Asian Journal of Control, Vol. 14, No. 3, pp. 1–4, May 2012
View this article online at wileyonlinelibrary.com. DOI: 10.1002/asjc.274

–Brief Paper–

CONSTRUCTION OF NONPATHOLOGICAL LYAPUNOV FUNCTIONS FOR DISCONTINUOUS SYSTEMS WITH CARATHEODORY SOLUTIONS

Alireza Faraji Armaki, Naser Pariz, and Rajab Asgharian

ABSTRACT

This paper presents a method based on the Nonpathological Lyapunov theorem for constructing Lyapunov function (LF) for discontinuous time invariant dynamical systems with Caratheodory solutions. The origin is stable, if the method constructs a Nonpathological Lyapunov Function (NPLF) for the system.

Key Words: Stability analysis, discontinuous systems, Caratheodory solutions, nonpathological Lyapunov theorem, nonsmooth Lyapunov functions.

I. INTRODUCTION

A nonpathological Lyapunov theorem to construct nonpathological Lyapunov function (NPLF) for discontinuous time invariant dynamical systems with Caratheodory solutions is proposed in [1]. The notion of Caratheodory solutions is accepted as a good notion of solution for discontinuous systems [1–3]. In this paper, we take advantage of a notion of derivative for locally Lipschitz continuous functions which are nonpathological introduced in [4], and then improved and applied by [1]. Many methods are proposed for constructing piecewise LFs [5]. This paper constructs piecewise smooth NPLF for Caratheodory systems based on the documents of [1].

Consider a nonlinear time invariant discontinuous dynamical system with Caratheodory solutions $\dot{x}=f(x),\ \dot{x}_i=f_i(x),\ i=1,2,\ldots,n$,

$$x\in D\subseteq R^n,\quad f:D\to R^n,\quad f(0)=0\in D \tag{1}$$

such that $f(.)$ is Lebesgue measurable, locally bounded and has bounded variations in an open set D where is a neighborhood of the origin, and for any initial condition $x_0=x(0)\in D$ at least a solution, $\varphi(t),\ t\in I=[0,T_\varphi)$ exists.

Definition 1. Assume D is partitioned into several n-dimensional regions $R_j,\ j\in\{1,2,\ldots,m\}$, using three kinds of $(n-1)$ dimensional hyperplanes (boundaries), which are coordinate hyperplanes $x_i=0$, nullclines $\dot{x}_i=f_i(x)=0$ and nonsmooth hyperplanes $H_g(x)=0$, and common set (vertex) of all regions is the origin. Let a nonsmooth hyperplane $H_g(x)=0,\ g\in\{1,2,\ldots,g_m\}$, $H_g(0)=0$, be a continuous set of nonsmooth points of $f_i(.)$ of (1), and it has bounded variations in D. Suppose, if a nullcline or a nonsmooth hyperplane is on a coordinate hyperplane, then the coordinate hyperplane is considered. Similarly, if a nonsmooth hyperplane is on a nullcline, then the nullcline is considered. The common boundary of two neighboring regions R_j, R_k is $S_{jk}=R_j\cap R_k$, and R_{jk_p} are all neighboring regions of R_j with LFs $v_{jk_p}(.)$. Each R_j has p_m boundaries $S_{jk_p},\ p\in\{1,\ldots,p_m\},\ p_m\geq n$.

Definition 2. $Q_L,\ L\in\{1,\ldots,2^n\}$ denotes an orthant, and $R^{Q_L}\subset Q_L,\ h\in\{1,\ldots,h_m\}$ denotes regions in Q_L, and v^{Q_L} denotes LF of corresponding R^{Q_L}.

Manuscript received November 2, 2009; revised March 19, 2010; accepted May 17, 2010.

Alireza Faraji Armaki is the corresponding author (e-mail: arfaraji@kashanu.ac.ir).

The authors are with the Electrical Engineering Department, Ferdowsi University of Mashhad, Mashhad, Iran.

Definition 3. Let, in each region R_j, the system (1) be stable and a time invariant smooth LF $v_j(.)$, exists and an open connected set $B \subset D$, so that $\forall j$, $0 \in B_j = B \cap R_j$. A primary LF is a time invariant smooth parametric function $v_j(.)$ for R_j that satisfies (2). A proper/special LF is a primary LF which satisfies (3) on some/all common boundaries of its region.

$$\forall j, v_j : R_j \subset D \subset R^n \to R,\ \ v_j(0)=0,\ \ \dot{v}_j(0)=0$$

$$\forall x \in B_j, x \neq 0 : (v_j(x) > 0 \text{ and } \dot{v}_j(x) \leq 0), \tag{2}$$

$$\forall x \in S_{jk} = (R_j \cap R_k \cap B) : v_j(x) = v_k(x) \tag{3}$$

II. CONSTRUCTION OF NPLF

Lemma 1. Let the common boundary of R_j, R_k be a discontinuity hyperplane $S_{jk} : H_g(.) = 0$, which is not on the coordinate hyperplanes and nullclines. A primary LF is selected for them, *i.e.* $v_j(.) = v_k(.)$. This LF is not differentiable on their common boundary, but the LF is locally Lipschitz continuous on the common boundary.

Proof. By assumption $v_j(.)$ is a smooth function in the interior of two regions, then $\nabla v(.)$ is defined in interior of them. $\forall x \in S_{jk}$, the system is nonsmooth and $\dot{v}_j(x) = \nabla v_j(x).\dot{x}$ is not differentiable, and then $\dot{v}_j(.)$ isn't defined. Since each smooth function is continuously differentiable in the interior of the two regions, and $\forall x \in S_{jk} \Rightarrow v_j(x) = v_k(x)$. So, from the Lipschitz definition, $\forall x \in B_j, x_0 \in S_{jk} \Rightarrow |v_j(x) - v_j(x_0)| \leq L_1 \|x - x_0\|$ and since $v_j(.) = v_k(.)$, then $\forall x \in B_k, x_0 \in S_{jk} \Rightarrow |v_j(x) - v_j(x_0)| \leq L_2 \|x - x_0\|$. Therefore $\forall x \in (B_j \cup B_k), x_0 \in S_{jk} \Rightarrow |v_j(x) - v_j(x_0)| \leq \max\{L_1, L_2\} \|x - x_0\|$, then $v_j(.)$ is locally Lipschitz continuous. □

Lemma 2. Let the common boundary of R_j, R_k be a nullcline $S_{jk} : \dot{x}_i = 0$, which is not on the coordinate hyperplanes. Let $|f_{ji}(.)|$, $|f_{ki}(.)|$ be primary LFs in R_j, R_k respectively, provided that, $f_{ji}(.) - f_{ki}(.) = f_i(.)$. Then $v_j(.) = \alpha_{jk}|f_{ji}(.)|$, $v_k(.) = \alpha_{jk}|f_{ki}(.)|$, $\alpha_{jk} \in R^+$ are proper LFs for R_j, R_k, respectively.

Proof. Since $f_{ji}(.) - f_{ki}(.) = f_i(.)$ and $\forall x \in S_{jk} \Rightarrow f_i(x) = \dot{x}_i = 0 \Rightarrow f_{ji}(x) - f_{ki}(x) = 0 |f_{ji}(x)| = |f_{ki}(x)| \Rightarrow v_j(x) = v_k(x)$, that satisfy (3), hence, they are proper LFs for R_j, R_k.

Note 1: The lowest order of all LFs should be equal together; this is a necessary condition for satisfying (3) on all coordinate hyperplanes. In many cases, a good selection of LFs is $f_{ji}(x) = \sum_{l=1}^{n} a_{l1} x_l + h.o.t._1$, $f_{ki}(x) = \sum_{l=1}^{n} a_{l2} x_l + h.o.t._2$, $f_{ji}(x) - f_{ki}(x) = f_i(x)$ where, $h.o.t._1$ and $h.o.t._2$ are high order terms of $f_i(.)$. □

Lemma 3. Let R_j be a region whose boundaries are the coordinate hyperplanes, *i.e.* this region is an orthant. Let the common boundary of R_j and R_{jk_p} be $S_{jk_p} : x_i = 0$. Suppose $v_{jk_p}(.)$ for more or each p, have been selected using the previous Lemmas. Let each $v_{jk_p}(x)|_{x_i=0}$ be a primary LF for R_j, then $v_j(x) = \sum_{i=1}^{n} b_{k_p} v_{jk_p}(x)|_{x_i=0}, b_{k_p} \in R^+$ is a primary LF for R_j. Furthermore, if system (1) is two-dimensional and $b_{k_1} = b_{k_2} = 1$, then it is a proper LF for it.

Proof. Since $v_{jk_p}(.)$ is a primary LF for R_{jk_p} and $S_{jkp} \subset R_{jk_p}$, then $v_{jk_p}(x)|_{x_i=0}$ is a primary LF for $S_{jkp} : x_i = 0$. By assumption, if each $v_{jk_p}(x)|_{x_i=0}$ is a primary LF for R_j, then, $b_{k_p} v_{jk_p}(x)|_{x_i=0}$ is a primary LF for R_j. Since $v_j(x)$ satisfies (2), it is a primary LF for R_j. For a two-dimensional system with $b_{k_1} = b_{k_2} = 1$, since, $v_j(x)|_{x_1=0} = v_{k_1}(x)|_{x_1=0}$, $v_j(x)|_{x_2=0} = v_{k_2}(x)|_{x_2=0}$ and $v_j(x) = v_{k_1}(x)|_{x_1=0} + v_{k_2}(x)|_{x_2=0}$ satisfies (3) on two boundaries of R_j, it is a proper LF for it. □

Lemma 4. Let $R_j \subset Q_J$ and $R_k \subset Q_K$ be two neighboring regions whose common boundary is $S_{jk} : x_i = 0$. Suppose v^{Q_J}, v^{Q_K} denote LFs of the regions that are in two quadrants Q_J and Q_K (see Definition 2), have been selected by the previous Lemmas. Suppose $d_j(x) = v_j(x)|_{x_i=0}$, $d_k(x) = v_k(x)|_{x_i=0}$ and $d_{jk}(x) = d_j(x) - d_k(x)$, if appropriate parameters of $v_j(.)$, $v_k(.)$ are specified in the below cases, then (3) is satisfied on $S_{jk} : x_i = 0$, and they will be proper LFs:

(a) If $d_{jk}(x) = 0$, then (3) is satisfied on $S_{jk} : x_i = 0$.
(b) Else if the lowest order of LFs doesn't exist in $d_{jk}(.)$, by adding $d_{jk}(.)$ with each v^{Q_K} (or $-d_{jk}(.)$ with each v^{Q_J}), then (3) is satisfied on $S_{jk} : x_i = 0$.
(c) Else if, the lowest order statements of LFs exist in $d_{jk}(.)$, then $d(.)$, $-d'(.)$ are selected as primary LFs in Q_K, Q_J respectively, such that $d_{jk}(.) = d(.) - d'(.)$. Then by adding $d(.)$ with LFs in v^{Q_K}, and $-d'(.)$ with LFs in v^{Q_J}, then (3) is satisfied on $S_{jk} : x_i = 0$.

Proof. (a) If $d_{jk}(.) = 0 \Rightarrow \forall x \in S_{jk} \Rightarrow d_j(x) = d_k(x) \Rightarrow v_j(.)|_{x_i=0} = v_k(.)|_{x_i=0}$, *i.e.* $v_j(.)$, $v_k(.)$ are continuous on $S_{jk} : x_i = 0$, and satisfy (3).

(b) Note that, the lowest order of all LFs is equal together. If the lowest order of $v_j(.)$, $v_k(.)$ is deleted and doesn't exist in $d_{jk}(.)$, since the value of $d_{jk}(.)$ in the neighborhood of the origin is smaller than the value of each of the LFs v^{Q_K} or v^{Q_J}, the new LF of each region in Q_K (or Q_J) are constructed, by adding $d_{jk}(.)$

with each LF, v^{Q_K} (or $-d_{jk}(x)$ with each LF, v^{Q_J}). Therefore, (3) is satisfied on $S_{jk}: x_i = 0$, because the new $d_{jk}(.) = 0$ (*i.e.* difference between the new $v_j(.)$ and new $v_k(.)$ equals zero).

(c) If the lowest order of $v_j(.)$, $v_k(.)$ exists in $d_{jk}(.)$, we select $d(.)$, $-d'(.)$, that are primary LFs in Q_K, Q_J respectively, such that $d_{jk}(.) = d(.) - d'(.)$. Since the sum of two primary LFs in a region is a new primary LF for it using (2), new LFs v^{Q_K} are constructed by adding $d(.)$ with each LF v^{Q_K}. Similarly, new LFs in v^{Q_J} are obtained by adding $-d'(.)$ with each LF v^{Q_J}, and new $d_{jk}(.) = 0$, then (3) is satisfied on $S_{jk}: x_i = 0$. □

Lemma 5. Suppose a function $V: D \to R$ in (4) is constructed by the special LF $v_j(.)$, which satisfies (2) $\forall x \in B_j = B \cap R_j$, and satisfies (3) on the boundaries of the regions in open set B.

$$V(x) = \sum_{j=1}^{m} v_j(x) \cdot \Psi_j(x), \quad \Psi_j(x) = \begin{cases} 1 & x \in B_j \\ 0 & x \notin B_j \end{cases}, \qquad \dot{V}_f(x) = \sum_{j=1}^{m} \dot{v}_j(x) \cdot \Psi_j(x) \tag{4}$$

$\Psi_j(.)$ is a characteristic function. $V(.)$ is a nonsmooth continuous positive definite function, and $\forall x \in A_V = (\{0\} \cup \bigcup_{j=1}^{m} \widehat{B}_j) \subset B$, $\widehat{B}_j = B \cap \mathring{R}_j$, where the derivative of $V(.)$, *i.e.* $\dot{V}_f(.)$ is defined ($\mathring{R}_j$ is interior of R_j) and $\dot{V}_f(.) \leq 0$ almost every where.

Proof. From (2) and (4), we have $\forall j, v_j(0) = 0 \Rightarrow V(0) = 0$, $V(.)$ is continuous at the origin, and $\forall x \in B_j$, $x \neq 0: v_j(x) = V(x) > 0$; therefore, $V(.)$ is positive-definite. $V(.)$ isn't generally differentiable on the boundaries, we have $\forall x \in \widehat{B}_j, \dot{v}_j(x) = \dot{V}_f(x) \leq 0$. Also $\forall_j, \dot{v}_j(0) = \dot{V}_f(0) = 0$, then $\forall x \in A_V = (\{0\} \cup \bigcup_{j=1}^{m} \widehat{B}_j) \subset B$ then $\dot{v}(x) = \dot{V}_f(x) \leq 0$ is defined, so it is differentiable at every point of trajectory x outside the set N which is a null measure set, since its elements are the countable points that are intersection of the trajectory of answer and the countable boundaries, we have from (3) on the boundaries, $\forall x \in S_{jk}: V(x) = v_j(x) = v_k(x)$. Then $V(.)$ is continuous everywhere, but its derivative $\dot{V}_f(.)$ is defined almost everywhere and it is negative semi definite. □

Lemma 6. Let function $V(.)$ in (4) be constructed by the special LFs in Lemma 6. Then it is a locally Lipschitz continuous function.

Proof. From Lemma 5, $\dot{V}_f(.)$ is defined at every point in the interior of each region, then $V(.)$ is locally Lipschitz continuous in these points. But $\dot{V}_f(.)$ isn't generally defined on the boundaries. The special LFs are continuous on the boundaries, $\forall x_o \in S_{jk}: V(x_o) = v_j(x_o) = v_k(x_o)$. From the Lipschitz definition, $\forall x \in B_j \Rightarrow |V(x) - V(x_0)| = |v_j(x) - v_j(x_0)| \leq L_1 \|x - x_0\|$ and $\forall x \in B_k \Rightarrow |V(x) - V(x_0)| = |v_k(x) - v_k(x_0)| \leq L_2 \|x - x_0\|$, so $\forall x \in (B_j \cup B_k) \Rightarrow |V(x) - V(x_0)| \leq \max\{L_1, L_2\} \|x - x_0\|$. Hence, $V(.)$ is locally Lipschitz on the set $B = \cup B_j$, since each point of B has a neighborhood, such that $f(.)$ satisfies the Lipschitz condition for all points of it with some Lipschitz constants. □

Lemma 7. $V(.)$ in (4) is constructed by the special LFs using Lemma 5, then it is an NP function.

Proof. By assumption, each $v_j(.)$ was selected smooth by (2), so $\forall x \in \widehat{B}_j, \dot{v}_j(x) = \dot{V}_f(x) \leq 0$ it is continuously differentiable, then one-order partial derivatives of it exist and they are continuous. Since $\forall x \in \widehat{B}_j, \dot{V}_f(x) = \nabla V(x).\dot{x}$, $\nabla V(.)$ exists and it is continuous and hence $\lim_{x_l \to x} \nabla V(x_l)$ exists. Therefore, $\forall x_l, x \in \widehat{B}_j \Rightarrow \lim_{x_l \to x} \nabla V(x_l) = \nabla V(x)$, and $\partial_C V(x) = co\{\lim_{x_l \to x} \nabla V(x_l), x_l \to x, \ x_l \notin N\} = \nabla V(x)$. Since $\forall x \in \widehat{B}_j, \dot{V}_f(x) = \nabla V(x).\dot{x}$, and $\partial_C V(x) = \nabla V(x)$ then $\partial_C V(\varphi(t)) = \nabla V(\varphi(t))$ is orthogonal to $\dot{\varphi}(t)$ within each region. Therefore, $\nabla V(\varphi(t))$ is defined at every point of trajectory $\varphi(.)$ outside the set N whose measure is zero, since its elements are the countable points that are intersection of the trajectory and the countable boundaries. $V(.)$ is locally Lipschitz continuous via Lemma 6, and for every absolutely continuous function $\varphi(t)$ of (1) for almost every $t \geq 0$ the set Clark's gradient $\partial_C V(\varphi(t))$ is a subset of an affine subspace orthogonal to $\dot{\varphi}(t)$ almost every where. Moreover $V(.)$ is locally Lipschitz by Lemma 6, then by Definition in [1], $V(.)$ is NP. □

Proposition 1. Let $V: D \to R$ be a positive definite, locally Lipschitz continuous and NP function, and $\varphi: [0, T_\varphi) \to R^n$ be any solution of (1) with an initial condition $x_0 = \varphi(0) \subset D$, then $V \circ \varphi: [0, T_\varphi) \to [0, +\infty)$ is differentiable almost every where. If $(\mathrm{d}/\mathrm{d}t)V(\varphi(t)) \leq 0$ for almost every t, then $T_\varphi = +\infty$, and the origin is Lyapunov stable and $V(.)$ is an NPLF.

Proof. Since $V \circ \varphi$ is absolutely continuous then it is differentiable almost everywhere (see Lemma 1 of [1]). Given $\varepsilon > 0$, choose $r \in (0, \varepsilon]$ such that $B_r = \{x \in R^n, \|x\| \leq r\} \subset D$. Let $\alpha = \min_{\|x\|=r} V(x)$, then $\alpha > 0$, since $V(.)$ is positive definite. Take $\beta \in (0, \alpha)$

and let $B_\beta=\{x\in B_r, V(x)\le\beta\}$ which is a compact set in the interior of B_r. By assumption for any $x_0=\varphi(0)\in B_\beta\subset D$, at least a solution $\varphi(t)$, $t\in I=[0,T_\varphi)$ of (1) exists. The set B_β has the property that any trajectory with the initial condition $\varphi(0)\in B_\beta$ starting in it is bounded, *i.e.* $\varphi(t)\in B_\beta$. Furthermore $(\mathrm{d}/\mathrm{d}t)V(\varphi(t))\le 0$ for almost every t, and $V\circ\varphi$ is non-increasing, that is $V(\varphi(t))\le V(\varphi(0))\le\beta$, for every $t\in[0,T_\varphi)$. This follows from: if $t_3\ge t_2\ge t_1\ge 0$ and $(\mathrm{d}/\mathrm{d}t)V(\varphi(t))$ isn't differentiable at t_2 and it is differentiable at t_3, t_1 and $V\circ\varphi$ is absolutely continuous, then $V(\varphi(t_2))-V(\varphi(t_1))=\int_{t_1}^{t_2^-}\frac{\mathrm{d}}{\mathrm{d}s}V(\varphi(s))+\int_{t_2^+}^{t_3}\frac{\mathrm{d}}{\mathrm{d}s}V(\varphi(s))$, $(\mathrm{d}/\mathrm{d}t)V(\varphi(t))\le 0$. By Remark 1 of [1], since $\varphi(t)\in B_\beta$ is bounded, one can take $T_\varphi=+\infty$, so $V\circ\varphi$ is non-increasing for any $t\in[0,+\infty)$. Since $V(.)$ is continuous and $V(0)=0$, there is $\delta>0$, such that $\|x\|\le\delta\Rightarrow V(\varphi(t))<\beta$, and $B_\delta\subset B_\beta\subset B_r$. So $\varphi(0)\in B_\delta\Rightarrow\varphi(0)\in B_\beta$ and then $\varphi(t)\in B_\beta\Rightarrow\varphi(t)\in B_r$ for any $t\ge 0$, or $|\varphi(0)|<\delta\Rightarrow|\varphi(t)|<r\le\varepsilon$, $\forall t\ge 0$. It shows that the origin is Lyapunov stable and $V(.)$ is an NPLF. □

Proposition 2. Let $V:D\to R$ be a positive definite and NP function, and $A_V=B-N$ be a set of points of system (1) where the gradient of $V(.)$ exists, and elements of set N are countable points. If $\forall x\in A_V$, $\dot V_f(x)\le 0$, then any solution of (1) is bounded with any initial condition $x_0=x(0)\in B$, thus $V(.)$ is an NPLF for the system and the origin is Lyapunov stable.

Proof. By assumption, $\nabla V(x)$ exists except to $\forall x\in N$. Since $x=\varphi$, in attention to Proposition 1, $\forall x\in A_V$, $\dot V_f(x)\le 0$ exists and $V\circ\varphi:[0,T_\varphi)\to[0,+\infty)$ is differentiable, and $\forall x\in N$ does not have derivative. Let for any t, $\varphi(t)$ be on $x=\varphi(t)$, if in this point $\nabla V(x)$ doesn't exist, then $V(\varphi(t))$ is not differentiable at this t. According to Proposition 1, for any solution $\varphi(.)$ of (1) with the initial condition $x_0=\varphi(0)\in B$, $V\circ\varphi$ is non-increasing and $\varphi(.)\in B$ is bounded. So, by Proposition 1, $(\mathrm{d}/\mathrm{d}t)V(\varphi(t))\le 0$ for almost every $t\ge 0$, then $V(.)$ is an NPLF and the origin is Lyapunov stable. □

Theorem 1. Consider the system (1). Suppose according to Definition 1, an open set D is divided into several regions R_j, and $V:D\to R$ in (4) is constructed by the special LFs $v_j(.)$, which satisfy (2) in B almost every where, and satisfy (3) on all boundaries of the regions in B. Then $V(.)$ is an NPLF for (1) and the origin is Lyapunov stable.

Proof. By Lemma 5, $V(.)$ is positive definite and $\dot V_f(x)=\sum_{j=1}^{m}\dot v_j(x)\Psi_j(x)\le 0$ is defined in A_V, whose points are interior of all regions in B, therefore it is differentiable for the points in B almost every where. Furthermore, $V(.)$ is continuous on the boundaries in B, then it is an NP, by Lemma 7. So, by Proposition 2, $V(.)$ is an NPLF for (1) and the origin is stable. □

III. CONCLUSION

In this paper, a method was proposed for constructing Lyapunov functions for Caratheodory systems based on the nonpathological Lyapunov theorem. The stability analysis of the origin by this method was possible. If a nonpathological LF was constructed in the neighborhood of the origin, the system is Lyapunov stable.

REFERENCES

1. Bacciotti, A. and F. Ceragioli, “Nonpathological Lyapunov functions and discontinuous Caratheodory systems,” *Automatica*, Vol. 42, pp. 453–458 (2006).
2. Clarke, F. H., Y. S. Ledyaev, and R. J. Stern, “Asymptotic stability and smooth Lyapunov functions,” *J. Differ. Equ.*, Vol. 149, pp. 69–114 (1998).
3. Grzanek, M. A., Michalak, and A. Rogowski, “A nonsmooth Lyapunov function and stability for ODE's of Caratheodory type,” *Nonlinear Anal.*, Vol. 69, pp. 337–342 (2008).
4. Shevitze, D. and B. Paden, “Lyapunov stability theory of nonsmooth systems,” *IEEE Trans. Autom. Control*, Vol. 39, No. 9, pp. 1910–1914 (1994).
5. Zhai, G., B. Hu, K. Yasuda, and A. N. Michel, “Piecewise Lyapunov functions for switched systems with average dwell time,” *Asian J. Control*, Vol. 2, pp. 192–197 (2000).