Doubly truncated mean and variance residual lifetime in reliability

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Studying the reliability concepts of the components of a system or a device based on conditional random variables are important and usual, because, most of the real observations are left, right or doubly truncated censored data. In many situations, we only have information about between two lifetime points, so we should study the reliability measures under the condition of doubly truncated random variables. One of the important and applicable reliability concepts, that recently has gathered the most attention of the researchers, is the variance residual life.

In this paper, we try to study some of the reliability properties of the variance residual life based on doubly censored data which has not been studied precisely yet. Its monotonicity properties and relation with doubly truncated mean residual life and doubly truncated residual coefficient of variation are discussed. Furthermore, the lower (upper) bound for it under some conditions is obtained. Finally, some examples due to this subject are mentioned.

Keywords: Doubly truncated mean residual life, Doubly truncated residual coefficient of variation, Doubly truncated variance residual life, extended exponential family of distributions, Generalized failure rate.

1. Introduction

Reliability has recently been very important in many aspects of real life situations. Initially, reliability was motivated by its application in the field of military and space technology. Now, it has become very important in production planning, design, operation and maintenance modeling.

The applications of classes of life distributions can be seen in reliability, engineering, biological science, maintaining and biometrics. Therefore, statisticians and reliability analysts are interested in modeling survival data and using classifications of life distributions based on some aspects of aging. (see for example, Barlow and Proschan (1981), Zacks (1992), Lai and Xie (2006)).

If X denote the lifetime of a unit, then the random variable ${}_xX_y = X - x|x \leq X \leq y$ is called the doubly truncated residual lifetime. Note that the well-known random variable, $X_x = X - x|X \geq x$, which is called residual lifetime in reliability, is the spacial case of ${}_xX_y$ when y tends to ∞.

In reliability, few works have been done based on ${}_xX_y$. Navarro and Ruiz (1996) generalized the failure rate function for doubly truncated random variables. Ruiz and Navarro (1995) characterized the discrete distributions using expected values based on ${}_xX_y$. Later, Sankaran and Sunoj (2004) defined and obtained some properties of the doubly truncated mean residual life ($dMRL$) in continuous lifetime distributions. Recently, many authors such as Ruiz and Navarro (1994, 1995), Su and Huang (2000), Ahmad (2001), Betensky and Martin (2003), Navarro and Ruiz (2004), Sankaran and Sunoj (2004), Bairamov and Gebizlioglu (2005), Poursaeed and Nematollahi (2008) and Sunoj et al. (2009), studied the properties of the conditional expectations of doubly truncated random variables in various areas like order statistics and k-out-of-n systems.

The variance and coefficient of variation of the random variable X_x, which are called variance residual life (VRL) and residual coefficient of variation (RCV) functions respectively, have gathered the attention of most researchers in reliability at the recent years. Karlin (1982) has studied the monotonic behavior of VRL when the density is log-convex or log-concave. Gupta (1987, 2006) studied the monotonicity of VRL and the associated aging classes of life distributions. Launcer (1984) and Gupta et al. (1987) discussed the class of life distributions having decreasing (increasing) variance residual life. The role and properties of the variance residual life and the residual coefficient of variation in reliability have been discussed considerably for continuous lifetime random variables by various authors such as Gupta and Kirmani (1998, 2000, 2004), El-Arishi (2005), Al-Zahrani and Stoyanov (2008) and Abu-Youssef (2004, 2007, 2009).

The variance and coefficient of variation of the random variable ${}_xX_y$, which we called them doubly truncated variance residual life ($dVRL$) and doubly truncated residual coefficient of variation ($dRCV$) functions respectively, have many applications in areas like biometry, actuarial science, burn-in concept and reliability.

In this paper, we study the doubly truncated variance residual life. Its relationship with doubly truncated mean residual life and doubly truncated residual coefficient of variation are obtained. Also, its monotonicity and the associated ageing classes of distributions are discussed. Some characterization results of the class of the increasing (decreasing) $dVRL$ are presented and the upper (lower) bound for $dVRL$ under some conditions is obtained. Furthermore, for some distributions such as, exponential and Weibull, the behavior of $dVRL$ and $dMRL$ are presented.

2. Preliminaries

Let X be a non-negative continuous random variable with cumulative distribution function (cdf), $F(x)$ and probability density function (pdf), $f(x)$.

Navarro and Ruiz (1996) defined the generalized failure rate (GFR) to the doubly truncated random variable ${}_xX_y$ by

$$h_1(x,y) = \lim_{h\to 0^-}\left[\frac{p_r(x \le X \le x+h|x \le X \le y)}{h}\right] = \frac{f(x)}{F(y)-F(x)},$$

and

$$h_2(x,y) = \lim_{h\to 0^-}\left[\frac{p_r(y \le X \le y+h|x \le X \le y)}{h}\right] = \frac{f(y)}{F(y)-F(x)},$$

Later, Sankaran and Sunoj (2004) defined doubly truncated mean residual life $(dMRL)$ for continuous lifetime distributions, which is denoted here by $\alpha(x,y)$, as,

$$\alpha(x,y) = E(X-x|x \le X \le y).$$

It has been shown that the generalized failure rate (GFR) and $dMRL$ determine the distribution uniquely.

3. Doubly truncated variance residual life

We define the continuous doubly truncated variance residual life function $(dVRL)$ as,

$$v(x,y) = Var(X_{x,y}) = Var(X-x|x \le X \le y),$$

such that $E(X^2) < \infty$.

Definition 3.1. A random variable X is said to be,

(i) increasing in doubly truncated mean residual life $(IdMRL)$ if $\alpha(x,y)$ is increasing in $x \ge 0$.
(ii) increasing in doubly truncated variance residual life $(IdVRL)$ if $v(x,y)$ is increasing in $x \ge 0$.

The ageing classes of decreasing doubly truncated mean residual life $(DdMRL)$ and decreasing doubly truncated variance residual life $(DdVRL)$ are defined analogously.

Using part by part integration method, we can see that $v(x,y)$ and $\alpha(x,y)$ are related as follows,

$$v(x,y) = \frac{(x^2-y^2)\overline{F}(y) + 2\int_x^y t\overline{F}(t)dt}{\overline{F}(x)-\overline{F}(y)} - 2x\alpha(x,y) - \alpha^2(x,y). \quad (1)$$

In the next theorem, we obtain an upper and a lower bound for $v(x,y)$, when X has $DdMRL$ and $IdMRL$ property, respectively.

Theorem 3.1. *If the non-negative continuous random variable X, has $DdMRL$ property, then,*

$$v(x,y) < \alpha^2(x,y); \quad 0 \le x < y, \tag{2}$$

and for $x = y$ the above inequality changed to equality.

Proof. In continuous case, we have,

$$\alpha(x,y) = \frac{\int_x^y \overline{F}(t)dt - (y-x)\overline{F}(y)}{\overline{F}(x) - \overline{F}(y)}, \tag{3}$$

hence,

$$\begin{aligned}\int_x^y [\overline{F}(t) - \overline{F}(y)]\alpha(t,y)dt &= \int_x^y [\int_t^y \overline{F}(z)dz - (y-t)\overline{F}(y)]dt \\ &= \int_x^y z\overline{F}(z)dz - x\int_x^y \overline{F}(z)dz - \int_x^y (y-t)\overline{F}(y)dt,\end{aligned}$$

using (1), it implies that,

$$\begin{aligned}\frac{2}{\overline{F}(x) - \overline{F}(y)} \int_x^y [\overline{F}(t) - \overline{F}(y)]\alpha(t,y)dt &= \frac{(x^2 - y^2)\overline{F}(y) + 2\int_x^y z\overline{F}(z)dz}{\overline{F}(x) - \overline{F}(y)} - 2x\alpha(x,y) \\ &= \alpha^2(x,y) + v(x,y).\end{aligned}$$

So, we have,

$$\begin{aligned}v(x,y) - \alpha^2(x,y) &= \frac{2}{\overline{F}(x) - \overline{F}(y)} \int_x^y [\overline{F}(t) - \overline{F}(y)][\alpha(t,y) - \alpha(x,y)]dt \\ &\le 0,\end{aligned}$$

since $\alpha(x,y)$ is decreasing in x. Hence the result follows. □

Remark 3.1. When the non-negative continuous random variable X, has $IdMRL$ property, we can get a lower bound for $v(x,y)$ by reversing the inequality in (2).

Now, we investigate the connection between $DdVRL(IdVRL)$ and other classes of distributions.

Theorem 3.2. *If $\alpha(x,y)$ is increasing (decreasing) in x, then $v(x,y)$ is increasing (decreasing) in x, i.e., the $IdMRL(DdMRL)$ property is stronger than the $IdVRL(DdVRL)$ property.*

Proof. Using (1), we have,

$$\frac{\partial}{\partial x} v(x,y) = h_1(x,y)[v(x,y) - \alpha^2(x,y)]. \tag{4}$$

According to (4) and Theorem 3.1 the result follows. □

Remark 3.2. When the general failure rate $h_1(x, y)$ of F is increasing (decreasing) in x, then F has $DdVRL(IdVRL)$ property. Since, it can be seen that, when $h_1(x, y)$ is increasing (decreasing) in x, it implies $DdMRL(IdMRL)$ property.

Another reliability measure that has been recently considered and related to the variance and the mean residual life, is the residual coefficient of variation. So, in doubly truncated random variables we consider the residual coefficient of variation ($dRCV$) as,

$$\gamma(x, y) = \frac{\sqrt{v(x, y)}}{\alpha(x, y)}. \tag{5}$$

The monotonic behavior of $v(x, y)$ in terms of $\gamma(x, y)$ is considered in the following theorem.

Theorem 3.3. *$v(x, y)$ is increasing (decreasing) in x according as $\gamma^2(x, y) \geq (\leq)1$.*

Proof. The (4) can be written as,

$$\frac{\partial}{\partial x} v(x, y) = h_1(x, y)\alpha^2(x, y)[\gamma^2(x, y) - 1],$$

which provided the required result. □

The next theorem characterizes the monotonic behavior of the variance of the random variable $X_{x,y}$. A similar result for the variance of X_x is given by Gupta (1987).

Theorem 3.4. *The following statements are equivalent,*

(i) $v(x, y)$ *is increasing (decreasing) in* $x > 0$.
(ii) $\gamma^2(x, y) \geq (\leq)1$ *for all* $x \geq 0$.
(iii) $\Phi(x, y) = \frac{E[(X-x)^2 | x \leq X \leq y]}{E[X-x | x \leq X \leq y]}$ *is increasing (decreasing) in* $x > 0$.

Proof. The equivalence of (i) and (ii) is considered in Theorem 3.3. Note that,

$$\Phi(x, y) = \frac{v_c(x, y) + \alpha_c^2(x, y)}{\alpha_c(x, y)}, \tag{6}$$

so, derivation of right hand side of (6) w.r.t x and using the relation,

$$\frac{\partial}{\partial x}\alpha(x, y) = h_1(x, y)\alpha(x, y) - 1, \tag{7}$$

and (1) shows the equivalence of (ii) and (iii). □

Example 3.1. Let X be a non-negative continuous random variable with distribution function,

$$G(x) = 1 - e^{-\lambda k_\theta(x)}; \quad x > 0, \tag{8}$$

where $\lambda > 0$ and θ is a the value of parameter vector (which may contains λ) and $k_\theta(x)$ is a strictly increasing function of x with $k_\theta(0) = 0$ and $k_\theta(x) \to \infty$ as $x \to \infty$. The density function of X is,

$$g(x) = \lambda k'_\theta(x) e^{-\lambda k_\theta(x)}; \quad x > 0,$$

such that $k'_\theta(x)$ is the derivative of $k_\theta(x)$ with respect to x. Rezaei Roknabadi et al. (2009) introduced the class of distributions (8) and called it *extended exponential* and denoted it by $EE(\lambda, k_\theta)$. This class contains many well known continuous life time distributions such as exponential, Rayleigh, Weibull, Linear-exponential, Gomperts, Rue, Brittle-Fracture and Wear-out.

- For $k_\theta(x) = x$, X have exponential distribution with parameter λ and density function $g(x) = \lambda e^{-\lambda x}, x \geq 0$. In this case we have,

  $$\alpha(x, y) = \frac{1}{\lambda} - \frac{y - x}{e^{-\lambda(x-y)} - 1},$$
  $$v(x, y) = \frac{(x^2 - y^2 - \frac{2}{\lambda^2} - \lambda y)e^{-\lambda y} + (\frac{2}{\lambda^2} + \lambda x)e^{-\lambda x}}{e^{-\lambda x} - e^{-\lambda y}} - 2x\alpha(x, y) - \alpha^2(x, y).$$

  It can be seen that for all $\lambda \geq 0$, $\alpha(x, y)$ is decreasing in x. In Figure 1, the plots of $\alpha^2(x, y)$ and $v(x, y)$ for $\lambda = 0.001, 0.5, 1, 2$ are drawn. As you see this figure confirms the Theorems 3.1 and 3.2.
- For $k_\theta(x) = x^\theta$, X have Weibull distribution with parameters λ and θ and density function $g(x) = \lambda\theta x^{\lambda-1} e^{-\lambda x^\theta}, x \geq 0$. Using derivation w.r.t x, it can be seen that in this distribution for $\theta \geq 1$, $\alpha(x, y)$ is decreasing and for $\theta < 1$ first it is increasing and then decreasing. In Figure 2 the plots of $\alpha(x, y)$ are drawn for different values of λ and θ. Also, in Figure 3 you can see the agreement with Theorems 3.1 and 3.2.

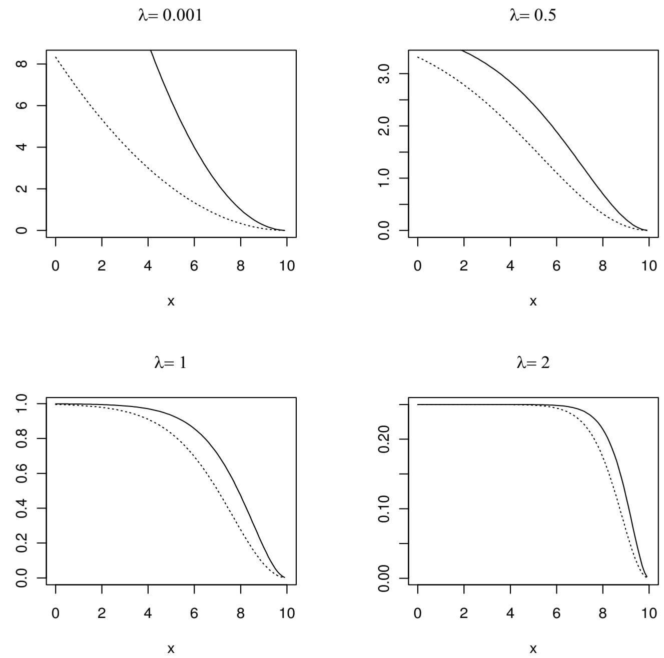

Fig. 1. *The plots of $v(x,y)$ (denoted by '.....') and $\alpha^2(x,y)$ (denoted by '____') in exponential distribution with parameter λ.*

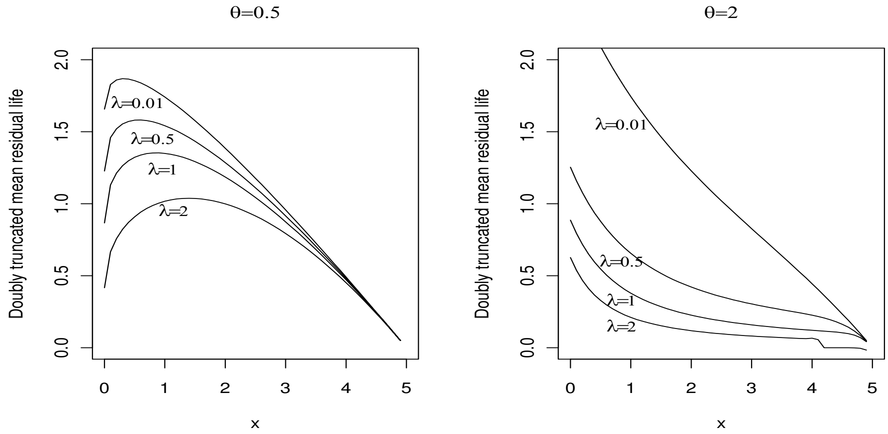

Fig. 2. *The plots of $\alpha(x,y)$ in Weibull distribution with parameters λ and θ.*

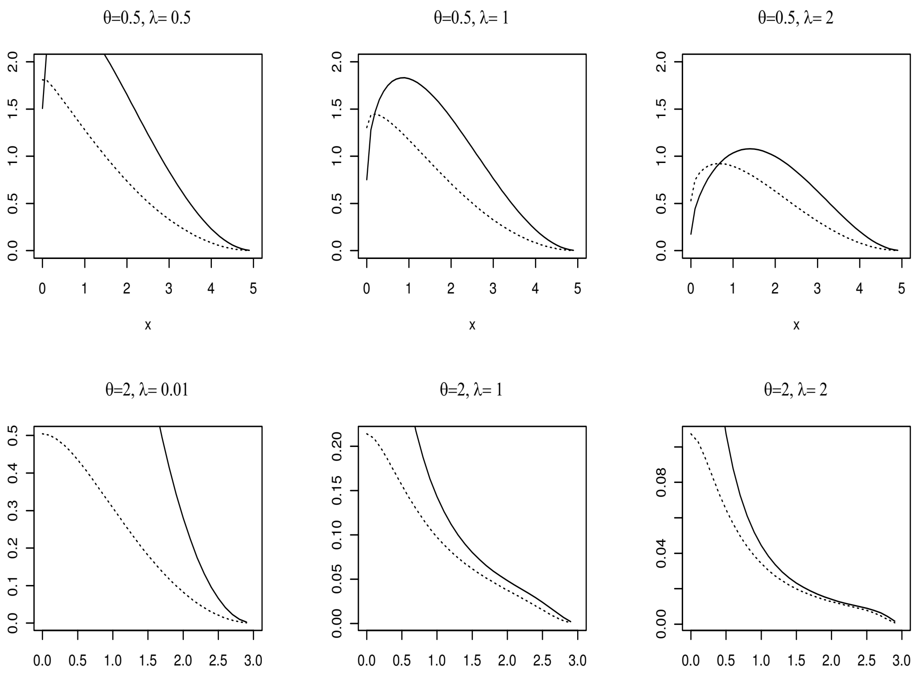

Fig. 3. *The plots of $v(x, y)$ (denoted by '.....')and $\alpha^2(x, y)$ (denoted by '_____') in Weibull distribution with parameters λ and θ.*

4. Summary and conclusions

In this paper, we generalized and obtained some reliability properties of variance residual life via doubly truncation. Furthermore, these measures are plotted and compared in exponential and Weibull distributions.

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