Abstract

Let $G$ be a finite group and $E_2(G)$ denote the probability that $[x, y, y] = 1$ for randomly chosen elements $x, y$ of $G$. We will obtain lower and upper bounds for $E_2(G)$ in the case where the sets $E_G(x) = \{y \in G : [y, x, x] = 1\}$ are subgroups of $G$ for all $x \in G$. Also the given examples illustrate that all the bounds are sharp.

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1 Introduction

For a given natural number $n$, the $n$-Engel degree of $G$, denoted by $E_n(G)$, is the probability that two randomly chosen elements $x, y$ of $G$ satisfy the $n$-Engel condition $[y, n x] = 1$, that is

$$E_n(G) = \frac{|\{(x, y) \in G \times G : [y, n x] = 1\}|}{|G|^2}.$$

The case $E_1(G)$, the commutativity degree of $G$ is extensively studied in the literature. We intend to study $E_2(G)$ and give some lower and upper bound in terms of $E_2(G)$. Note that the situation is too complicated for computing $E_n(G)$, when $n > 2$.  

88
2 Preliminary results

We begin with some elementary lemmas.

**Lemma 2.1** Let \( G \) be a group and \( x \in G \). Then the followings statements are equivalent:

(i) \( E_G(x) \leq G \);  
(ii) \( [E_G(x), x, E_G(x), x] = 1 \);  
(iii) \( [[E_G(x), x], [E_G(x), x]] = 1 \) that is \( E_G(x), x \) is abelian.

According to the above lemma we shall restrict ourselves to groups in variety \( V \) of all finite groups admitting the law \( [[x, y], [x, z]] = 1 \). This variety is studied by Macdonald [5] and Farrokhi and Moghaddam [2].

**Lemma 2.2** Let \( G \) be a finite group in variety \( V \) and \( x \in G \). Then

(i) \( E_G(x) \leq G \);  
(ii) \( |E_G(x)| = |C_G(x)||G_G(x) \cap x^G| \);  
(iii) \( |C_G(x)x^G : C_G(x)| = [G : E_G(x)] \);  
(iv) \( |C_G(x)x^G| = [G : C_G(x) \cap x^G] \) divides \( |G| \).

**Lemma 2.3** Let \( G \) be a finite group in variety \( V \) with an element \( x \) such that \( C_G(x)x^G = G \). Then

(i) \( G = [x, G] \rtimes C_G(x) \);  
(ii) If \( x \in L(G) \), then \( x \in Z(G) \).

3 Main theorems

Our main theorems give sharp lower and upper bounds for 2-Engel degree of a finite group.

**Theorem 3.1** Let \( G \) be a finite non 2-Engel group belonging to variety \( V \) and \( p = \min \pi(G) \). Then

\[
E_2(G) \leq \frac{1}{p} + \left( 1 - \frac{1}{p} \right) \frac{|L_2(G)|}{|G|}
\]

and if \( L_2(G) \leq G \), then

\[
E_2(G) \leq \frac{2p - 1}{p^2}.
\]

Moreover, both of the upper bounds are sharp at any prime \( p \).
Theorem 3.2 Let $G$ be a finite non-2-Engel group belonging to variety $V$ and $p = \min \pi(G)$. Then

$$E_2(G) \geq E_1(G) - (p - 1)\frac{|Z(G)|}{|G|} + (p - 1)\frac{k_G(L(G))}{|G|}$$

and if either $G$ is a $p$-group, or $G'$ is a cyclic 2-group or a generalized quaternion 2-group, then

$$E_2(G) \geq pE_1(G) - (p - 1)\frac{|Z(G)|}{|G|}.$$ 

Moreover, both of the lower bounds are sharp at any prime $p$.

References


