Extended Abstracts of 4-th Seminar on Linear Algebra. Appl.
and The Wavelets Workshop
7-9 March 2007, Vali-Asr University, Rafsanjan, Iran.

OPTIMAL APPRIXIMATION PROBLEM

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Abstract. In this paper which involve relativity between Inverse eigenvalue problem for Reflxive matrix and Optimal Approximation, we get the sufficient and necessary condition for solving such problem.Furthermore,the algorithm to compute the optimal approximate solution.

1. Introduction

First we introduce some symbols and notations.$OR^{n\times n}$ denote the set of all $n \times n$ orthogonal matrices.A^+ be the Moore-Penrose generalized inverse of matrix A, $||.||$ be the Frobenius norm of a matrix and for A, B$\in R^{n\times n}, (A, B) = tr(B^T A)$ denote the inner product of matrices A and B.

Definition 1.1. A matrix P is said to be generalized reflection matrix if P satisfies $P^T = P$ and $P^2 = I$(see[1,2]).

Definition 1.2. Let $P \in R^{n\times n}$ be a given generalized reflection matrix. A matrix $A \in R^{n\times n}$ is said to be an $n \times n$ reflexive matrix with respect to P if A satisfies

2000 *Mathematics Subject Classification.* Primary 15A57; Secondary 15A18.

Key words and phrases. Reflexive Matrix,Inverse eigenvalue problem,Optimal Approximation,Moore-Penrose generalized inverse of A.

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A=PAP;we denote the set of all $n \times n$ reflexive matrices by $R_r^{n\times n}(P)$ .A matrix $A \in R^{n\times n}$ is said to be an $n \times n$ anti-reflexive matrices with respect to P if A satisfies A=-PAP,the set of all $n \times n$ anti-reflexive matrices denoted by $R_a^{n\times n}(P)$.

The following Problems are studied.

Problem 1. Given X=$(x_1, x_2, ..., x_m) \in R^{n\times m}$, S $\subseteq R^{n\times n}$, Λ= diag$(\lambda_1, \lambda_2, ...\lambda_m) \in R^{m\times m}$, Find A$\in$ S such that AX=ΛX, where $\lambda_1, \lambda_2, ...\lambda_m$ are the eigenvalues of matrix A, x_i is a eigenvector of matrix A associated with $\lambda_i (i = 1, 2, ...m)$.

Problem 2. Given $\tilde{A} \in R^{n\times n}$, find $A^* \in S_E$ such that $||\tilde{A} - A^*|| = inf_{A\in S_E}||\tilde{A} - A||$ where S_E is the solution set of Problem 1.

These problems initially arise in the design and modification of mass-spring systems and dynamic structures have been applied in various areas.Many impotrant results have been achieved on the discussions of the above problems associated with many kinds of matrix set S, such as Jacobi matrices, symmetric(nonnegetive definite)matrices and so on.

2. Solvability conditions of Problem 1

let P$\in R^{n\times n}$ be a given generalized reflection matrix.We have the following lemmas:

Lemma 2.1. *(i) A$\in R_r^{n\times n}(P)$ iff PA=AP, (ii) A$\in R_a^{n\times n}(P)$ iff PA=-AP.*

Lemma 2.2. $R^{n\times n} = R_r^{n\times n}(P)\dot{+}R_a^{n\times n}(P)$

Lemma 2.3. *Given $\tilde{A} \in R^{n\times n}$,then there exist unique $\tilde{A}_1 \in R_r^{n\times n}$ and $\tilde{A}_2 \in R_a^{n\times n}(p)$ such that $\tilde{A} = \tilde{A}_1 + \tilde{A}_2$ and $(\tilde{A}_1, \tilde{A}_2)$=0 where*

$$\tilde{A_1} = \frac{\tilde{A}+P\tilde{A}P}{2},\ \tilde{A_2} = \frac{\tilde{A}-P\tilde{A}P}{2}.$$

A$\in R^{n\times n}(p)$ if and only if that there exist M$\in R^{r\times r}$, H$\in R^{(n-r)\times(n-r)}$ such that

$$A=U\begin{pmatrix} M & 0 \\ 0 & H \end{pmatrix} U^T.$$

set

$$P_1 = \frac{1}{2}(P + I) \qquad and \qquad P_2 = \frac{1}{2}(I - P) \tag{2.1}$$

It is easy to prove that P_1 and P_2 are orthogonal projection matrices,that is,

$$P_1^2 = P_1^T = P_1, \qquad P_1P_2 = 0, \qquad P_1 + P_2 = I.$$

Let rank(P_1)=r then rank(P_2)=n-r.And let

$$P_1 = U_1U_1^T, \quad P_2 = U_2U_2^T, \quad U_1 \in R^{n\times r}, \quad U_2 \in R^{n\times(n-r)}. \tag{2.2}$$

Set $U = (U_1, U_2)$ then

$$U \in OR^{n\times n}, \quad P = P_1 - P_2 = U_1U_1^T - U_2U_2^T. \tag{2.3}$$

Lemma 2.4. *Given $X_1 \in R^{r\times m}$, $\Lambda = diag(\lambda_1, \lambda_2, ...\lambda_m) \in R^{m\times m}$.Let the SVD of X_1 be*

$$X_1 = W \begin{pmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{pmatrix} V^T = W_1 \sum_1 V_1^T. \tag{2.4}$$

where $W=(W_1, W_2)\in OR^{r\times r}$,$V=(V_1, V_2)\in OR^{m\times m}$, $W_1 \in R^{r\times r_1}$, $V_1 \in R^{m\times r_1}$, $r_1 = rank(X_1)$, $\Sigma_1 = diag(\sigma_1, \sigma_2, ..., \sigma_{r_1}), \sigma_i > 0, 1 \le i \le r_1$.Then $AX_1=X_1\Lambda$ is solvable in $R^{r\times r}$ if and only if

$$X_1\Lambda X_1^+ = X_1\Lambda.$$

Theorem 2.5. *Given $X \in R^{n\times m}$, $\Lambda = diag(\lambda_1, \lambda_2, ...\lambda_m) \in R^{m\times m}$.Let*

$$U^TX = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, X_1 = U_1^TX \in R^{r\times m}, X_2 = U_2^TX \in R^{(n-r)\times m}. \tag{2.5}$$

Let the SVD of X_1 be (2.3) and the SVD of X_2 be

$$X_2 = Q \begin{pmatrix} \Sigma_2 & 0 \\ 0 & 0 \end{pmatrix} Z^T = Q_1\Sigma_2 Z_1^T, \tag{2.6}$$

Where $Q=(Q_1, Q_2)\in OR^{(n-r)\times(n-r)}$, $Z=(Z_1, Z_2)\in OR^{m\times m}$,$Q_1 \in R^{(n-r)\times r_2}$, $Z_1 \in R^{m\times r_2}$, $r_2 = rank(X_2)$,$\Sigma_2 = diag(\sigma_1, \sigma_2, ..., \sigma_{r_2})$,$\sigma_i > 0, i = 1, ..., r_2$. Then $A\Lambda=X\Lambda$ is solvable in $R_r^{n\times n}(P)$if and only if

$$X_1\Lambda X_1^+X_1 = X_1\Lambda, \qquad X_2\Lambda X_2^+X_2 = X_2\Lambda \tag{2.7}$$

and its general solution can be expressed as

$$A = A_0+U \begin{pmatrix} G_1W_2^T & 0 \\ 0 & G_2Q_2^T \end{pmatrix} U^T, \quad \forall G_1 \in R^{r\times(r-r_1)} and G_2 \in R^{(n-r)\times(n-r-r_2)}, \tag{2.8}$$

Where

$$A_0 = U \begin{pmatrix} X_1\Lambda X_1^+ & 0 \\ 0 & X_2\Lambda X_2^+ \end{pmatrix} U^T, \quad U \ is \ defined \ in \ (2.3). \tag{2.9}$$

3. The expression of the solution of Problem 2

When the solution set S_E of Problem 1 is nonempty , it is easy to verify that S_E is a closed convex set,therefore we have the following results.

Theorem 3.1. *Given $\tilde{A} \in R^{n\times n}$.Under the hypothesis of Theorem 1, if Problem 1 is solvable,then Problem 2 has unique solution A^*,which can be expressed as*

$$A^* = A_0 + U \begin{pmatrix} \tilde{A_{11}} W_2 W_2^T & 0 \\ 0 & \tilde{A_{22}} Q_2 q_2^T \end{pmatrix}, \tag{3.1}$$

Where A_0, U are the same as in (2.9) and (2.3),respectively

$$\tilde{A_{11}} = U_1^T(\tilde{A}_1 - A_0)U_1 \ , \ \tilde{A_{22}} = U_2^T(\tilde{A}_1 - A_0)U_2 \ , \ \tilde{A}_1 = \frac{\tilde{A} + P\tilde{A}P}{2}. \tag{3.2}$$

Now we give the procedure to compute the optimal approximate solution A^* of the Problem 2.

Algorithm

(1) Input Λ, X, P
(2) Calculate P_1, P_2, U_1, U_2, U according to (2.1) and (2.2)
(3) Compute X_1, X_2 according to (2.5)
(4) Find the SVDs of X_1, X_2 according to (2.4) and (2.6)
(5) if (2.7) holds, then continue; otherwise, go to 1
(6) Compute A_0 according to (2.9)
(7) According to (3.1) calculate A^*

Using the software "MATLAB" ,we obtain the unique solution A^* of Problem 2.

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