstrate the accuracy of the numerical approximations.*

Keywords: r-refinement; adaptive grids; monitor functions moving finite element method; equidistribution principles; element-free Galerkin and Petrov-Galerkin method; Burgers' and Korteweg-de Vries equation

Introduction

Numerical solutions of time dependent partial differential equations with moving shocks or steep fronts have been the subject of research for many years. These problems are characterized by small regions in which the solution oscillates. Examples of such problems are gas dynamics with low viscosity, boundary layers, crack propagation, material inter-penetration, and underwater bubble explosions. Numerical approximations in this small subregion require the use of special techniques such as adaptivity. There are three main adaptive methods, (i) h-refinement, in which nodes are added or deleted and error tolerances are balanced so that a suitable approximation can be found; (ii) p-refinement, in which the order of basis functions is changed locally or globally to increase the quality of the approximation; (iii) r-refinement, in which nodes are dynamically moved into the regions of high solution activity and concentrated there for sharper resolution of large gradients or solution variations. (iv) any combination of the previous three adaptive approaches.

Among r-adaptive methods, there are two main approaches for the numerical solution of time dependent PDE's: (1) explicit or "separate" r-adaptive methods, and (2) implicit or together r-adaptive methods. In the first approach, the mesh is computed independently, and then the approximate solution is calculated on the non-uniform mesh [1, 7, 9]. To the second category belong the moving finite elements method (MFE) developed by Keith Miller [21, 22], in which approximate mesh and approximate solution computed together dependently.

In the last decade, meshless methods are a relatively new numerical approach that have absorbed many researchers. These methods employ unstructured meshes and approximate PDEs using scattered data particles, and also have the ability to simulate unsteady problems with shocks. Two main advantages of meshless methods are: computational efficiency by avoiding mesh generation or re-meshing, and the degree of smoothing in the approximation. These methods retain some of the finite element properties such as locality and reduce computational effort for problems with complex domain and moving fronts. But there are two important disadvantages for meshless methods: computation of an inverse of a matrix in each step, and the lack of an interpolation property which shows itself in enforcement of the essential boundary conditions.

[1]Department of Mathematics, Sistan & Baluchestan University, (soheili@hamoon.usb.ac.ir).
[2]Department of Mathematics, Sistan And Baluchestan University, Iran, (ghorbany@hamoon.usb.ac.ir).

In 1968, Donald Shepard [25] presented a meshless interpolation for irregularly-spaced data points. After introducing the MLS method by Lancaster and Salkauskas in 1981 [19], Nayroles et al. [24] employed a local form of this approximation for the numerical solution of some PDE's using nonsingular weight functions with compact support; they obtained good results in spite of the significant disadvantage of having to find the complete derivative of their basis functions. Nayroles et al. named their method the diffuse element method (DEM), and it has the following properties: (1) locality of the finite element method, (2) increases the degree of smoothing in the approximation, (3) avoids the time-consuming mesh generation process, (4) the basis function derivatives are not complete, (5) essential boundary condition cannot be satisfied exactly, (6) approximation is based on irregular distribution of the nodes, (7) the smoothing degree of the approximation is directly depend on the basis, instead of the weights. In 1994, Belytschko and others [6] generalized the DEM and introduced the element-free Galerkin (EFG) method. Some of the properties of this method are: (1) high accuracy, (2) use of the complete derivatives, (3) applying essential boundary conditions using Lagrange multipliers because of the lack of a Kronecker delta property for meshless methods, and (4) relative to the DEM, the EFG method is computationally more intensive although it requires a smaller number of nodes.

In 2004, Ghorbany and Soheili presented the moving element-free Petrov-Galerkin viscous method (MEFPGVM) [16]. This method is a moving meshless method which links the MFE [21, 22] and EFG methods [6]; *i.e.*, the MLS meshless basis with the complete or EFG derivative is used in the MFE method instead of a fixed finite element (FE) basis such as hat functions. Use of the Galerkin and Petrov-Galerkin methods tends toward a weak or generalized solution with lower regularity and larger approximation space. The test space consists of piecewise cubic Hermite polynomial functions belonging to the space $C_0^1(\Omega)\bigcap H_0^3(\Omega)$. The time-dependent coordinates of the nodes require that variable approximation and test spaces be used. *The aim in combining two or more methods is to exploit the advantages of each method.* Here, for better adaption at the initial time, we employed an explicit adaptive method using a suitable monitor function. Effect of this stage is lower pressure of the nodes for adaptivity or adaptivity to the solution from initial. The penalty term is added to the MFE method in order to: (1) prevent colliding of the nodes; (2) avoid concentration of all nodes within the shock region; (3) prevent excessive node velocity; (4) by calibrating the penalty parameters, velocity of the nodes must be almost equated to the velocity of the shock; (5) ensure that the mass matrix is non-singular, positive definite, and well-conditioned. Viscosity is a decreasing function of the nodal distances. This new algorithm has the advantages of the MEFPGVM method, and combines it with penalty parameters and initial adaptivity; we therefore refer to this method as the modified Moving Element-Free Petrov-Galerkin Viscous Method or modified MEFPGVM.

The paper is as follows: Sections 1-4 explain the explicit adaptive and implicit adaptive methods, meshless interpolation, and DEM, EFG and EFPG methods. Section 5 introduces our new method. Section 6 presents several examples, including a 1–D Burgers' equation and a third order Korteweg-de Vries equation (with larger Reynolds number than in previous work [16]).

1 Explicit Adaptive Method

In these methods, the computational mesh and particle coordinates are computed separately, based either on the equality of certain quantities in each element for finite element methods, or on the supports of the weights and base functions in the case of meshless methods. The algorithm proceeds at each time level with the adaptive mesh being dynamically computed using the solution at that time level. The distribution of the elements and nodes in any given sub-domain is based on equality of some quantity or measure of the solution, usually called a "monitor function," within each element, interval or support. Some examples of the adaptive mesh strategies commonly used in practice are those which employ monitor functions based on:

1. Equality of the

$$\int_{x_0}^{x_1} f(x,t)\,dx = \int_{x_1}^{x_2} f(x,t)\,dx = \cdots = \int_{x_{np-1}}^{x_{np}} f(x,t)\,dx = 1/n_p \int_{x_0}^{x_{np}} f(x,t)\,dx \tag{1}$$

where $f(x,t)$ is a monitor function. In practice, this integral constraint is approximated using a

quadrature rule. The generalized arclength monitor function, which contains both the first derivative $|u_x(x,t)|$ and the second derivative $|u_{xx}(x,t)|$, takes the following form:

$$f(x,t) = (\alpha + c_1 |u_x(x,t)|^{c_{p_1}} + c_2 |u_{xx}(x,t)|^{c_{p_2}})^{1/c_p}, \tag{2}$$

with $c_p > 1$, $c_{p_1} \geq 0$, $c_{p_2} \geq 0$, $\alpha > 0$, $c_1 \geq 0$ and $c_2 \geq 0$. By carefully combining the first the and the second derivative such as, $(1 + c_1|u_x(x,t)|^{\theta})\,(1 + c_2|u_{xx}(x,t)|^{1-\theta})$, with $0 \leq \theta \leq 1$, another class of monitor functions can be built (usually $\theta = 1/3$, $1/2$ or $2/3$). Use of this type of monitor function allows mesh control to be dependent on both the gradient and curvature of the solution, so that within oscillatory solution features, nodes become close to each other, since any increase of the monitor function results closer distance between the nodes. The additive constant 1 appears in both derivative terms to prevent degeneracy of the monitor function when either or both of the derivatives is close to zero and its duty is the same as the α that introduced in Eq. (2). The best choice of monitor function is one which incorporates some specific information about the behavior of the solution to the problem. Finding a suitable monitor function for a given problem is therefore not obvious, and is often somewhat of an art.

2. Equality of the local errors, by requiring that each of the local errors, or the numerical integration errors for the Galerkin methods, are equal within each element. Application of this approach for a time-dependent problem means that moving a fixed number of elements or nodes approximately minimizes the discretization error per time step. This approach is known to asymptotically equidistribute the error, *i.e.*, it leads to a mesh where the error is equal on every element or interval. Equality of the dicretization or truncation errors in each interval or each element tends also to the minimize the global error [10, 27].

3. Equality of the residual, restricted in each subinterval, element, or local support of each particle for meshless methods. This method is similar to the weighted residuals method for finite elements [27]. The residual is typically oscillatory and is therefore not suitable for direct application; however, approximation of its weighted square integral or discretization in finite volume form within the support of each particle provides better results [5].

Here, we make use of (2) as a monitor function only at time $t = 0$. Increasing constant c_1 tends to concentrate the nodes at part of the domain where the solution has a large gradient, while increasing c_2 concentrates nodes at the top and bottom of the shock regions and where, that the curvature is large. Some of the references on explicit adaptive methods are [7, 9, 15].

2 Implicit Adaptive Method

In 1981, Miller [21, 22] presented a 1–D Moving Finite Element (MFE) methods or moving piecewise linear finite element method for problems with moving shocks. Discretizations for this type of problem based on a uniform mesh can lead to spurious solutions, and so a lot can be gained through the use of adaptive methods. In 1983, Herbst and Schoombie [17] proved the existence and type of equidistribution in a 1–D, implicit, moving piecewise linear finite element method. Then, Mitchell [23] in 1986 employed the piecewise $C_0^1(\Omega)$ cubic Hermite polynomials instead of the usual base function as the test functions, *i.e.* in his method, the residual is orthogonalized to the cubic Hermite base functions with compact support. Here, we employed this type of Petrov-Galerkin method by using two types of piecewise cubic Hermite polynomials suitable for our example.

Our test example and model problem in general form is,

$$\begin{aligned} &u_t(x,t) = \mathbf{A}u(x,t) \quad x \in \Omega = (0,1), \quad t \in [0,T], \\ &u(x,t) = g(x,t), \quad x \in \partial\Omega = \{0,1\}, \quad u(x,0) = h(x), \quad x \in \bar{\Omega} \end{aligned} \tag{3}$$

where the operator $\mathbf{A}$ contains only space derivatives, and typically encompasses the nonlinear and convection terms. For the 1–D Burgers' equation, the operator $\mathbf{A}$ is given by

$$\mathbf{A}u = -u\,u_x + (1/Re)\,u_{xx} = -(u^2/2)_x + (1/Re)\,u_{xx}, \tag{4}$$

while for third order 1–D Korteweg-de Vries equation

$$\mathbf{A}u = -u\,u_x - (1/Re)\,u_{xxx}, \tag{5}$$

where $Re \gg 1$. The time-dependent functions describing the node locations are $0 = x_0(t) < x_1(t) < \cdots < x_{n_p-1}(t) < x_{n_p}(t) = 1$. As stated at the initial time level *i.e.*, $t = 0$, the nodes are equidistributed using the monitor function (2), and employing an iterative process based on discretizing (1) using Simpsons rule:

$$\begin{aligned}
&(x_i^{[\nu+1]}(0) - x_{i-1}^{[\nu+1]}(0))(f(x_{i-1}^{[\nu]}(0),0) + 4f(x_{i-1/2}^{[\nu]}(0),0) + f(x_i^{[\nu]}(0),0)) = \\
&(x_{i+1}^{[\nu+1]}(0) - x_i^{[\nu+1]}(0))(f(x_i^{[\nu]}(0),0) + 4f(x_{i+1/2}^{[\nu]}(0),0) + f(x_{i+1}^{[\nu]}(0),0)), \\
&for \quad i = 1,2,\cdots,n_p-1, \quad \nu = 0,1,\cdots,s_0.
\end{aligned} \tag{6}$$

Assume that $\tilde{u}(\cdot,t) \in \{\phi_j(\cdot,t)\}_{j=0}^{n_p}$ is the approximation of the $u(x,t)$ in the following form

$$\tilde{u}(x,t) = \sum_{j=0}^{n_p} u_j(t)\phi_j(x,t), \tag{7}$$

where $\{(x_j(t),u_j(t))\}_{j=0}^{n_p}$ are unknowns. In the MFE method, the base functions $\phi_j(x,t)$ are usually taken to be hat functions on the moving nodes. However, in the MEFPGVM, the base functions are built using MLS together with their EFG derivatives, instead of hat functions (see Section 4). The partial derivative $\tilde{u}_t(x,t)$ is given by

$$\tilde{u}_t(x,t) = \sum_{j=0}^{n_p} (\dot{u}_j(t)\,\phi_j(x,t) + \dot{x}_j(t)\,\psi_j(x,t)), \tag{8}$$

where the dot denotes an ordinary time derivative [21, 22, 23]. Equation (8) implies that $\tilde{u}_t(\cdot,t) \in \{\phi_j(\cdot,t), \psi_j(\cdot,t)\}_{j=0}^{n_p}$ where $\psi_j(x,t)$ are given by

$$\psi_j(x,t) = \partial\tilde{u}/\partial x_j = -\tilde{u}_x(x,t)\,\phi_j(x,t). \tag{9}$$

By substituting equation (9) into equation (8), then the time derivative can be rewritten in the following useful form

$$\tilde{u}_t(x,t) = \sum_{j=0}^{n_p} (\dot{u}_j(t) - \tilde{u}_x(x,t)\,\dot{x}_j(t))\,\phi_j(x,t). \tag{10}$$

Under some conditions, the set of basis functions $\{\phi_j(.,t),\psi_j(.,t)\}_{j=0}^{n_p}$, is linearly independent. For example collinearity between three immediate nodes tend to singularity of the system [21, 22]. By substituting equation (7) into the PDE (3), we find the residual in the following form

$$\mathbf{R}(x,t) \equiv \sum_{j=0}^{n_p} (\dot{u}_j(t)\,\phi_j(x,t) + \dot{x}_j(t)\,\psi_j(x,t)) - \mathbf{A}\tilde{u}(x,t) \not\equiv 0. \tag{11}$$

Minimizing the $L^2(\Omega)$ norm of the time-dependent residual

$$\mathbf{J}(t) = \|\mathbf{R}(.,t)\|^2_{L^2(\Omega)}, \tag{12}$$

with respect to the graph characteristic velocities, $\{(\dot{x}_j(t),\dot{u}_j(t))\}_{j=0}^{n_p}$, tend to the Galerkin method *i.e.*, the following inner product,

$$\begin{aligned}
&(\mathbf{R}(\cdot,t),\phi_i(\cdot,t)) = 0, \\
&(\mathbf{R}(\cdot,t),\psi_i(\cdot,t)) = 0, \qquad i = 0,1,\ldots,n_p,
\end{aligned} \tag{13}$$

which means orthogonalization of the residual on the approximation space, so $\mathbf{R} \in (\{\phi_j,\psi_j\}_{j=0}^{n_p})^{\perp}$, or by the Petrov-Galerkin method, *i.e.*, orthogonalizing the residual on another space different from the approximation space that have: (1) suitable regularity; (2) efficiency for the computational work and (3) their supports are equal to the base functions, the ODE system will be found. Applying these inner products requires mollification and special care taken for higher order terms. Here, the test function space is the set of piecewise cubic Hermite functions $\{S_j(\cdot,t),T_j(\cdot,t)\}_{j=0}^{n_p}$. For Burgers' equation, these functions take the following form,

$$\begin{aligned}
&S_i(x,t) = \ell_i^2(x,t)\,(3 - 2\ell_i(x,t)), \\
&T_i(x,t) = \ell_i^2(x,t)\,(x - x_i(t)), \qquad i = 0,1,\ldots,n_p,
\end{aligned} \tag{14}$$

while for the Korteweg-de Vries equation,

$$\begin{aligned} S_i(x,t) &= \ell_i^3(x,t)\,(4-3\ell_i(x,t)),\\ T_i(x,t) &= \ell_i^3(x,t)\,(x-x_i(t)), \qquad i=0,1,\ldots,n_p \end{aligned} \tag{15}$$

where $l_i(x,t)$'s are hat functions, special case of the Lagrange polynomial. Following these definitions, the ODE system and the mass and stiffness matrix is computed by the following inner products

$$\begin{aligned} &(\mathbf{R}(\cdot,t),S_i(\cdot,t)) = 0,\\ &(\mathbf{R}(\cdot,t),T_i(\cdot,t)) = 0, \qquad i=0,1,\ldots,n_p. \end{aligned} \tag{16}$$

The equation (16) means that $\mathbf{R} \in (\{S_j,T_j\}_{j=0}^{n_p})^{\perp}$. The Galerkin and Petrov-Galerkin methods both tend to the generalized or weak solution with smaller regularity and 2×2 block matrix. This block matrix is symmetric for the Galerkin method and nonsymmetric for the Petrov Galerkin method. For a stronger and smoother test space, the residual will be closer to zero and tend to the better answer.

The final stiff ODE system takes the following form

$$\underline{\mathbf{A}}(t)\,\dot{\underline{\mathbf{u}}}(t) = \underline{\mathbf{u}}(t) \tag{17}$$

where $\underline{\mathbf{A}}(t)$ is a $2(n_p-1)\times 2(n_p-1)$ matrix or a tri-diagonal 2×2 block mass matrix,

$$\underline{\mathbf{u}}(t) = [u_1(t),x_1(t),u_2(t),x_2(t),\ldots,u_{n_p-1}(t),x_{n_p-1}(t)]^T, \tag{18}$$

and a dot denotes the time derivative. The MFE method can be defined in gradient weighted form by multiplying the residual (11) into the term $1/\sqrt{1+\tilde{u}_x^2(x,t)}$ which is constant in each element. It seems that this gradient weighted form of the residual is more sensitive with respect to shocks and abrupt changes in the solution. This method is named the Gradient Weighted Residual Moving Finite Element (GWMFE) method. The GWMFE method changes the integration equation functional (12) defined on domain Ω into the boundary integral of the projection of the residual $\mathbf{R}(x,t)\cos(\theta)$, where θ is the slope of line in each element [8, 14].

3 Meshless Interpolation and Meshless Approximation

Assume that we are given irregularly-spaced data $\{(x_j(t),u_j(t))\}_{j=0}^{n_p}$, distributed in a domain Ω. The irregularity of the nodal distribution has a direct effect on the condition number of the system. When finding an interpolation and fitting irregular data in 2–D, a FE method with triangular elements is the best choice, especially for a complex domain. To increase the smoothing degree of the interpolation, both the size of the system and the numbers of degrees of freedom will be increased. The Moving Least Square method (MLS) is a typical meshless interpolation or approximation that is very effective for scattered data. For meshless methods, raising the smoothing degree does not increase the number of unknowns. Let $\{p_i(\cdot)\}_{i=1}^{m}$ be an m-dimensional basis such that, $n_p \gg m$ and $p_1(x)\equiv 1$. This basis usually consists of complete polynomials such as $p_i(x)=x^{i-1}$, $i=1,2,\cdots,m$. So, $p_i \in C^{\infty}(\Omega)$. The complete polynomial basis is the simplest choice, and one can also use other functions such as trigonometric, exponential, singular functions – the choice depends on the PDE and its solution. In particular, if we want to have better approximation, then the basis should have built such that, into it there exists some local characteristic and some information about the PDE; of course, it can also be of variable dimension for adaptivity [12, 13].

Let $\mathbf{P}(x)=[p_1(x),p_2(x),\cdots,p_m(x)]^T$, and let the local approximation of u for an arbitrary fixed point $y\in\Omega$, should be

$$\tilde{u}_y(x,t) = \mathbf{P}^T(x)\,\mathbf{a}(y,t). \tag{19}$$

The coefficient vector $\mathbf{a}(y,t)=[a_1(y,t),\cdots,a_m(y,t)]^T$ is unknown and must be found. The members of the set $\{u_j(t)\}_{j=0}^{n_p}$ are known for interpolation or approximation and are unknown for the numerical solution of a PDE. The MLS method needs a weight with local or global support and with or without singularity point at its center, for using in discrete weighted $L^2(\Omega)$-norm. Suppose that $r=r_i(t)=|x-x_i(t)|/\rho_i(t)$, where $\rho_i(t)$ is the radius of support centered at $x_i(t)$ for $i=0,1,\ldots,n_p$. Then some of the singular and nonsingular weight functions in standard or radial form are

- $w(r)=1/r^{2k}$, which is an inverse square weight usually with $k=1$ or $k=2$. The use of this weight is easy with low computational time and have computational difficulty in order to singularity at its center. So in order to correct its singularity a small number must be added to its denominator.

- $w(r) = \exp^{(-cr^{2k})}$,
  is the Gaussian weight in which $c > 0$ is a parameter near 8 or 9 and k is a positive integer, usually equal to 2.
- $w(r) = \begin{cases} 2/3 - 4r^2 + 4r^3 & r \le 1/2, \\ 4/3 - 4r + 4r^2 - 4/3r^3 & 1/2 < r \le 1, \\ 0 & r > 1, \end{cases}$
  This weight is the cubic spline weight and has necessary smoothing degree for second order problems.
- $w(r) = \begin{cases} 1 - 6r^2 + 8r^3 - 3r^4 & r \le 1, \\ 0 & \text{otherwise} \end{cases}$
  which is the quintic weight.

The singular weight functions tend to the interpolation, but have difficulties in computational work with them. The nonsingular weight functions do not satisfy the interpolation property but can be used to construct good approximations; however, the lack of an interpolation property by using nonsingular weights as a disadvantages, will show itself on exertion boundary condition.

For a given problem, one can use both nonsingular weights for internal nodes and singular weights for boundary nodes. Also, by dividing each of the nonsingular weights by any power of r or $r+\varepsilon$, a singular weight can be built. *The accuracy of a meshless method is directly dependent on the choice of its weight function*. Furthermore, one can arrange many type of systems by using many type of weights, this shows flexibility of meshless methods similar to the various form of finite difference methods. The weights must have the following properties,

1. For each j, $j = 0,1,2,\ldots,n_p$, there exists a subset $\{x_{j_l}(t)\}_{l=1}^{n_j}$ of the particles $\{x_j(t)\}_{j=0}^{n_p}$ in the local support $\Omega_j \bigcap \Omega$ of the weight $w_j(x,t) = w(|x - x_j(t)|/\rho_j(t)) = w(r_j(t))$ such that $m < n_j < c_w$, where the constant c_w is an upper bound for the particles that are in the support of the weights. For better conditioning, these particles must have a minimum distance from the boundary of support.
2. If Ω_j is the support of the weight $w_j(x,t)$, then $\Omega \subset \bigcup_{j=0}^{n_p} \Omega_j$. The support of the weights have nonempty intersection, overlap each other and must cover the domain, but in FE methods, the elements are disjoint, at least continuous on the boundary and only patched together.
3. For each $x \in \Omega$, there are $j_1, j_2, \cdots, j_k$ indices for which $x \in \bigcap_{l=1}^{k} \Omega_{j_l}$ and $m < k < c_w$ [3].
4. The 2–D and 3–D weights are built by using a tensor product of the 1–D weights with rectangular support or in radial form with circular or spherical support.

After some explanations about the proper weights, we explain the MLS method. By minimizing the following discrete weighted inner product or weighted $L^2(\Omega)$-norm of the error functional, we have

$$\begin{aligned} \mathbf{J}(\mathbf{a}(\mathbf{y},\mathbf{t})) &= \|u(\cdot,t) - \tilde{u}_y(\cdot,t)\|^2_{w(y-\cdot,t)}, \\ &= \sum_{j=0}^{n_p} w_j(y,t)\,(u_j(t) - \tilde{u}_y(x_j(t),t))^2, \\ &= \sum_{j=0}^{n_p} w_j(y,t)\,(u_j(t) - \mathbf{P}^T(x_j(t))\,\mathbf{a}(y,t))^2, \\ &= (\mathbf{U}(t) - \mathbf{B}(t)^T\,\mathbf{a}(y,t))^T\,\mathbf{W}(y)\,(\mathbf{U}(t) - \mathbf{B}(t)^T\,\mathbf{a}(y,t)), \end{aligned} \tag{20}$$

for a fixed point $y \in \Omega$ with respect to the coefficient vector $a_i(y,t)$, $i = 1,2,\cdots,m$, we obtain the following system

$$\mathbf{A}(y,t)\,\mathbf{a}(y,t) = \mathbf{F}(y,t)\,\mathbf{U}(t), \tag{21}$$

where

$$\begin{aligned} \mathbf{A}(y,t) &= \mathbf{B}(t)\,\mathbf{W}(y,t)\,\mathbf{B}(t)^T, \\ \mathbf{F}(y,t) &= \mathbf{B}(t)\,\mathbf{W}(y,t), \\ \mathbf{B}(t) &= \{x_j(t)^{i-1}\}, \quad i = 1,2,\ldots,m, \quad j = 0,1,\ldots,n_p, \\ \mathbf{W}(y,t) &= \mathrm{diag}(w_0(y,t), w_1(y,t), \ldots, w_{n_p}(y,t), \\ \mathbf{U}(t) &= [u(x_0(t)), u(x_1(t)), \ldots, u(x_{n_p}(t))]^T. \end{aligned}$$

The rank of the matrix $\mathbf{F}(y,t)$ for each point $y \in \Omega$ is equal to m because, referring to the property 3 of the weights, there are m particles in the support of each weight for which the corresponding m columns of $\mathbf{F}(y,t)$ are linear independent. There are more than m positive elements on the diagonal of the weight matrix $\mathbf{W}(y,t)$. Substituting the coefficient vector $\mathbf{a}(y,t)$ from system (21) into the local approximation (19) gives the following form of the local approximation

$$\tilde{u}_y(x,t) = \mathbf{P}^T(x)\,\mathbf{A}^{-1}(y,t)\,\mathbf{F}(y,t)\,\mathbf{U}(t) = \mathbf{\Phi}_y^T(x,t)\,\mathbf{U}(t), \tag{22}$$

Since the fixed point $y \in \Omega$ is arbitrary, by mysterious moving y into the point $x \in \Omega$ the global approximation will be defined as follows,

$$\tilde{u}(x,t) = \lim_{y \longrightarrow x} \tilde{u}_y(x,t) = \mathbf{\Phi}^T(x,t)\,\mathbf{U}(t) = \sum_{j=0}^{n_p} \phi_j(x,t)\,u_j(t). \tag{23}$$

where the vector basis function is equal to

$$\mathbf{\Phi}^T(x,t) = \mathbf{P}^T(x)\,\mathbf{A}^{-1}(x,t)\,\mathbf{F}(x,t). \tag{24}$$

The $\phi_j(x,t)$ s, $j = 0,1,\cdots,n_p$ are the elements of the vector basis (24) and can be found directly from the following formula,

$$\phi_j(x,t) = \mathbf{P}^T(x)\,\mathbf{A}^{-1}(x,t)\,\mathbf{F}_j(x,t), \quad j = 0,1,\ldots,n_p, \tag{25}$$

where $\mathbf{F}_j(x,t) = \mathbf{P}(x_j(t))w_j(x,t)$, the j-th column of the matrix $\mathbf{F}(x,t)$. For these set of bases interpolation property don't satisfy or bases have not Kronecker delta property. In the special case when $m = 1$, the approximation decreases to the Shepard interpolation, which is an average approximation or convex combination of the partition of unity and normalized positive base functions [19, 25]. If $w_j(\cdot,t) \in C^{l_1}(\Omega)$ and $P_i(\cdot) \in C^{l_2}(\Omega)$ then $\phi_j(\cdot,t) \in C^{\min(l_1,l_2)}(\Omega)$. So if $u(\cdot,t) \in C^{l+1}(\overline{\Omega})$ then $\tilde{u}(\cdot,t)$ is of order ρ^{l+1} [3]. The support of $\phi_j(x,t)$ and support of its derivatives is equal to the support of $w_j(x,t)$. It can be shown that $\{p_i(\cdot)\}_{i=1}^m \subset \{\phi_j(\cdot,t)\}_{j=0}^{n_p}$. This relation is equivalent to the reproducing or consistency property of the MLS method [11, 19]. If $\phi_j(\cdot,t) \in C^1(\Omega)$ then $\exists\, c_{b_1}, c_{b_2}$ such that $\| \phi_j(\cdot,t) \|_{L^\infty(\Omega)} < c_{b_1}$ and $\| \partial\phi(\cdot,t)/\partial x \|_{L^\infty(\Omega)} < c_{b_2}/\rho$, where $\rho = \max_j\{\rho_j(t)\}$, [2, 3].

4 Diffuse Element Method (DEM), Element-Free Galerkin Method (EFG)

All of the meshless fittings such as MLS method can be applied to the numerical solution of PDEs. Lancaster and Salkauskas [19] developed the MLS method as the generalization of the Taylor series method. Nayroles et al., [24] introduced the DEM as a smart change of polynomials defined on an element in FE literature to a neighborhood. Elements such as triangles changed to the neighborhoods and supports for the weights. The size of these supports can be a constant or a variable and the bases built by the MLS method are completely meshless. Nayroles et al. [24] used the local and incomplete derivative

$$\partial\mathbf{\Phi}^T(x,t)/\partial x = (\mathbf{P}^T)'(x)\,\mathbf{A}^{-1}(x,t)\,\mathbf{F}(x,t), \tag{26}$$

or derivative of (19) or (22) with respect to x. For large numbers of particles this type of derivative and consequently the DEM, gives satisfactory approximations. The smoothing degree of the DEM depends only on the polynomial basis and not the smoothing degree of the weights. Belytschko et al. [6, 11], used the complete derivative of the MLS bases or differentiation of the global form of the approximation (23) in the following form,

$$\begin{aligned} \partial\mathbf{\Phi}(x,t)/\partial x \;=\; & (\mathbf{P}^T)'(x)\,\mathbf{A}^{-1}(x,t)\,\mathbf{F}(x,t) + \mathbf{P}^T(x)\,\mathbf{A}^{-1}(x,t)\,(\partial\mathbf{F}(x,t)/\partial x - \\ & \partial\mathbf{A}(x,t)/\partial x\,\mathbf{A}^{-1}(x,t)\,\mathbf{F}(x,t)), \end{aligned} \tag{27}$$

Elements of the vector derivatives can be found by differentiating the element bases (25). The use of the complete derivatives in numerical solution of PDEs gives an increasing accuracy, specially for problems with high activity such as crack and shock. Their method named Element-Free Galerkin method (EFG).

5 Modified Moving Element-Free Petrov-Galerkin Viscous Method (MEFPGVM)

In 2004, Ghorbany and Soheili [16] introduced the Moving Element-Free Petrov-Galerkin Viscous Method (MEFPGVM). In this method, instead of hat functions with movable nodes as in the MFE method, we used meshless functions with movable nodes that have been constructed using the MLS method with EFG derivatives. Test functions are different from the approximation space and are $C_0^1(\Omega)\bigcap H_0^3(\Omega)$ piecewise cubic Hermite functions such as (14) and (15) with variable local compact support, such that their supports are equal to the support of the MLS base functions. This is a jump from an implicit adaptive MFE method to the meshless methods and then coming back by using the mesh dependent test functions. This new method is a powerful and flexible method for the numerical solution of time dependent PDE's with high computational cost and the aim of using these test functions are decreasing computational costs.

The MEFPGVM proceeds as follows. First, the approximation (23) is set in the model or test problem (3), then the the residual functional (12) is appended by the following penalty and regularizing term

$$\sum_{j=0}^{n_p-1} (\varepsilon_j(t)\,(\dot{x}_{j+1}(t)-\dot{x}_j(t))-\eta_j(t))^2, \tag{28}$$

for controlling node movement, and this makes the following residual functional

$$\mathbf{J}(t) = \|\mathbf{R}(\cdot,t)\|^2_{L^2(\Omega)} + \sum_{j=0}^{n_p-1} (\varepsilon_j(t)\,(\dot{x}_{j+1}(t)-\dot{x}_j(t))-\eta_j(t))^2. \tag{29}$$

The residual $\mathbf{R}(x,t)$ is similar to the equation (11) in which $\{\phi_j(\cdot,t)\}_{j=0}^{n_p}$ are the MLS basis functions. The $\tilde{u}_t(x,t)$ is similar to the (8) with different basis functions. Here, the functions $\psi_j(x,t)$ will appear similar to the first part of the equation (9). The smoothing degree of these functions is equal to the $\phi_j(x,t)$, with equality for their supports. In the MFE method $\psi_j(x,t)$ are discontinuous [21]. After some investigation,

$$\begin{aligned}
\psi_j(x,t) &= \partial\tilde{u}(x,t)/\partial x_j, \\
&= -\partial w(r_j(t))/\partial x_j(t)/\sum_{j=0}^{n_p} w(r_j(t))\,(\tilde{u}(x,t)-u_j(t)), \\
&= \partial w(r_j(t))/\partial x/\sum_{j=0}^{n_p} w(r_j(t))\,(\tilde{u}(x,t)-u_j(t)), \\
&\quad j=0,1,\cdots,n_p.
\end{aligned} \tag{30}$$

The set $\{\phi_j(\cdot,t),\psi_j(\cdot,t)\}_{j=0}^{n_p}$ is linearly independent, but this fact is difficult to show. The computation of the derivatives $\partial\psi_j(x,t)/\partial x$ is similar to the equation (27). In the penalty part, $\varepsilon_j(t)$ represents a viscosity term and is a decreasing function of the nodal distance. Here we used the following definition

$$\varepsilon_j(t) = \varepsilon\,(x_{j+1}(t)-x_j(t))/((x_{j+1}(t)-x_j(t))-\delta), \quad j=0,1,2,\ldots,n_p-1, \tag{31}$$

The small positive parameters ε and δ are determined experimentally. The parameter ε, is an amount for normal condition or uniform nodal distribution, and the parameter δ is used to prevent collision of the nodes. When the nodes become closer to each other, the viscosity function increases and makes a viscous region for the nodes and hence hinders mesh movement. The function $\eta_j(t)$ is called a spring nodal force function. Nodes in the inactive region must be at a greater distance from the shock region. The function $\eta_j(t)$ have this responsibility [21, 22, 17]. Here, we did not use the spring nodal force functions, and instead, we used only of the viscosity function (31). In addition, the penalty functions and their parameters must be calibrated so that the velocity of the nodal motions be nearly equal to the velocity of the shock area. By minimizing the functional (29) with respect to the velocities $\dot{x}_j(t)$, $\dot{u}_j(t)$, $j=0,1,\cdots,n_p$, an ODE system with time derivative is found, similar to the system (17). This method together with initial adaptivity that explained in the Section 1, is named as the modified Moving Element-Free Petrov-Galerkin Viscous Method (MEFPGVM).

6 Computational Experiments

The modified MEFPGVM, is used for time dependent PDEs, including hyperbolic, parabolic, nonlinear and with convection term, especially those with shocks and other types of active areas. Our experiments showed that application of this method to problems without shocks gives high accuracy approximations. For problems with sharp gradients, most of the errors occur within the shock region. We applied this method on two well-known 1–D problems with shocks: Burgers' equation with almost and the Korteweg-de Vries equation. We used Mathematica to code the algorithm and did not use any packages for numerical quadratures or the ODE solution.

Example 1 *The 1–D hyperbolic time dependent Burgers' equation, is an equation that has a nonlinear term which gives rise to a shock. By applying the FDM, FEM, DEM, EFG method and many other numerical methods to this equation the coefficient matrix of the nonlinear part is almost skew symmetric. Furthermore, using a uniform distribution of nodes without adaptivity and using only fixed type of these methods the approximation is very oscillatory in the active area and near the shocks. As time proceeds, the shock moves from left to the right and will be very sharp specially for high Reynolds number and small diffusion part. The shock part appears by two reasons:* (i) *the Reynolds number which decreases diffusion part of the problem, and* (ii) *nonlinear part of the problem. The model problem was introduced in equation (4). Sometimes, in the numerical methods, convex combination of the two forms of the nonlinear part* i.e., $u\,u_x$ *and* $(u^2/2)_x$ *is used. Here we use of the Burgers' equation with the following exact solution,*

$$u(x,t) = (\mu+\lambda+(\mu-\lambda)\exp(\lambda\xi/\varepsilon)/(1+\exp(\lambda\xi/\varepsilon)), \tag{32}$$

$\xi = x-\mu\,t-\beta$, $\lambda = 0.4$, $\beta = 0.16$, $\mu = 0.5$ *and* $\varepsilon = 1/Re$, *where Re is Reynolds number. In this example,* $Re = 100$, $dt = 0.01$ *as the time step which is almost high and final time is* $T = 0.7$ *second. Figure (1) shows the approximation of the Burgers' equation, the exact solution and nodal points at time levels* $0.2, 0.3,$ *to* 0.7 *seconds, from left to the right respectively. Figure (2) shows pointwise error plot at* $t = 0.7$ *or* $|\tilde{u}(x,0.7) - u(x,0.7)|$ *for* $x \in [0,1]$ *and nodal motion of* $x_i(t)$ *for* $i = 0,1,\cdots,n_p$ *with respect to the time* $t \in [0,0.7]$.

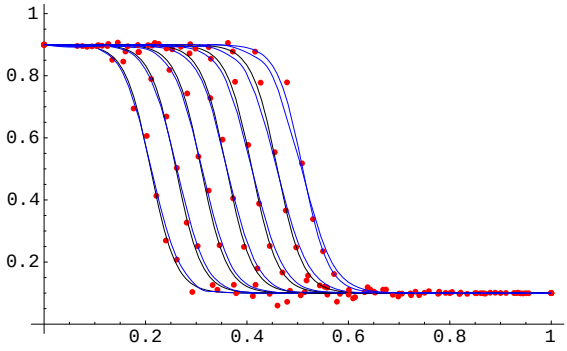

Figure 1: Exact solution, approximation and particles coordinates at 0.1, 0.2 to 0.7 seconds.

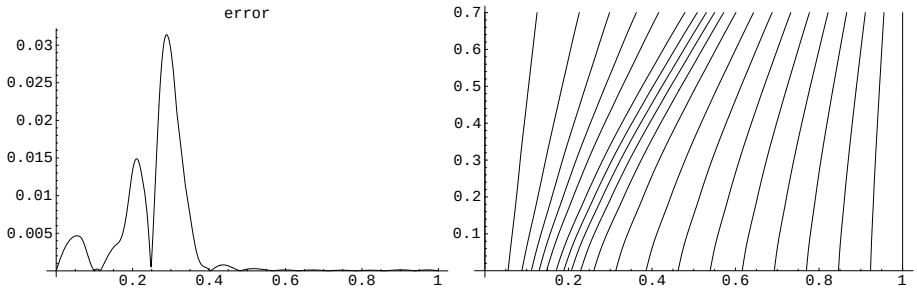

Figure 2: Pointwise error at 0.7 seconds and nodal motions with respect to the time from 0.0 to 0.7 seconds.

Here, the initial adaptivity initiated by the uniform nodal distributions as the beginning, monitor function (2) with parameters, $\alpha = 0.5$, $c_1 = 0.5$, $c_2 = 0.05$ *and the iterative procedure (6). Number of points are* $n_p = 20$

and initial distribution after using the adaptivity procedure is as follows, { 0.0, 0.0555, 0.0886, 0.1111, 0.1296, 0.14768, 0.1705, 0.1887, 0.20705, 0.22912, 0.2607, 0.3133, 0.38597, 0.4625, 0.5393, 0.61608, 0.6929, 0.7696, 0.846, 0.9232, 1.0 }. *One can see the beauty of the smooth nodal motions calibrated and moved with the shock for better handling, approximation and adaption. The constant parameters used in the viscosity functions are* $\varepsilon = 0.06$, $\delta = 0.01$. *The weight function we used for this example is the spline weight function with the following variable radius of support,* $r_j(t) = \rho \max(x_j(t) - x_{j-1}(t), x_{j+1} - x_j(t))$, *where* $\rho = 1.51$ *is the dilation parameter. As stated, the test functions are those introduced in equation (14) and also* $\mathbf{P}^T(x) = [1, x, x^2]$, *which means* $p_i(x) = x^{i-1}, \quad i = 1,2,3$.

Example 2 *The 1–D Korteweg-de Vries equation is another nonlinear time dependent PDE of order three which has a moving sharp area and shock in which there is both a convex and concave part. This equation is* $u_t + uu_x + 1/Re\, u_{xxx} = 0$, *together with exact solution* $u(x,t) = 3c\,Sech^2(k(x - ct - \beta))$, *in which* $c = 0.25$, $\beta = 0.4$, $k = \sqrt{Re\, c/4}$ *and* $Re = 1000$. *The parameters used in the monitor function (2) are* $\alpha = 1.0$, $c_1 = 0.0$, $c_2 = 0.05$. *In this problem,* $n_p = 15$, $m = 3$, $dt = 0.01$, $\rho = 2.6$ *and the weight function is Gaussian with* 8 *as its parameter. Initial distribution after using the adaptivity procedure is as follows,* { 0.0, 0.0835, 0.1579, 0.2208, 0.2771, 0.34512, 0.38775, 0.42566, 0.47564, 0.53824, 0.5953, 0.66115, 0.73858, 0.823536, 0.9113626, 1.0 }. *Figure (3) shows the graph of the approximation together with points* $(x_i(t), u_i(t))$ *for* $i = 0, 1, \cdots, n_p$ *where* $t = 0.1,\ 0.2,\ \cdots,\ 0.7$ *seconds. The points are not on the approximation curve because of lack of the Kronecker delta property for meshless methods. Figure (4)*

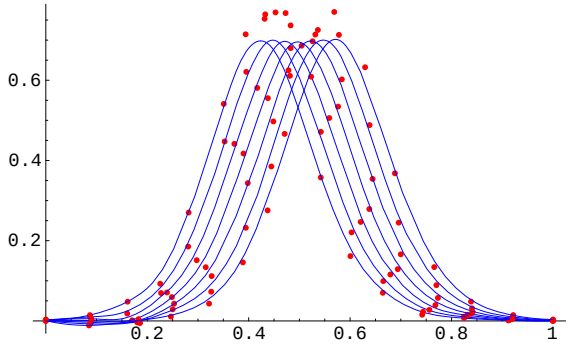

Figure 3: Approximation and the particles of the Korteweg-de Vries equation.

shows the nodal motion and pointwise error at the final time. The test functions are (15). It is noted that

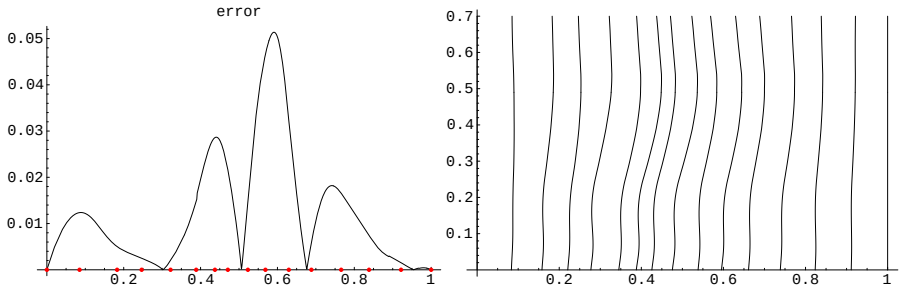

Figure 4: Error at final time step 0.7 Sec. and smooth motion of nodes for the Korteweg-de Vries equation

the second derivatives of the test functions $T_j(x,t)$ *at* $x_j(t)$ *are discontinuous. Therefore, for computing the inner product* $(\tilde{u}_{xxx}(.,t), T_j(.,t))$ *for* $j = 1,2,\cdots,n_p - 1$ *after using Green's theorem, use of the left and right integration in the intervals* $(x_{j-1}(t), x_j(t))$ *and* $(x_j(t), x_{j+1}(t))$ *respectively, is necessary.*

7 Conclusions

In a previous paper [16], the authors introduced MEFPGVM as a typical implicit adaptive meshless method that seems to have disappear equidistribution principle. This method is a time dependent EFPG method together with time dependent and unknown particle coordinates.

In this paper, there are some new subjects: (i) explicit equidistribution using the monitor function (2) at $t = 0$. In this monitor function, the effect of the first derivative is the concentration of the nodes on high slope part of the curve and the effect of the second derivative is concentration of the nodes on the corners, steep parts, extremum points or high curvature parts of the curve. α is a constant for preventing of degeneration, when the derivatives are zero. If α is very small, then particles from the area with the zero derivatives moves into the high slope parts of the solution. Usually, stresses is on the constant c_2 or the second derivative part of the monitor function, because spurious part of the solution is near the shock or near the sharp part, and by increasing value of the c_2, the nodes are concentrated there, from $t = 0$; (ii) a decreasing viscosity function introduced in the regularizing term and penalty (28); (iii) give more computational experiments together with stronger shocks than the previous paper; (iv) new test functions (15) suitable for the Korteweg-de Veries example; (v) a new definition for ψ_j in equation (30). Our experiments show that the Gaussian weight function together with variable radius of support is a good weight by its infinite regularity. We calibrated radius of support, such that almost four or five nodes be in the support of this weight. However, in that part of the region that the problem has more activity, and more concentration of the nodes, it had better radius of support be greater than another parts and contains more particles, for preventing of ill-conditioning. Here, there are two dark part, error estimation and enforcement of boundary conditions that were explained.

Some of the conclusions can be itemized as follows:

- This method can be extended to another meshless methods and to the higher dimensions.
- Mathematical analysis of this method similar to the other meshless methods such as existence, uniqueness, consistency, stability, convergency etc., is poor.
- The computational experiments can help us for giving a conjecture that in this method an equidistribution principle exists and finding it similar to the work that introduced in [17], is a good area for research.
- The computational volume of this method is very high, but can be optimized.
- Use of the piecewise cubic Hermite test functions, needs definition of a typical mesh and elements in the problem domain.
- Matrix inverse computations in EFG method is a time consuming work together with remarkable errors. Therefore, a matrix free method can cure this difficulty.
- The numerical quadratures used for the Galerkin method give high errors into the computational work and need employing of a mesh. Pure meshless method can be done by applying the collocation method, with some cares on the particle distributions.
- The truncation errors that is entered by discretizing the function u, using FD method on the time derivatives must be almost equated by another errors.

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