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A NOVEL MODAL SERIES REPRESENTATION APPROACH TO SOLVE A CLASS OF NONLINEAR OPTIMAL CONTROL PROBLEMS

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Abstract. *This paper presents a new approach to solve a class of nonlinear optimal control problems which have a quadratic performance index. In this approach, the nonlinear two-point boundary value problem (TPBVP), derived from the Pontryagin's maximum principle, is transformed into a sequence of linear time-invariant TPBVP's. Solving the proposed linear TPBVP sequence in a recursive manner leads to the optimal control law and the optimal trajectory in the form of uniformly convergent series. Hence, to obtain the optimal solution, only the techniques of solving linear ordinary differential equations are employed. In order to use the proposed method in practice, a control design algorithm with low computational complexity and fast convergence rate is presented. Through the finite iterations of algorithm, a suboptimal control law is obtained for the nonlinear optimal control problem. Finally, numerical examples are included to demonstrate efficiency, simplicity and high accuracy of the proposed method.*

Keywords: Nonlinear optimal control problem, Pontryagin's maximum principle, Modal series, Suboptimal control

1. **Introduction.** One of the most active subjects in control theory is the optimal control which has a wide range of practical applications not only in all areas of physics but also in economy, aerospace, chemical engineering, robotic, etc. [1-4]. For linear time-invariant systems, optimal control theory and its application have been developed perfectly [5,6]. However, optimal control of nonlinear systems is much more challenging which has been studied extensively for decades [7,8].

One familiar scheme for the optimal control of nonlinear systems is the state-dependent Riccati equation (SDRE) method [9]. In the SDRE, a direct parameterization is used to transform the nonlinear system into a linear structure with state-dependent coefficients. Then, a matrix Riccati algebraic equation with state-dependent coefficients is solved at each sample state along the trajectory to obtain a nonlinear state feedback control law. Although this method has been widely used in various applications, its major limitation is that it needs solving a sequence of matrix algebraic equations which may take long computing time and large memory space. Moreover, this method produces a suboptimal control law even when the analytic solution of SDRE is available.

Another method, called the approximating sequence of Riccati equations (ASRE), has been introduced in [10]. Although the ASRE is attractive from practical aspects, it suffers from computational complexity due to the need for solving recursively a sequence of linear quadratic time-varying matrix Riccati differential equations.

In order to determine the optimal control law, there is another approach using dynamic programming [11]. This approach leads to the Hamilton-Jacobi-Bellman (HJB) equation that is hard to solve in most cases. An excellent literature review on the methods for solving the HJB equation is provided in [12] where a successive Galerkin approximation (SGA) method is also considered. In the SGA, a sequence of generalized HJB equations is solved iteratively to obtain a sequence of approximations approaching the solution of HJB equation. However, the proposed sequence may converge very slowly or even diverge.

The optimal control law can be derived using the Pontryagin's maximum principle [13]. Unfortunately, for the nonlinear optimal control problem (OCP), this approach leads to a nonlinear two-point boundary value problem (TPBVP) that in general can not be solved analytically. Therefore, many researches have been devoted to find an approximate solution for nonlinear TPBVP's [14]. Over recent years, some better results have been obtained. For instance, a new successive approximation approach (SAA) has been proposed in [15] where instead of directly solving the nonlinear TPBVP, a sequence of nonhomogeneous linear time-varying TPBVP's is solved iteratively to obtain a sequence of approximations approaching the optimal control law. The sensitivity approach, proposed in [16], is a similar one to the SAA. It only requires solving iteratively a sequence of nonhomogeneous linear time-varying TPBVP's to determine the infinite series presenting the optimal control law. However, solving time-varying equations is much more difficult than solving time-invariant ones.

The Modal series method, which is recently developed in the nonlinear system analysis [17-22], was initially introduced in [17]. It provides the solution of autonomous nonlinear systems in terms of fundamental and interacting modes. This solution, which is called modal series, yields a good deal of physical insight into the system behavior. In contrast with the perturbation method [23], the modal series method does not depend on the small/large physical parameters in system model. In addition, unlike the traditional non-perturbation methods, such as Lyapunov's artificial small parameter method [24] and Adomian's decomposition method [25], the solution series obtained via the modal series method converges uniformly to the exact solution.

In this paper, we propose a new approach to solve nonlinear OCP's by extending the Modal series method. In this approach, the optimal control law and the optimal trajectory are determined in the form of uniformly convergent series with easy-computable terms. Moreover, to obtain the optimal solution, only a sequence of linear time-invariant TPBVP's is solved in a recursive manner.

The paper is organized as follows. Section 2 describes the problem statement. The optimal control design strategy and convergence analysis are elaborated in Section 3. Section 4 explains how to use the results of Section 3 in practice. In this section, in order to obtain a suboptimal control law, an efficient algorithm with low computational complexity and fast convergence rate is presented. Section 5 introduces two numerical examples verifying validity of the proposed method. Finally, conclusions and future works are given in Section 6.

2. **Statement of the Problem.**

Many nonlinear systems of practical importance are affine in the control; for example, we can mention many fed-batch processes [26], aircraft systems [2], chemical reactors [27], etc. Besides, optimal control of such systems has received growing attention due to its tremendous applications including autopilot design [28], chemical reactor control [3], ducted fan control [29], control of aeroelastic systems [30], etc. Therefore, consider a nonlinear OCP described by:

$$\begin{aligned} \min \quad & J = \frac{1}{2}\int_{t_0}^{t_f} \left(x^T(t)Qx(t) + u^T(t)Ru(t)\right) dt \\ s.t. \quad & \begin{cases} \dot{x}(t) = F(x(t)) + G(x(t))u(t), \quad t \in [t_0, t_f] \\ x(t_0) = x_0, \quad x(t_f) = x_f \end{cases} \end{aligned} \tag{1}$$

where $x \in R^n$ and $u \in R^m$ are respectively the state vector and the control vector, $F : R^n \to R^n$ is a nonlinear analytic vector function where $F(0) = 0$, $G : R^n \to R^{n\times m}$ is a nonlinear analytic mapping, $x_0 \in R^n$ is the initial state vector and $x_f \in R^n$ is the final state vector, $Q \in R^{n\times n}$ and $R \in R^{m\times m}$ are positive semi-definite and positive definite matrices, respectively.

Remark 2.1. *Although the above imposed conditions on the OCP (1) may seem to be conservative from the theoretic aspect, they are satisfied by many practical applications such as control of a continuous stirred-tank chemical reactor [5], F8 fight aircraft control [2], optimal maneuvers of a rigid asymmetric spacecraft [31], etc.*

According to the Pontryagin's maximum principle, the optimality conditions are obtained as the following nonlinear TPBVP:

$$\begin{cases} \dot{x}(t) = F(x(t)) - G(x(t))R^{-1}G^T(x(t))\lambda(t), \quad t \in [t_0, t_f] \\ \dot{\lambda}(t) = -Qx(t) - \left(\dfrac{\partial F(x(t))}{\partial x(t)}\right)^T \lambda(t) + \begin{bmatrix} \lambda^T(t)\frac{\partial G(x(t))}{\partial x_1(t)}R^{-1}G^T(x(t))\lambda(t) \\ \vdots \\ \lambda^T(t)\frac{\partial G(x(t))}{\partial x_n(t)}R^{-1}G^T(x(t))\lambda(t) \end{bmatrix}, \quad t \in [t_0, t_f] \\ x(t_0) = x_0, \quad x(t_f) = x_f \end{cases} \tag{2}$$

where $\lambda \in R^n$ is the co-state vector and $x_i(t)$ is the i^{th} element of vector $x(t)$. Also, the optimal control law is given by:

$$u^*(t) = -R^{-1}G^T(x(t))\lambda(t), \quad t \in [t_0, t_f]. \tag{3}$$

Let define $\Psi(x(t), \lambda(t))$ and $\bar{\Psi}(x(t), \lambda(t))$ as:

$$\begin{cases} \Psi(x(t), \lambda(t)) \stackrel{\Delta}{=} F(x(t)) - G(x(t))R^{-1}G^T(x(t))\lambda(t) \\ \bar{\Psi}(x(t), \lambda(t)) \stackrel{\Delta}{=} -Qx(t) - \left(\dfrac{\partial F(x(t))}{\partial x(t)}\right)^T \lambda(t) + \begin{bmatrix} \lambda^T(t)\frac{\partial G(x(t))}{\partial x_1(t)}R^{-1}G^T(x(t))\lambda(t) \\ \vdots \\ \lambda^T(t)\frac{\partial G(x(t))}{\partial x_n(t)}R^{-1}G^T(x(t))\lambda(t) \end{bmatrix}. \end{cases} \tag{4}$$

Note that $\Psi : R^n \times R^n \to R^n$ and $\bar{\Psi} : R^n \times R^n \to R^n$ are nonlinear analytic vector functions since F and G are assumed to be analytic. In the light of (4), problem (2) can be rewritten in a compact form as follows:

$$\begin{cases} \dot{x}(t) = \Psi(x(t), \lambda(t)), \quad t \in [t_0, t_f] \\ \dot{\lambda}(t) = \bar{\Psi}(x(t), \lambda(t)), \quad t \in [t_0, t_f] \\ x(t_0) = x_0, \quad x(t_f) = x_f. \end{cases} \tag{5}$$

3. **Optimal Control Design Strategy and Convergence Analysis.** Unfortunately, problem (2) (and equivalently (5)) is a nonlinear TPBVP that in general can not be solved analytically except for a few simple cases. Here, we extend the Modal series method to solve this problem. To do so, first we need Taylor series expansion of the nonlinear non-polynomial terms. Therefore, if Ψ and $\bar{\Psi}$ in (5) are not polynomial, they should be expanded in the Taylor series around the operating point $(x^o, \lambda^o) = (0, 0)$ which yields:

$$\begin{cases} \dot{x}(t) = & A_{10}x(t) + A_{01}\lambda(t) \\ & +\dfrac{1}{2!}\begin{bmatrix} x^T(t)H_{20}^1x(t) \\ \vdots \\ x^T(t)H_{20}^n x(t) \end{bmatrix} + \begin{bmatrix} x^T(t)H_{11}^1\lambda(t) \\ \vdots \\ x^T(t)H_{11}^n\lambda(t) \end{bmatrix} + \dfrac{1}{2!}\begin{bmatrix} \lambda^T(t)H_{02}^1\lambda(t) \\ \vdots \\ \lambda^T(t)H_{02}^n\lambda(t) \end{bmatrix} + \cdots \\ \dot{\lambda}(t) = & \bar{A}_{10}x(t) + \bar{A}_{01}\lambda(t) \\ & +\dfrac{1}{2!}\begin{bmatrix} x^T(t)\bar{H}_{20}^1x(t) \\ \vdots \\ x^T(t)\bar{H}_{20}^n x(t) \end{bmatrix} + \begin{bmatrix} x^T(t)\bar{H}_{11}^1\lambda(t) \\ \vdots \\ x^T(t)\bar{H}_{11}^n\lambda(t) \end{bmatrix} + \dfrac{1}{2!}\begin{bmatrix} \lambda^T(t)\bar{H}_{02}^1\lambda(t) \\ \vdots \\ \lambda^T(t)\bar{H}_{02}^n\lambda(t) \end{bmatrix} + \cdots \end{cases} \tag{6}$$

where x and λ belong to the convergence domain of Taylor series in (6) which is denoted by $\Phi \times \Omega$ where $\Phi \subseteq R^n$ and $\Omega \subseteq R^n$ are subsets of the state space and co-state space, respectively. Also, the coefficients in (6) are as follows:

$$\begin{array}{llll} A_{10} = \left.\frac{\partial \Psi}{\partial x}\right|_{\substack{x=0\\ \lambda=0}}, & \bar{A}_{10} = \left.\frac{\partial \bar{\Psi}}{\partial x}\right|_{\substack{x=0\\ \lambda=0}}, & A_{01} = \left.\frac{\partial \Psi}{\partial \lambda}\right|_{\substack{x=0\\ \lambda=0}}, & \bar{A}_{01} = \left.\frac{\partial \bar{\Psi}}{\partial \lambda}\right|_{\substack{x=0\\ \lambda=0}}, \\ H_{20}^j = \left.\frac{\partial^2 \Psi_j}{\partial x^2}\right|_{\substack{x=0\\ \lambda=0}}, & \bar{H}_{20}^j = \left.\frac{\partial^2 \bar{\Psi}_j}{\partial x^2}\right|_{\substack{x=0\\ \lambda=0}}, & H_{11}^j = \left.\frac{\partial^2 \Psi_j}{\partial \lambda \partial x}\right|_{\substack{x=0\\ \lambda=0}}, & \\ \bar{H}_{11}^j = \left.\frac{\partial^2 \bar{\Psi}_j}{\partial \lambda \partial x}\right|_{\substack{x=0\\ \lambda=0}}, & H_{02}^j = \left.\frac{\partial^2 \Psi_j}{\partial \lambda^2}\right|_{\substack{x=0\\ \lambda=0}}, & \bar{H}_{02}^j = \left.\frac{\partial^2 \bar{\Psi}_j}{\partial \lambda^2}\right|_{\substack{x=0\\ \lambda=0}} & \end{array} \tag{7}$$

where Ψ_j and $\bar{\Psi}_j$ are the j^{th} components of vector functions Ψ and $\bar{\Psi}$, respectively.

Remark 3.1. *For many large-scale systems, the analytic partial derivatives as in (7) are difficult to be obtained symbolically. For these systems, it may be needed to compute derivatives numerically as in [17]. In addition, in the presence of such systems, complexity of computations brought by the curse of dimensionality adds another difficulty. Future works are focused on improving the method in order to overcome these difficulties.*

Theorem 3.1. *The solution of nonlinear TPBVP (2) (and equivalently (5)) can be expressed as* $x(t) = \sum\limits_{i=1}^{\infty} g_i(t)$ *and* $\lambda(t) = \sum\limits_{i=1}^{\infty} h_i(t)$ *where the* i^{th} *order terms* $g_i(t)$ *and* $h_i(t)$ *for* $i \geq 1$ *are achieved by solving recursively a sequence of linear time-invariant TPBVP's.*

Proof: The solution of problem (2) (and equivalently (5)) for arbitrary boundary condition $x_b = (x_0, x_f)$ and for all $t \in [t_0, t_f]$ can be expressed as:

$$\begin{cases} x(t) = \Lambda(x_b, t) \\ \lambda(t) = \bar{\Lambda}(x_b, t) \end{cases} \tag{8}$$

where $\Lambda : R^{2n} \times R \to R^n$ and $\bar{\Lambda} : R^{2n} \times R \to R^n$ are analytic vector functions since Ψ and $\bar{\Psi}$ (defined in (4)) are analytic [32]. Since Λ and $\bar{\Lambda}$ are analytic vector functions and x_b is assumed to be arbitrary, we can expand (8) as Maclaurin series with respect to x_b which yields:

$$x(t) = \Lambda(x_b, t) = \underbrace{\left.\frac{\partial \Lambda(x_b, t)}{\partial x_b}\right|_{x_b=0} x_b}_{g_1(t)} + \underbrace{\frac{1}{2!}\begin{bmatrix} x_b^T \left(\left.\frac{\partial^2 \Lambda_1(x_b,t)}{\partial x_b^2}\right|_{x_b=0} \right) x_b \\ \vdots \\ x_b^T \left(\left.\frac{\partial^2 \Lambda_n(x_b,t)}{\partial x_b^2}\right|_{x_b=0} \right) x_b \end{bmatrix}}_{g_2(t)} + \cdots$$

$$\lambda(t) = \bar{\Lambda}(x_b, t) = \underbrace{\left.\frac{\partial \bar{\Lambda}(x_b, t)}{\partial x_b}\right|_{x_b=0} x_b}_{h_1(t)} + \underbrace{\frac{1}{2!}\begin{bmatrix} x_b^T \left(\left.\frac{\partial^2 \bar{\Lambda}_1(x_b,t)}{\partial x_b^2}\right|_{x_b=0} \right) x_b \\ \vdots \\ x_b^T \left(\left.\frac{\partial^2 \bar{\Lambda}_n(x_b,t)}{\partial x_b^2}\right|_{x_b=0} \right) x_b \end{bmatrix}}_{h_2(t)} + \cdots \tag{9}$$

where Λ_j and $\bar{\Lambda}_j$ are respectively the j^{th} components of vector functions Λ and $\bar{\Lambda}$, $g_i(t)$ and $h_i(t)$ are vector functions which their components are linear combination of all terms depending on the multiplication of i elements of vector x_b. For example, components of $g_2(t)$ and $h_2(t)$ contain linear combination of all terms of the form $x_{b,k}x_{b,l}$ for $k, l \in \{1, 2, \ldots, 2n\}$ where $x_{b,j}$ is the j^{th} element of x_b. Moreover, since Λ and $\bar{\Lambda}$ are analytic vector functions, existence and uniformly convergence of the Maclaurin series in (9) are guaranteed. Let $\Phi_1 \times \Phi_2$ be the convergence domain of Maclaurin series in (9) for all $t \in [t_0, t_f]$ where $\Phi_1 \subseteq R^n$ and $\Phi_2 \subseteq R^n$ are subsets of the initial state space and final state space, respectively. Assume that $\Theta_1 = \Phi \cap \Phi_1$ and $\Theta_2 = \Phi \cap \Phi_2$ are non-empty and let the boundary condition be εx_b, i.e., $x(t_0) = \varepsilon x_0$ and $x(t_f) = \varepsilon x_f$ where ε is an arbitrary scalar parameter such that $\varepsilon x_b \in \Theta_1 \times \Theta_2$. Since Θ_1 and Θ_2 are assumed to be non-empty, such parameter exists. Moreover, this parameter only simplifies the calculations and its value does not have any significance. Similarly to (9), we can write:

$$\begin{cases} x_\varepsilon(t) = \Lambda(\varepsilon x_b, t) = \varepsilon g_1(t) + \varepsilon^2 g_2(t) + \cdots \\ \lambda_\varepsilon(t) = \bar{\Lambda}(\varepsilon x_b, t) = \varepsilon h_1(t) + \varepsilon^2 h_2(t) + \cdots \end{cases} \tag{10}$$

Since $\varepsilon x_b \in \Theta_1 \times \Theta_2$, (10) must satisfy (6). Satisfying (10) in (6) and rearranging terms with respect to the order of ε yield:

$$\begin{cases} \varepsilon \dot{g}_1(t) + \varepsilon^2 \dot{g}_2(t) + \cdots = \varepsilon \left(A_{10} g_1(t) + A_{01} h_1(t) \right) \\ \quad + \varepsilon^2 \left(\begin{array}{l} A_{10} g_2(t) + A_{01} h_2(t) \\ + \dfrac{1}{2!} \begin{bmatrix} g_1^T(t) H_{20}^1 g_1(t) \\ \vdots \\ g_1^T(t) H_{20}^n g_1(t) \end{bmatrix} + \begin{bmatrix} g_1^T(t) H_{11}^1 h_1(t) \\ \vdots \\ g_1^T(t) H_{11}^n h_1(t) \end{bmatrix} + \dfrac{1}{2!} \begin{bmatrix} h_1^T(t) H_{02}^1 h_1(t) \\ \vdots \\ h_1^T(t) H_{02}^n h_1(t) \end{bmatrix} \end{array} \right) + \cdots \\ \varepsilon \dot{h}_1(t) + \varepsilon^2 \dot{h}_2(t) + \cdots = \varepsilon \left(\bar{A}_{10} g_1(t) + \bar{A}_{01} h_1(t) \right) \\ \quad + \varepsilon^2 \left(\begin{array}{l} \bar{A}_{10} g_2(t) + \bar{A}_{01} h_2(t) \\ + \dfrac{1}{2!} \begin{bmatrix} g_1^T(t) \bar{H}_{20}^1 g_1(t) \\ \vdots \\ g_1^T(t) \bar{H}_{20}^n g_1(t) \end{bmatrix} + \begin{bmatrix} g_1^T(t) \bar{H}_{11}^1 h_1(t) \\ \vdots \\ g_1^T(t) \bar{H}_{11}^n h_1(t) \end{bmatrix} + \dfrac{1}{2!} \begin{bmatrix} h_1^T(t) \bar{H}_{02}^1 h_1(t) \\ \vdots \\ h_1^T(t) \bar{H}_{02}^n h_1(t) \end{bmatrix} \end{array} \right) + \cdots \end{cases} \tag{11}$$

Since (11) must hold for any ε as long as $\varepsilon x_b \in \Theta_1 \times \Theta_2$, terms with the same order of ε on each side must be equal. This procedure yields:

$$\varepsilon : \begin{cases} \dot{g}_1(t) = A_{10} g_1(t) + A_{01} h_1(t) \\ \dot{h}_1(t) = \bar{A}_{10} g_1(t) + \bar{A}_{01} h_1(t) \end{cases} \tag{12a}$$

$$\varepsilon^2 : \begin{cases} \dot{g}_2(t) = A_{10}g_2(t) + A_{01}h_2(t) \\ \qquad + \frac{1}{2!}\begin{bmatrix} g_1^T(t)H_{20}^1 g_1(t) \\ \vdots \\ g_1^T(t)H_{20}^n g_1(t) \end{bmatrix} + \begin{bmatrix} g_1^T(t)H_{11}^1 h_1(t) \\ \vdots \\ g_1^T(t)H_{11}^n h_1(t) \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} h_1^T(t)H_{02}^1 h_1(t) \\ \vdots \\ h_1^T(t)H_{02}^n h_1(t) \end{bmatrix} \\ \dot{h}_2(t) = \bar{A}_{10}g_2(t) + \bar{A}_{01}h_2(t) \\ \qquad + \frac{1}{2!}\begin{bmatrix} g_1^T(t)\bar{H}_{20}^1 g_1(t) \\ \vdots \\ g_1^T(t)\bar{H}_{20}^n g_1(t) \end{bmatrix} + \begin{bmatrix} g_1^T(t)\bar{H}_{11}^1 h_1(t) \\ \vdots \\ g_1^T(t)\bar{H}_{11}^n h_1(t) \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} h_1^T(t)\bar{H}_{02}^1 h_1(t) \\ \vdots \\ h_1^T(t)\bar{H}_{02}^n h_1(t) \end{bmatrix} \end{cases} \tag{12b}$$

and so on.

Note that (12a) is a system of homogeneous linear time-invariant ODE's. From (12a), $g_1(t)$ and $h_1(t)$ can be easily obtained. Assume that $g_1(t)$ and $h_1(t)$ have been obtained from (12a) in the first step. Then, $g_2(t)$ and $h_2(t)$ can be easily obtained from (12b) in the second step since (12b) is a system of nonhomogeneous linear time-invariant ODE's. Notice that, the nonhomogeneous terms of (12b) are calculated using the solution of (12a). Continuing as above, $g_i(t)$ and $h_i(t)$ for $i \geq 2$ can be easily obtained in the i^{th} step only by solving a system of nonhomogeneous linear time-invariant ODE's. In addition, at each step nonhomogeneous terms are calculated using the information obtained from previous steps. Therefore, solving the presented sequence is a recursive process.

To obtain the boundary condition for the above-mentioned sequence, set $t = t_0$ and $t = t_f$ in the first equation of (10) as follows:

$$\begin{cases} \varepsilon x_0 = x_\varepsilon(t_0) = \Lambda(\varepsilon x_b, t_0) = \varepsilon g_1(t_0) + \varepsilon^2 g_2(t_0) + \cdots \\ \varepsilon x_f = x_\varepsilon(t_f) = \Lambda(\varepsilon x_b, t_f) = \varepsilon g_1(t_f) + \varepsilon^2 g_2(t_f) + \cdots \end{cases} \tag{13}$$

Comparing the coefficients of the same order terms with respect to ε yields:

$$\begin{cases} g_1(t_0) = x_0 \\ g_1(t_f) = x_f \end{cases} \quad \text{and} \quad \begin{cases} g_i(t_0) = 0 \\ g_i(t_f) = 0 \end{cases} \quad \text{for } i \geq 2. \tag{14}$$

Thus, the proof is complete.

Corollary 3.1. *For the nonlinear OCP (1), the optimal trajectory and the optimal control law are determined as follows:*

$$\begin{cases} x^*(t) = \sum_{i=1}^{\infty} g_i(t) \\ u^*(t) = -R^{-1}G^T\left(\sum_{i=1}^{\infty} g_i(t)\right)\left(\sum_{i=1}^{\infty} h_i(t)\right) \end{cases} \tag{15}$$

where $g_i(t)$ and $h_i(t)$ for $i \geq 1$ are obtained only by solving recursively the presented linear TPBVP sequence in the proof of Theorem 3.1.

As it can be interpreted from Corollary 3.1, the proposed method only requires solving a sequence of linear time-invariant TPBVP's; on the other hand, the other approximate methods such as SDRE, ASRE, SGA, SAA, etc., need solving a sequence of matrix differential (or algebraic) equations or a sequence of generalized HJB equations or a sequence of linear time-varying TPBVP's [9,10,12,15,16]. Therefore, the proposed method takes less computing time and memory space in comparison with the above-mentioned approximate methods.

The following theorem shows uniformly convergence of the obtained solution series in (15) to the optimal solution:

Theorem 3.2. *Define sequences* $\left\{x^{(k)}(t)\right\}_{k=1}^{\infty}, \left\{\lambda^{(k)}(t)\right\}_{k=1}^{\infty}$ *and* $\left\{u^{(k)}(t)\right\}_{k=1}^{\infty}$ *as follows:*

$$\begin{cases} x^{(k)}(t) \triangleq \sum\limits_{i=1}^{k} g_i(t) \\ \lambda^{(k)}(t) \triangleq \sum\limits_{i=1}^{k} h_i(t) \\ u^{(k)}(t) \triangleq -R^{-1}G^T(x^{(k)}(t))\lambda^{(k)}(t). \end{cases} \tag{16}$$

Then, for the nonlinear OCP (1), the sequences $\left\{x^{(k)}(t)\right\}_{k=1}^{\infty}$ *and* $\left\{u^{(k)}(t)\right\}_{k=1}^{\infty}$ *converge uniformly to the optimal trajectory and the optimal control law, respectively.*

Proof: According to the proof of Theorem 3.1, Maclaurin series $\sum\limits_{i=1}^{\infty} g_i(t)$ and $\sum\limits_{i=1}^{\infty} h_i(t)$ converge uniformly to the exact solution of nonlinear TPBVP (2), i.e., $x(t)$ and $\lambda(t)$, respectively. That is:

$$\begin{cases} x(t) = \lim\limits_{k\to\infty} \sum\limits_{i=1}^{k} g_i(t) \Leftrightarrow x^{(k)}(t) \xrightarrow{U} x(t) \\ \lambda(t) = \lim\limits_{k\to\infty} \sum\limits_{i=1}^{k} h_i(t) \Leftrightarrow \lambda^{(k)}(t) \xrightarrow{U} \lambda(t). \end{cases} \tag{17}$$

Besides, the control sequence $\left\{u^{(k)}(t)\right\}_{k=1}^{\infty}$ depends on the state vector sequence $\{x^{(k)}(t)\}_{k=1}^{\infty}$ through an analytic mapping and on the co-state vector sequence $\left\{\lambda^{(k)}(t)\right\}_{k=1}^{\infty}$ through a linear operator. So, we have:

$$\begin{aligned} u^*(t) &= -R^{-1}G^T(x(t))\lambda(t) = -R^{-1}G^T\left(\sum_{i=1}^{\infty} g_i(t)\right)\left(\sum_{i=1}^{\infty} h_i(t)\right) \\ &= -R^{-1}G^T\left(\lim_{k\to\infty}\sum_{i=1}^{k} g_i(t)\right)\left(\lim_{k\to\infty}\sum_{i=1}^{k} h_i(t)\right) \\ &= \lim_{k\to\infty}\left\{-R^{-1}G^T\left(\sum_{i=1}^{k} g_i(t)\right)\left(\sum_{i=1}^{k} h_i(t)\right)\right\} \\ &= \lim_{k\to\infty}\left\{-R^{-1}G^T(x^{(k)}(t))\lambda^{(k)}(t)\right\} = \lim_{k\to\infty} u^{(k)}(t), \end{aligned} \tag{18}$$

i.e., the control sequence $\left\{u^{(k)}(t)\right\}_{k=1}^{\infty}$ converges uniformly to the optimal control law and the proof is complete.

Remark 3.2. *Let the solution of nonlinear OCP (1) with the boundary condition* $x_b \in \Theta_1 \times \Theta_2$ *be available as in (15). To find the solution for any other boundary condition of the form* εx_b*, as long as* $\varepsilon x_b \in \Theta_1 \times \Theta_2$*, there is no need to repeat the recursive process. The optimal solution can be easily obtained as:*

$$\begin{cases} x^*(t) = \sum\limits_{i=1}^{\infty} \varepsilon^i g_i(t) \\ u^*(t) = -R^{-1}G^T\left(\sum\limits_{i=1}^{\infty} \varepsilon^i g_i(t)\right)\left(\sum\limits_{i=1}^{\infty} \varepsilon^i h_i(t)\right). \end{cases} \tag{19}$$

4. **Suboptimal Control Design Strategy.** In this section, we explain how to use the results of previous section in practice. In fact, obtaining the optimal control law as in (15) is almost impossible since (15) contains infinite series. Therefore, in practical applications, by replacing ∞ with a finite positive integer N in (15), an N^{th} order suboptimal control law is obtained as follows:

$$u_N(t) = -R^{-1}G^T\left(\sum_{i=1}^{N} g_i(t)\right)\left(\sum_{i=1}^{N} h_i(t)\right). \tag{20}$$

The integer N in (20) is generally determined according to a concrete control precision. For example, every time $g_i(t)$ and $h_i(t)$ are obtained from the presented linear TPBVP sequence, we let $N = i$ and calculate the N^{th} order suboptimal control law from (20). Then, the following quadratic performance index (QPI) can be calculated:

$$J^{(N)} = \frac{1}{2}\int_{t_0}^{t_f} \left(x^T(t)Qx(t) + u_N^T(t)Ru_N(t)\right) dt \tag{21}$$

where $u_N(t)$ has been obtained from (20) and $x(t)$ is the corresponding state trajectory obtained from applying $u_N(t)$ to the original nonlinear system in (1) with $x(t_0) = x_0$. The N^{th} order suboptimal control law has desirable accuracy if for two given positive constants $e_1 > 0$ and $e_2 > 0$, the following conditions hold jointly:

$$\left|\frac{J^{(N)} - J^{(N-1)}}{J^{(N)}}\right| < e_1 \tag{22}$$

$$\|x(t_f) - x_f\| < e_2 \tag{23}$$

where $\|\cdot\|$ is a suitable norm on R^n and $x(t_f)$ is the value of corresponding state trajectory at the final time t_f. If the tolerance error bounds $e_1 > 0$ and $e_2 > 0$ be chosen small enough, the N^{th} order suboptimal control law will be very close to the optimal control law $u^*(t)$, the value of QPI in (21) will be very close to its optimal value J^*, and the boundary condition will be satisfied tightly.

In order to obtain an accurate enough suboptimal control law, we present an iterative algorithm with low computational complexity as follows:

Algorithm:

Step 1. Let $i = 1$.

Step 2. Calculate the i^{th} order terms $g_i(t)$ and $h_i(t)$ from the presented linear TPBVP sequence in the proof of Theorem 3.1.

Step 3. Let $N = i$ and obtain the N^{th} order suboptimal control law $u_N(t)$ from (20), apply it to the original nonlinear system with $x(t_0) = x_0$ to obtain the corresponding state trajectory $x(t)$, and then calculate $J^{(N)}$ according to (21).

Step 4. If (22) and (23) hold together for the given small enough constants $e_1 > 0$ and $e_2 > 0$, go to Step 5; else replace i by $i + 1$ and go to Step 2.

Step 5. Stop the algorithm; $u_N(t)$ is the desirable suboptimal control law.

As it can be understood from Theorem 3.2, the above algorithm has a relatively fast convergence rate. Therefore, only a few iterations of the algorithm are required to get a desirable accuracy. This fact reduces the size of computations, effectively.

5. **Numerical Examples.** In this section, two illustrative examples are employed to show efficiency, simplicity and high accuracy of the proposed method.

Example 5.1. *Consider the following nonlinear OCP [33]:*

$$\begin{aligned} &Min \quad J = \textstyle\int_0^1 u^2(t)dt \\ &s.t. \quad \begin{cases} \dot{x}(t) = 0.5x^2(t)\sin(x(t)) + u(t), \quad t \in [0,1] \\ x(0) = 0, \quad x(1) = 0.5. \end{cases} \end{aligned} \tag{24}$$

According to the Pontryagin's maximum principle, the following nonlinear TPBVP should be solved:

$$\begin{cases} \dot{x}(t) = 0.5x^2(t)\sin(x(t)) - 0.5\lambda(t) \\ \dot{\lambda}(t) = -\lambda(t)x(t)\sin(x(t)) - 0.5\lambda(t)x^2(t)\cos(x(t)) \\ x(0) = 0, \quad x(1) = 0.5 \end{cases} \tag{25}$$

TABLE 1. Simulation results of Example 5.1 at different iteration times

i (Iteration time)	$J^{(i)}$	$\frac{J^{(i)}-J^{(i-1)}}{J^{(i)}}$	$\lVert x(t_f) - x_f \rVert$
1	0.25	—	1.601×10^{-2}
2	0.25	0	1.601×10^{-2}
3	0.2349	6.43×10^{-2}	4.477×10^{-4}
4	0.2349	0	4.477×10^{-4}
5	0.2346	1.3×10^{-3}	2.8×10^{-5}

and the optimal control law is given by:

$$u^*(t) = -0.5\lambda(t). \tag{26}$$

Based on the proposed method, instead of directly solving (25), a sequence of linear time-invariant TPBVP's is solved in a recursive manner as follows:

$$\varepsilon : \begin{cases} \dot{g}_1(t) = -0.5h_1(t) \\ \dot{h}_1(t) = 0 \\ g_1(0) = 0, \quad g_1(1) = 0.5 \end{cases} \Rightarrow \begin{cases} g_1(t) = 0.5t \\ h_1(t) = -1 \end{cases} \tag{27a}$$

$$\varepsilon^2 : \begin{cases} \dot{g}_2(t) = -0.5h_2(t) \\ \dot{h}_2(t) = 0 \\ g_2(0) = 0, \quad g_2(1) = 0 \end{cases} \Rightarrow \begin{cases} g_2(t) = 0 \\ h_2(t) = 0 \end{cases} \tag{27b}$$

$$\varepsilon^3 : \begin{cases} \dot{g}_3(t) = -0.5h_3(t) + 0.5g_1^3(t) \\ \dot{h}_3(t) = -1.5h_1(t)g_1^2(t) \\ g_3(0) = 0, \quad g_3(1) = 0 \end{cases} \Rightarrow \begin{cases} g_3(t) = 0 \\ h_3(t) = 0.125t^3 \end{cases} \tag{27c}$$

and so on.

In order to obtain an accurate enough suboptimal control law, we applied the algorithm, proposed in Section 4, with the tolerance error bounds $e_1 = 2 \times 10^{-3}$ and $e_2 = 5 \times 10^{-5}$. In this case, convergence is achieved after 5 iterations, i.e., $\left| \frac{J^{(5)}-J^{(4)}}{J^{(5)}} \right| = 1.3 \times 10^{-3} < 2 \times 10^{-3}$ and $\lVert x(t_f) - x_f \rVert = |x(1) - 0.5| = 2.8 \times 10^{-5} < 5 \times 10^{-5}$. In addition, a closed-form expression for $u_5(t)$ is obtained as follows:

$$u_5(t) = -0.5 \sum_{i=1}^{5} h_i(t) = \frac{895}{1792} - \frac{1}{16}t^3 + \frac{1}{384}t^5 + \frac{1}{256}t^6. \tag{28}$$

Simulation results including the QPI values, the QPI relative errors and the final state errors at different iteration times are listed in Table 1.

Simulation curves have been obtained by applying (28) to the original nonlinear system in (24) with $x(0) = 0$. Also, we obtained the simulation curves by directly solving (25) using the collocation method [14]. Results of both methods are very close to each other as shown in Figures 1 and 2. This confirms that the proposed method yields excellent results. Furthermore, in contrast with the collocation method, computing procedure of our method is very straightforward that can be done by pencil-and-paper only.

Problem (24) has also been solved by J. E. Rubio in [33] via the measure theory in which to find an acceptable solution, a linear programming problem with 1000 variables and 20 constraints should be solved. Therefore, applying Rubio's measure theoretical method is impossible without using computer. Table 2 provides the simulation results of Example 5.1 obtained using our method and measure theoretical method. Comparing the results shows the efficiency of proposed method for solving nonlinear OCP (24).

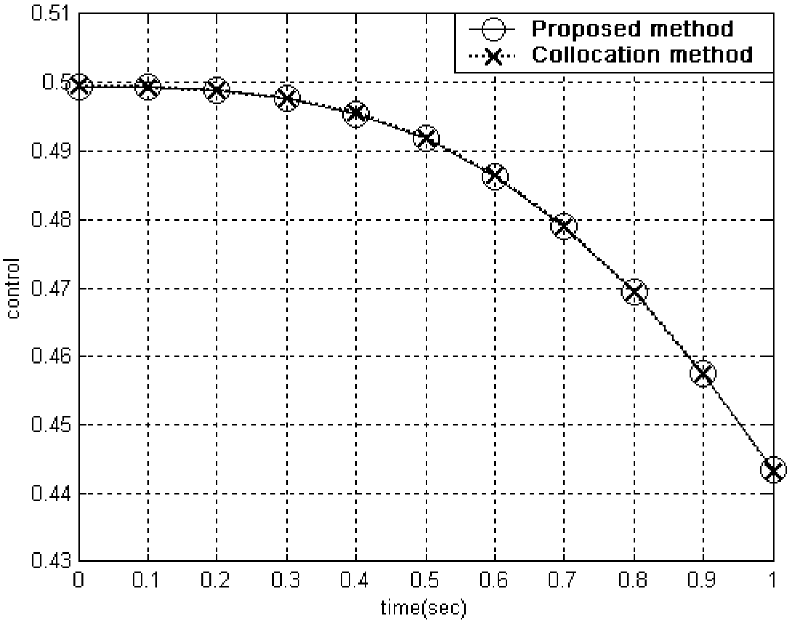

FIGURE 1. Simulation curves of the suboptimal control in Example 5.1

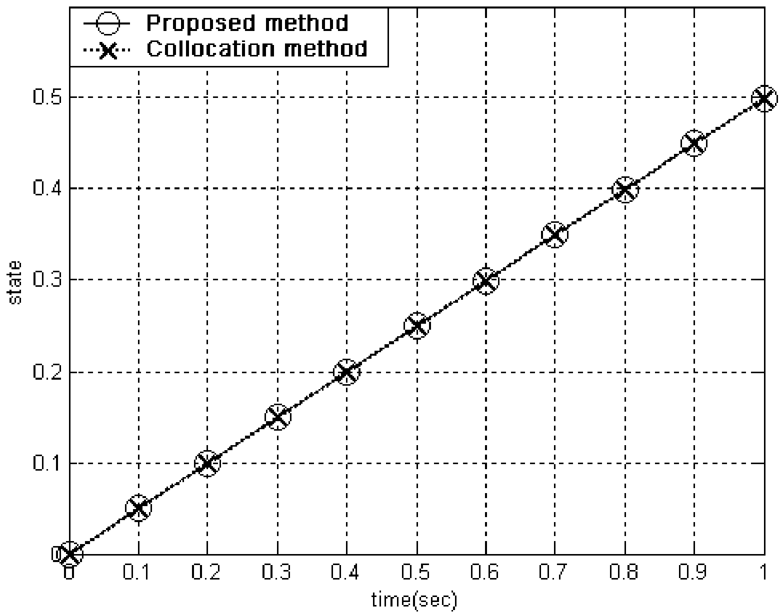

FIGURE 2. Simulation curves of the state trajectory in Example 5.1

Example 5.2. *In this example, a model of F8 fight aircraft is considered as follows [2]:*

$$\begin{cases} \dot{x}_1 = -0.877x_1 + x_3 + 0.47x_1^2 - 0.088x_1x_3 - 0.019x_2^2 \\ \qquad + 3.846x_1^3 - x_1^2x_3 + \left(-0.215 + 0.28x_1^2\right)u \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = -4.208x_1 - 0.396x_3 - 0.47x_1^2 - 3.564x_1^3 + \left(-20.967 + 6.265x_1^2\right)u \end{cases} \tag{29}$$

TABLE 2. Results of the proposed method and measure theoretical method in Example 5.1

Method	Performance index value	Final state error
Proposed method with $N = 5$	0.2346	2.8×10^{-5}
Measure theoretical method	0.2425	4.3×10^{-3}

TABLE 3. Simulation results of Example 5.2 at different iteration times

i (Iteration time)	$J^{(i)}$	$\left\lvert\frac{J^{(i)}-J^{(i-1)}}{J^{(i)}}\right\rvert$	$\\|x(t_f) - x_f\\|$
1	0.2497	—	1.55×10^{-2}
2	0.2421	3.1×10^{-2}	1.22×10^{-2}
3	0.2381	1.6×10^{-2}	1.57×10^{-3}
4	0.2371	4.2×10^{-3}	1.04×10^{-3}
5	0.2365	2.5×10^{-3}	1.01×10^{-4}

where x_1 is the deviation of angle attack, x_2 is the flight path angle, x_3 is the rate of change in flight path angle, and the control u is the deviation of tail deflection angle. The QPI to be minimized is given by:

$$J = \int_0^1 \left(x_1^2 + x_2^2 + x_3^2 + 10u^2\right) dt. \tag{30}$$

In addition, the following boundary conditions should be satisfied:

$$\begin{cases} x_1(0) = x_2(0) = x_3(0) = 0 \\ x_1(1) = 0.22, \quad x_2(1) = 0.25, \quad x_3(1) = 0.3. \end{cases} \tag{31}$$

As previous example, in order to obtain an accurate enough suboptimal control law, we applied the proposed algorithm with the tolerance error bounds $e_1 = 5 \times 10^{-3}$ and $e_2 = 2 \times 10^{-4}$. Simulation results at different iteration times are summarized in Table 3. From Table 3, it is observed that convergence is achieved after 5 iterations, i.e., $\left\lvert \frac{J^{(5)}-J^{(4)}}{J^{(5)}} \right\rvert = 2.5 \times 10^{-3} < 5 \times 10^{-3}$ and $\|x(t_f) - x_f\| = 1.01 \times 10^{-4} < 2 \times 10^{-4}$. This indicates that $J^{(5)} = 0.2365$ is very close to its optimal value J^* and the boundary conditions are satisfied tightly.

Simulation curves of the computed 5^{th} order suboptimal control law, i.e., $u_5(t)$, and the corresponding state trajectories obtained from applying $u_5(t)$ to the nonlinear system (29) with $x_1(0) = x_2(0) = x_3(0) = 0$ are shown in Figure 3.

As mentioned before, using the method proposed in [16] needs solving a sequence of linear time-varying TPBVP's. Applying this method up to 5^{th} iteration results a minimum of $J = 0.2458$. On the other hand, our proposed method only requires solving linear time-invariant TPBVP's and results a minimum of $J = 0.2365$ in 5 iterations. Therefore, our method gets together both accuracy and simplicity in comparison with the above-mentioned method.

6. **Conclusions.** In this paper, a novel practical approach has been introduced to solve a class of nonlinear OCP's. In this approach, by introducing a recursive process, the optimal control law and the optimal trajectory are determined in the form of uniformly convergent series with easy-computable terms. The proposed method avoids directly solving the nonlinear TPBVP or the HJB equation. In addition, despite of the other approximate methods such as SDRE, ASRE, SGA, SAA, etc. [9,10,12,15,16], it avoids solving

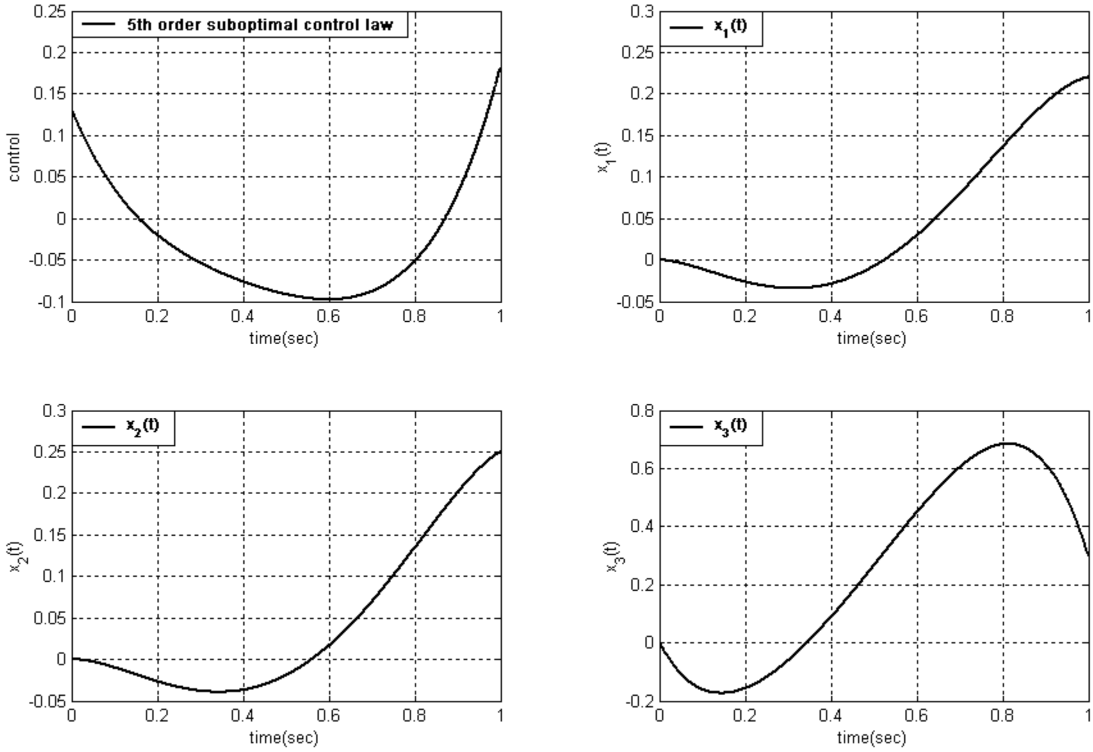

FIGURE 3. Simulation curves of the 5^{th} order suboptimal control law and the state trajectories in Example 5.2

a sequence of matrix differential (or algebraic) equations or a sequence of generalized HJB equations or a sequence of linear time-varying TPBVP's. It only requires solving a sequence of linear time-invariant TPBVP's. Therefore, in view of computational complexity, the proposed method is more practical than the above-mentioned approximate methods.

Future works are focused on improving the proposed method for the optimal control of nonlinear interconnected large-scale systems where the complexity of computations brought by the curse of dimensionality is effectively reduced. Future studies can also be focused on extending the method for more general nonlinear OCP's than ones which were considered in this paper.

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