NORM CONTINUITY OF WEAKLY QUASI-CONTINUOUS MAPPINGS

BY

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Abstract. Let $Q$ be the class of Banach spaces $X$ for which every weakly quasi-continuous mapping $f : A \to X$ defined on an $\alpha$-favorable space $A$ is norm continuous at the points of a dense $G_\delta$ subset of $A$. We will show that this class is stable under $c_0$-sums and $\ell^p$-sums of Banach spaces for $1 \leq p < \infty$.

1. Introduction. In 1974, I. Namioka [16] proved that every weakly continuous mapping $f$ from a countably Čech-complete space $A$ into a Banach space $X$ is norm continuous at the points of a dense $G_\delta$ subset of $A$. It was conjectured that Namioka’s result remains valid for any $\alpha$-favorable space $A$. However, in 1985, M. Talagrand [19] gave an example of a weakly continuous nowhere norm continuous mapping defined on an $\alpha$-favorable space. Therefore the following problem naturally arises:

Under what conditions on a Banach space $X$, every weakly quasi-continuous mapping from an $\alpha$-favorable $A$ into $X$ is norm continuous at each point of a dense $G_\delta$ subset of $A$?

During the past four decades similar problems have been considered by several mathematicians: see e.g. [1, 4, 11, 12], [14]–[19] and the references therein.

A Banach space $X$ is said to have the property $Q$ if every quasi-continuous mapping $f$ defined on an $\alpha$-favorable space $A$ into $(X, \text{weak})$ is norm-continuous on a dense $G_\delta$ subset of $A$. It is known that $\ell^\infty$ and $\ell^\infty/c_0$ do not have the property $Q$ [9]. However, the class of Banach spaces with the property $Q$ properly contains all Banach spaces which are weakly $\sigma$-fragmentable [5, 8]. It follows that this class includes all weakly Lindelöf Banach spaces and Banach spaces with an equivalent Kadec norm [6, 9, 13].

In [8] and [9] a game characterization of Banach spaces $X$ with the property $Q$ was obtained. In fact it was shown that the absence of winning

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strategy for one of the players in the fragmenting game on a Banach space $X$ guarantees the property $Q$, and that $\ell^\infty$ is not in the class $Q$.

In this paper, we use this characterization to show that if each member of a family $\{X_\gamma\}_{\gamma \in \Gamma}$ of Banach spaces satisfies the property $Q$, then so do the $c_0$-sum and $\ell^p$-sum of the family for $1 \leq p < \infty$.

2. Preliminaries. Let $A$ and $X$ be topological spaces. Following Kempiaty [7], a mapping $f : A \to X$ is said to be quasi-continuous at a point $a_0 \in A$ if for every open neighborhood $U$ of $f(a_0)$, there exists an open set $V \subset A$ such that $a_0 \in \overline{V}$ (the closure of $V$ in $A$) and $f(V) \subset U$.

The mapping $f$ is called quasi-continuous if it is quasi-continuous at each point of $A$.

Next, we introduce the following topological game on a topological space $A$, which is a version of the classical Banach–Mazur game [2, 3]:

Two players $\alpha$ and $\beta$ alternately select nonempty open subsets of $A$. Player $\beta$ begins the game by selecting a nonempty open set $V_1 \subset A$. In response, $\alpha$ selects some nonempty open subset $W_1 \subset V_1$. Inductively, player $\beta$'s $n$th move is a nonempty open subset $V_n \subset W_{n-1}$ followed by $\alpha$'s $n$th move, a nonempty open subset $W_n$ of $V_n$. Proceeding in this fashion, the players generate a sequence $(V_n, W_n)_{n=1}^\infty$ which is called a play. Player $\alpha$ wins the play $(V_n, W_n)_{n=1}^\infty$ if $\bigcap_{n \geq 1} V_n = \bigcap_{n \geq 1} W_n \neq \emptyset$. Otherwise $\beta$ wins. A partial play is a finite sequence of sets consisting of the first few moves of a play. A strategy $s$ for player $\alpha$ is a rule which determines $\alpha$'s move at each stage based on the game played so far. An $s$-play is a play in which $\alpha$ selects his moves according to the strategy $s$. The strategy $s$ is said to be a winning strategy for the player $\alpha$ if every $s$-play is won by $\alpha$. A strategy for $\beta$ can be defined similarly by switching the sides. $\alpha$ is called $\alpha$-favorable if there exists a winning strategy for $\alpha$ in $A$.

Krom [10] and Raymond [18] have shown independently that a topological space is Baire if and only if it is $\beta$-unfavorable. Hence every $\alpha$-favorable space is a Baire space.

Let $\tau$ and $\tau'$ be two topologies on a set $X$. The topological game $G(X, \tau, \tau')$ is played by two players $\Sigma$ and $\Omega$ as follows:

$\Sigma$ starts a game by taking a nonempty subset $A_1$ of $X$. Then $\Omega$ selects a nonempty relatively $\tau$-open subset $B_1$ of $A_1$. In general if the selection $B_n$ of player $\Omega$ is already specified, $\Sigma$ makes the next move by choosing an arbitrary nonempty set $A_{n+1}$ contained in $B_n$. Continuing, the players produce a sequence of nonempty sets $A_1 \supseteq B_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq B_n \supseteq \cdots$, which is called a play and will be denoted by $p := (A_i, B_i)_{i \geq 1}$. The winning rule is connected with the topology $\tau'$. Player $\Omega$ is said to win a play $p := (A_i, B_i)_{i \geq 1}$ if the set $\bigcap_{n \geq 1} A_n$ is either empty or contains exactly one point
$x$ and for every $\tau'$-open neighborhood $U$ of $x$, there is some positive $n$ with $B_n \subset U$. Otherwise $\Sigma$ wins.

A partial play is a finite sequence which consists of the first few moves $A_1 \supseteq B_1 \supseteq A_2 \supseteq \cdots \supseteq B_n$ of the players. A strategy for either of the players in $G(X, \tau, \tau')$ can be defined as in the Banach–Mazur game.

The game $G(X, \tau, \tau')$ (or the space $X$) is called $\Sigma$-unfavorable if there does not exist a winning strategy for player $\Sigma$.

We have the following connection between the topological game and dense $\tau'$-continuity of $\tau$-quasi-continuous mappings:

**Theorem 2.1** ([8, 9]). Let $\tau$, $\tau'$ be two $T_1$ topologies on a set $X$. Suppose that for every $\tau'$-open set $U$ and every point $x \in U$ there exists a $\tau'$-neighborhood $V$ of $x$ such that $V^\tau \subset U$. Then the following conditions are equivalent:

(i) The game $G(X, \tau, \tau')$ is $\Sigma$-unfavorable.

(ii) Every quasi-continuous mapping $f : Z \to (X, \tau)$ from an $\alpha$-favorable space $Z$ into $(X, \tau)$ is $\tau'$-continuous at all points of a subset which is of second category in every nonempty open subset of $Z$.

In particular, when $\tau'$ is a metrizable topology, the set of $\tau'$-continuity points is a dense $G_\delta$ subset of $Z$.

Let $X$ be a Banach space. By applying the above result when $\tau'$ is the norm topology and $\tau$ is the weak topology on $X$, we get the following result:

**Corollary 2.2.** The following assertions are equivalent:

(a) $X$ does not have the property $\mathcal{Q}$.

(b) There exists a strategy $\sigma$ for player $\Sigma$ in the game $G(X, \text{weak, } \| \cdot \|)$ such that for each $\sigma$-play $(A_i, B_i)_{i \geq 1}$,

$$\bigcap_{n \geq 1} A_n \neq \emptyset \quad \text{and} \quad \text{norm-diam}(A_n) > \varepsilon \text{ for each } n \in \mathbb{N}$$

for some $\varepsilon > 0$.

3. $c_0$-sums of Banach spaces and the property $\mathcal{Q}$. Let $\{(X_\gamma, \| \cdot \|_\gamma) : \gamma \in \Gamma\}$ be a family of Banach spaces. The $c_0$-sum of this family, denoted by $c_0\{X_\gamma : \gamma \in \Gamma\}$, is the set of all $x \in \prod_{\gamma \in \Gamma} X_\gamma$ such that for each $\varepsilon > 0$, the set $\{\gamma : \|x(\gamma)\|_\gamma \geq \varepsilon\}$ is finite. This set equipped with the norm

$$\|x\|_\infty = \sup\{\|x(\gamma)\|_\gamma : \gamma \in \Gamma\}$$

is a Banach space. Throughout this section, we will assume that $X$ is the Banach space $c_0\{X_\gamma : \gamma \in \Gamma\}$, where $\{(X_\gamma, \| \cdot \|_\gamma) : \gamma \in \Gamma\}$ is a family of Banach spaces.
Lemma 3.1. Given $\varepsilon > 0$, player $\Omega$ has a strategy $s$ in $X$ such that for every $s$-play $(A_i, B_i)_{i \geq 1}$ either $\bigcap_{i=1}^{\infty} A_i = \emptyset$ or for some $n_0 \in \mathbb{N}$ and a finite subset $F \subset \Gamma$,

\[(3.1) \quad \{\gamma \in \Gamma : \|x(\gamma)\| > \varepsilon\} \subset F \quad \text{for all } x \in A_{n_0}.\]

Proof. Let $A_1$ be the first choice of player $\Sigma$. Then the following cases may happen:

(i) For each $x \in A_1$ and $\gamma \in \Gamma$, $\|x(\gamma)\|_\gamma \leq \varepsilon$. In this case put $F_1 = \emptyset$ and define $B_1 = s(A_1) = A_1$.

(ii) For some $x_1 \in A_1$ and $\gamma_1 \in \Gamma$, $\|x_1(\gamma_1)\|_{\gamma_1} > \varepsilon$. Then define

$$B_1 = s(A_1) = \{x \in A_1 : \|x(\gamma_1)\|_{\gamma_1} > \varepsilon\}$$

as the first move of $\Omega$ and let $F_1 = \{\gamma_1\}$.

In step $n \geq 2$, when the partial play $p_n = (A_1, \ldots, A_n)$ and finite subsets $F_1 \subset \cdots \subset F_{n-1}$ of $\Gamma$ have already been selected, we consider the following possibilities:

(i) For each $x \in A_n$,

$$\{\gamma \in \Gamma : \|x(\gamma)\|_{\gamma} > \varepsilon\} \subset F_{n-1}.$$ 

In this situation, we define $s(A_1, \ldots, A_n) = B_n$ and $F_n = F_{n-1}$.

(ii) There are some $x_n \in A_n$ and $\gamma_n \in \Gamma - F_{n-1}$ such that $\|x_n(\gamma_n)\|_{\gamma_n} > \varepsilon$.

Let

$$B_n = s(A_1, \ldots, A_n) = \{x \in A_n : \|x(\gamma_n)\|_{\gamma_n} > \varepsilon\}$$

be the next move of $\Omega$ and define $F_n = F_{n-1} \cup \{\gamma_n\}$.

In this way, by induction on $n$, a strategy $s$ for player $\Omega$ is defined. If for some $n_0 \in \mathbb{N}$, (i$_{n_0}$) is satisfied, then for $F = F_{n_0}$, (3.1) holds. Suppose that for each $n \in \mathbb{N}$, (ii$_n$) holds. We will show that $\bigcap_{n \geq 1} A_n = \emptyset$. On the contrary, let $x \in \bigcap_{n \geq 1} A_n$. Define $F_x = \{\gamma \in \Gamma : \|x(\gamma)\|_{\gamma} > \varepsilon\}$. Since $x \in A_n$ for all $n \in \mathbb{N}$,

$$\|x(\gamma_n)\|_{\gamma_n} > \varepsilon \quad (n \in \mathbb{N}).$$

Therefore $F_x$ contains the infinite set $\bigcup_{n \geq 1} F_n$. However, by the definition, $F_x$ is finite. This contradiction shows that in this case $\bigcap_{n \geq 1} A_n = \emptyset$. $\blacksquare$

Lemma 3.2. Suppose that $X_1, \ldots, X_n$ are Banach spaces with the property $Q$. Then $G(\prod_{i=1}^{n} X_i, \text{weak}, \|\cdot\|_\infty)$ is $\Sigma$-unfavorable.

Proof. Let $f$ be a quasi-continuous mapping from an $\alpha$-favorable space $A$ to $\prod_{i=1}^{n} X_i$. Then for each $1 \leq i \leq n$, $\pi_i \circ f : A \to X_i$ is quasi-continuous, where $\pi_i$ denotes the canonical projection map to the $i$th coordinate. Since each $X_i$ has the property $Q$, there is a dense $G_\delta$ subset $D_i$ of $A$ such that
$\pi_i \circ f|_{D_i}$ is norm continuous. Define $D = \bigcap_{i=1}^n D_i$. Clearly $f$ is $\| \cdot \|_\infty$-continuous on the dense $G_\delta$ set $D$. Hence the result follows from Theorem 2.1. \end{proof}

**Theorem 3.3.** Let $\{X_\gamma : \gamma \in \Gamma\}$ be a family of Banach spaces with the property $Q$. Then $c_0\{X_\gamma : \gamma \in \Gamma\}$ has the property $Q$ as well.

**Proof.** On the contrary, suppose that $c_0\{X_\gamma : \gamma \in \Gamma\}$ does not have the property $Q$. Then by Corollary 2.2, player $\Sigma$ has a strategy $\sigma$ such that for each $\sigma$-play $(A_i, B_i)_{i \geq 1}$,

$$\bigcap_{n \geq 1} A_n \neq \emptyset \quad \text{and} \quad \text{norm-diam}(A_n) > \varepsilon$$

for some $\varepsilon > 0$ and all $n \geq 1$. By Lemma 3.1, $\Omega$ has a strategy $s$ such that for each $s$-play $(A_i, B_i)_{i \geq 1}$, either $\bigcap_{n \geq 1} A_n = \emptyset$ or there is a finite subset $F$ of $\Gamma$ such that

$$\|x(\gamma)\|_\gamma \leq \varepsilon \quad \text{for all} \ x \in A_{n_0} \quad \text{and} \quad \gamma \in \Gamma - F.$$  

However, by (3.2), $\bigcap_{n \geq 1} A_n \neq \emptyset$. Therefore, we may assume that (3.3) holds. We define a strategy $s'$ for player $\Omega$ as follows:

For $1 \leq m < n_0$, let $s'(A_1, \ldots, A_n) = s(A_1, \ldots, A_n)$. Suppose that $n \geq n_0$ and the partial play $p_n = (A_1, \ldots, A_n)$ is specified. Let $\pi_F : c_0\{X_\gamma : \gamma \in \Gamma\} \rightarrow \prod_{\gamma \in F} X_\gamma$ be the canonical projection $\pi_F(\{x_\gamma\}_{\gamma \in F}) = \{x_\gamma\}_{\gamma \in F}$. Choose a relatively open subset $B_n$ of $A_n$ such that $\pi_F(B_n)$ is the response of player $\Omega$ to the partial play $(\pi_F(A_1), \ldots, \pi_F(A_n))$ according to the strategy whose existence is guaranteed by Lemma 3.2, and define $s'(A_1, \ldots, A_n) = B_n$.

In this way, a strategy $s'$ for $\Omega$ in $c_0\{X_\gamma : \gamma \in \Gamma\}$ is defined. By Lemma 3.2, $G(\prod_{\gamma \in F} X_\gamma, \text{weak}, \| \cdot \|_\infty)$ is $\Sigma$-unfavorable, hence there is a play $(\pi_F(A_n), \pi_F(B_n))_{n \geq 1}$ such that either $\bigcap_{n=1}^{\infty} \pi_F(A_n) = \emptyset$ or for some $n_0 \in \mathbb{N}$, $\|x(\gamma) - y(\gamma)\|_\gamma < \varepsilon$ for all $x, y \in A_{n_0}$ and $\gamma \in F$. Hence either $\bigcap_{n=1}^{\infty} A_n = \emptyset$ or $\| \cdot \|_\infty$-diam$(A_{n_0}) \leq \varepsilon$, by (3.3). This contradiction proves the theorem. \end{proof}

**4. The property $Q$ for $\ell^p$-sums of Banach spaces.** For $1 \leq p < \infty$, we use $\ell^p\{X_\gamma : \gamma \in \Gamma\}$ to denote the Banach space of all $x \in \prod_{\gamma \in \Gamma} X_\gamma$ for which the norm series

$$\|x\|_p = \left\{ \sum_{\gamma \in \Gamma} \|x(\gamma)\|_{\gamma}^p \right\}^{1/p}$$

converges.

**Lemma 4.1.** Let $\{X_\gamma : \gamma \in \Gamma\}$ be a family of Banach spaces and $\varepsilon > 0$. Then player $\Omega$ has a strategy $s$ in $\ell^p\{X_\gamma : \gamma \in \Gamma\}$ such that for each $s$-play
(A_i, B_i)_{i \geq 1}, either \bigcap_{i=1}^{\infty} A_i = \emptyset or there are some n_0 \in \mathbb{N} and a finite subset F of \Gamma such that

$$\sum_{\gamma \in \Gamma - F} \|x(\gamma)\|_\gamma^p \leq \varepsilon \quad \text{for all } x \in A_{n_0}.$$ 

Proof. Let player \( \Sigma \) start a game with a nonempty subset \( A_1 \) of \( \ell^p \{X_\gamma : \gamma \in \Gamma\} \). Then we distinguish the following two possibilities:

(i) For each \( x \in A_1 \), \( \sum_{\gamma \in \Gamma} \|x(\gamma)\|_\gamma^p \leq \varepsilon \). In this case, put \( F_1 = \emptyset \) and define \( B_1 = A_1 \) as the first choice of \( \Omega \).

(ii) There is \( x_1 \in A_1 \) such that \( \sum_{\gamma \in \Gamma} \|x_1(\gamma)\|_\gamma^p > \varepsilon \). By the definition, there is a finite subset \( F_1 \subset \Gamma \) such that \( \sum_{\gamma \in F_1} \|x_1(\gamma)\|_\gamma^p > \varepsilon \). We can assume \( x_1(\gamma) \neq 0 \) for each \( \gamma \in F_1 \), and choose some \( \delta_1 > 0 \) such that \( \|x_1(\gamma)\|_\gamma^p > \delta_1 \) for all \( \gamma \in F_1 \) and \( \sum_{\gamma \in F_1} \|x_1(\gamma)\|_\gamma^p > \varepsilon + n_1 \delta_1 \), where \( |F_1| = n_1 \). Define

\[
B_1 = s(A_1)
= \{x \in A_1 : \|x(\gamma)\|_\gamma > (\|x_1(\gamma)\|_\gamma^p - \delta_1)^{1/p} \text{ for all } \gamma \in F_1\}.
\]

Then for each \( x \in B_1 \), we have

$$\sum_{\gamma \in F_1} \|x(\gamma)\|_\gamma^p > \sum_{\gamma \in F_1} \|x_1(\gamma)\|_\gamma^p - n_1 \delta_1 > \varepsilon.$$ 

In step \( k \), when \( A_1, \ldots, A_{k-1} \) together with finite subsets \( F_1, \ldots, F_{k-1} \) of \( \Gamma \) have already been specified, we consider the following possibilities:

(i) \( \sum_{\gamma \in \Gamma - \bigcup_{i=1}^{k-1} F_i} \|x(\gamma)\|_\gamma^p \leq \varepsilon \) for each \( x \in A_k \). In this situation, let \( F_k = F_{k-1} \) and define \( B_k = s(A_k) = A_k \) as the next move of player \( \Omega \).

(ii) There is some \( x_k \in A_k \) such that \( \sum_{\gamma \in \Gamma - \bigcup_{i=1}^{k-1} F_i} \|x_k(\gamma)\|_\gamma^p > \varepsilon \).

By the definition, we can find a finite subset \( F_k \subset \Gamma - \bigcup_{i=1}^{k-1} F_i \) such that \( \sum_{\gamma \in F_k} \|x_k(\gamma)\|_\gamma^p > \varepsilon \). As before, we can assume that \( x_k(\gamma) \neq 0 \) for each \( \gamma \in F_k \). Let \( |F_k| = n_k \) and select \( \delta_k > 0 \) such that \( \|x_k(\gamma)\|_\gamma^p > \delta_k \) for all \( \gamma \in F_k \) and \( \sum_{\gamma \in F_k} \|x_k(\gamma)\|_\gamma^p > \varepsilon + n_k \delta_k \).

Define

\[
B_k = s(A_1, \ldots, A_k)
= \{x \in A_k : \|x(\gamma)\|_\gamma > (\|x_k(\gamma)\|_\gamma^p - \delta_k)^{1/p} \text{ for all } \gamma \in F_k\}
\]

as the response of \( \Omega \) to the partial play \( (A_1, \ldots, A_k) \). A similar argument to the one in (ii) can be used to prove that for each \( x \in B_k \), \( \sum_{\gamma \in F_k} \|x(\gamma)\|_\gamma^p > \varepsilon \).

In this way, by induction on \( k \), a strategy \( s \) for \( \Omega \) is defined. The following two cases may happen:
(1) There is some \( n_0 \in \mathbb{N} \) such that \((i_{n_0})\) happens. In this situation, for \( F = \bigcup_{i=1}^{n_0} F_i \), we have
\[
\sum_{\gamma \in \Gamma - F} \|x(\gamma)\|_\gamma^p \leq \varepsilon \quad \text{for all } x \in A_{n_0}.
\]

(2) For each \( n \in \mathbb{N} \), \((i_n)\) holds. In this case, we claim that \( \bigcap_{i=1}^{\infty} A_i = \emptyset \). On the contrary, suppose that \( x \in \bigcap_{i=1}^{\infty} A_i \). Then for each \( n \in \mathbb{N} \), we have
\[
\sum_{\gamma \in \Gamma} \|x(\gamma)\|_\gamma^p \geq \sum_{\gamma \in \bigcup_{1 \leq i \leq n} F_i} \|x(\gamma)\|_\gamma^p = \sum_{i=1}^{n} \sum_{\gamma \in F_i} \|x(\gamma)\|_\gamma^p > n\varepsilon.
\]
Hence \( x \notin \ell^p\{X_\gamma : \gamma \in \Gamma\} \). This contradiction proves our claim in this case. ■

The proof of the following lemma is similar to the proof of Lemma 3.2, hence it is omitted.

**Lemma 4.2.** Let \( X_1, \ldots, X_n \) be Banach spaces with the property \( Q \). Then \( G(\ell^p\{X_i : 1 \leq i \leq n\}, \text{weak}, \| \cdot \|_p) \) is \( \Sigma \)-unfavorable.

**Theorem 4.3.** If \( \{X_\gamma : \gamma \in \Gamma\} \) is a family of Banach spaces with the property \( Q \), then \( \ell^p\{X_\gamma : \gamma \in \Gamma\} \) has the property \( Q \).

**Proof.** Let \( \varepsilon > 0 \). By Lemma 4.1, player \( \Omega \) has a strategy \( s \) such that for each \( s \)-play \((A_i, B_i)_{i \geq 1}\), either \( \bigcap_{i \geq 1} A_i = \emptyset \) or there is \( n_0 \in \mathbb{N} \) such that for some finite subset \( F \) of \( \Gamma \),
\[
\sum_{\gamma \in \Gamma - F} \|x(\gamma)\|_\gamma^p \leq \frac{\varepsilon^p}{2^{p+1}} \quad \text{for all } x \in A_{n_0}.
\]

According to Corollary 2.2, we may assume that (4.1) holds. We define a strategy \( s' \) for player \( \Omega \) as follows:

For each \( 1 \leq n < n_0 \), let \( s'(A_1, \ldots, A_n) = s(A_1, \ldots, A_n) \). Suppose that for \( n \geq n_0 \), \( A_1, \ldots, A_n \) have been selected. Let \( \pi_F : \ell^p\{X_\gamma : \gamma \in \Gamma\} \rightarrow \prod_{\gamma \in F} X_\gamma \) be the canonical map and \( B_n = s'(A_1, \ldots, A_n) \) be a relatively open subset of \( A_n \) such that \( \pi_F(B_n) \) is the answer of player \( \Omega \) to \((\pi_F(A_1), \ldots, \pi_F(A_n))\) according to the strategy whose existence is guaranteed by Lemma 4.2. In this way, a strategy \( s' \) for \( \Omega \) is determined.

By Lemma 4.2, \( G(\ell^p\{X_\gamma : \gamma \in F\}, \text{weak}, \| \cdot \|_p) \) is \( \Sigma \)-unfavorable, so that there is a play \((\pi_F(A_i), \pi_F(B_i))_{i \geq 1}\) such that either \( \bigcap_{i \geq 1} \pi_F(A_i) = \emptyset \) or norm-diam \( \pi_F(A_n) < \varepsilon/2^{1/p} \). In the first case \( \bigcap_{i \geq 1} A_i = \emptyset \). In the latter case for each \( x, y \in A_{n_0} \), we have
\[
\|x - y\|_p^p \leq \sum_{\gamma \in F} \|x(\gamma) - y(\gamma)\|_\gamma^p + \sum_{\gamma \in \Gamma - F} \|x(\gamma) - y(\gamma)\|_\gamma^p.
\]
Since for each \( x, y \in A_{n_0} \),
\[
\sum_{\gamma \in F} \| x(\gamma) - y(\gamma) \|_\gamma^p \leq \{ \text{norm-diam} \, \pi_F(A_{n_0}) \}^p < \frac{\varepsilon^p}{2}
\]
and
\[
\left\{ \sum_{\gamma \in \Gamma - F} \| x(\gamma) - y(\gamma) \|_\gamma^p \right\}^{1/p} \leq \left\{ \sum_{\gamma \in \Gamma - F} \| x(\gamma) \|_\gamma^p \right\}^{1/p} + \left\{ \sum_{\gamma \in \Gamma - F} \| y(\gamma) \|_\gamma^p \right\}^{1/p}
\leq \frac{\varepsilon}{2(p+1)/p} + \frac{\varepsilon}{2(p+1)/p} = \frac{\varepsilon}{2^{1/p}},
\]
by (4.2), \( \| x - y \|_p \leq \varepsilon \) for each \( x, y \in A_{n_0} \). Therefore norm-diam\( (A_{n_0}) \leq \varepsilon \). Corollary 2.2 implies that \( \ell^p \{ X_\gamma : \gamma \in \Gamma \} \) has the property \( Q \). \( \blacksquare \)

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REFERENCES


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