Extended Abstracts of the 19^{th} Mathematical Seminar On the Analysis and its applications
19-20 February 2011, University of Mazandaran, Babolsar, Iran, pp 00-00

A NEW CLASS OF STOCHASTIC RUNGE-KUTTA METHODS FOR WEAK APPROXIMATION OF ITÔ SDE

A.R.SOHEILI[1] AND R.GHARECHAHI[2]*

Abstract. In the present paper,a new class of stochastic Runge-Kutta(SRK) methods for the weak approximation of the solution of Itô SDE systems is introduced.Order1 and order2 conditions for coefficients of explicit SRK methods are calculated by applying the coloured rooted tree analysis.

1. Introduction and Preliminaries

We consider the solution$(X_t)_{t\in I}$ of the autonomous d-dimensional Itô SDE system

$$dX_t = a(X_s)ds + b(X_s)dW_s, \qquad X_0 = x_0 \tag{1.1}$$

w.r.t, a one-dimensional wiener process $(W_t)_{t\in I}$.The drift and diffusion functions a_i,$b_i \in C_p^{2(p+1)}(R^d, R), i = 1, ..., d$ are assumed to satisfy the conditions of the Existence and Uniqueness Theorem[3]. In the following,let $I_h = \{t_0, t_1, ..., t_N\}$ be a discretization of the time interval $I = [t_0, T]$ with

$$0 \leq t_0 < t_1 < ... < t_N = T \tag{1.2}$$

and define $h_n = t_{n+1} - t_n$ for $n = 0, 1, ..., N - 1$.Let $h = max_{1\leq n\leq N-1} h_n$ be the maximum step size of I_h.

Definition 1.1. A sequence of approximation process $y = (y(t))_{t\in I_h}$ converges weakly with order P to X as $h \to 0$ if for each $f \in C_p^{2(p+1)}(R^d, R)$ there exist a constant c_f and a finite $\delta_0 > 0$ such that

$$|E(f(X_T)) - E(f(Y_T))| \leq c_f h^p \tag{1.3}$$

for each $h \in]0, \delta_0[$.

2000 *Mathematics Subject Classification.* Primary 65C30; Secondary 60H35, 60H10.
Key words and phrases. SRK methods,weak approximation,stochastic differential equations.
*R.Gharechahi .

The paper is organized as follows:In section2,a new class of SRK methods is introduced.Further more coefficiants for explicit second order SRK schemes are presented.Then,it closes with a numerical example in section3.

2. A new class of efficient stochastic Runge-Kutta methods

we introduce a new class of second order SRK methods for the weak approximation of the solution of the Itô SDE(1.1). We define the d-dimensional approximation process y with $y_n = y(t_n)$ for $t_n \in I_h$ by the following SRK method with $y_0 = x_0$ and

$$y_{n+1} = y_n + \sum_{i=1}^{s} \alpha_i a(H_i^{(0)}) h_n + \sum_{i=1}^{s} \beta_i^{(1)} b(H_i^{(1)}) I_{(1)} + \sum_{i=1}^{s} \beta_i^{(2)} b(H_i^{(1)}) \frac{I_{(1,1)}}{\sqrt{h_n}} \tag{2.1}$$

for $n = 0, 1, ..., N-1$ with supporting values

$$H_i^{(0)} = y_n + \sum_{j=1}^{i-1} A_{ij}^{(0)} a(H_j^{(0)}) h_n + \sum_{j=1}^{i-1} B_{ij}^{(0)} b(H_j^{(1)}) I_{(1)}$$

$$H_i^{(1)} = y_n + \sum_{j=1}^{i-1} A_{ij}^{(1)} a(H_j^{(0)}) h_n + \sum_{j=1}^{i-1} B_{ij}^{(1)} b(H_j^{(1)}) \sqrt{h_n}$$

The coefficients of such a method can be represented by the usual Butcher-Arrays which take the form

$A^{(0)}$	$B^{(0)}$	
$A^{(1)}$	$B^{(1)}$	
α^T	${\beta^{(1)}}^T$	${\beta^{(2)}}^T$

Applying the rooted tree analysis and to all rooted trees up to order 2.5, we can calculate the following complete order two conditions for the SRK methods (2.1).

Theorem 2.1. *if coefficients of the SRK methods (2.1) fulfill the equations*
1)$\alpha^T e = 1$ 2)${\beta^{(2)}}^T e = 0$ 3)$({\beta^{(1)}}^T e)^2 = 1$ 4)${\beta^{(1)}}^T B^{(1)} e = 0$
then the method converges with order 1.0 in the weak sense.In addition,if the equations
5)$\alpha^T (B^{(0)} e)^2 = \frac{1}{2}$ 6)$\alpha^T A^{(0)} e = \frac{1}{2}$ 7)${\beta^{(1)}}^T (B^{(1)}(B^{(1)}(B^{(1)} e))) = 0$
8)$({\beta^{(1)}}^T e)(\alpha^T B^{(0)} e) = \frac{1}{2}$ 9)$({\beta^{(1)}}^T e)({\beta^{(1)}}^T A^{(1)} e) = \frac{1}{2}$ 10)${\beta^{(1)}}^T (B^{(1)}(B^{(1)} e)) = 0$
11)$({\beta^{(1)}}^T e)({\beta^{(1)}}^T (B^{(1)} e)^2) = \frac{1}{2}$ 12)${\beta^{(1)}}^T (B^{(1)}(A^{(1)}(B^{(0)} e))) = 0$
13)$\alpha^T (B^{(0)} e)(A^{(1)}(B^{(0)} e)) = 0$ 14)${\beta^{(1)}}^T (B^{(1)} e)(A^{(1)}(B^{(0)} e))) = 0$
15)${\beta^{(1)}}^T (A^{(1)}(B^{(0)}(B^{(1)} e))) = 0$ 16)${\beta^{(1)}}^T (B^{(1)} e)(B^{(1)}(B^{(1)} e))) = 0$
17)${\beta^{(1)}}^T (B^{(1)} e)^3 = 0$ 18)${\beta^{(1)}}^T (A^{(1)}(B^{(0)} e)) = 0$ 19)$\alpha^T (B^{(0)}(B^{(1)} e)) = 0$
20)${\beta^{(1)}}^T (B^{(1)}(A^{(1)} e) = 0$ 21)${\beta^{(1)}}^T ((B^{(1)} e)(A^{(1)} e)) = 0$ 22)${\beta^{(2)}}^T A^{(1)} e = 0$
23)${\beta^{(2)}}^T (B^{(1)}(B^{(1)} e)) = 0$ 24)${\beta^{(2)}}^T (B^{(1)} e)^2 = 0$ 25)${\beta^{(2)}}^T (A^{(1)}(B^{(0)} e)^2) = 0$
26)${\beta^{(2)}}^T B^{(1)} e = 1$ 27)${\beta^{(1)}}^T (B^{(1)}(B^{(1)} e)^2) = 0$ 28)${\beta^{(2)}}^T (A^{(1)}(B^{(0)} e)) = 0$
are fulfilled,then the SRK method (2.1) converges with order 2.0 in the weak sense.

Remark 2.2. Due to Theorem 2.1, we have to solve 28 equations.Considering the order conditions 1-4 of Theorem 2.1,we can easily calculate order two SRK methods converging with order one in the weak sense.Further,if we calculate order two

SRK methods with $s \geq 3$ stages,there are some degrees of freedom in choosing the coefficients.Especially,it is possible to calculate SRK methods converging with some higher order if it applied to a deterministic ODE.For example,if the weights α_i and the coefficients $A_{ij}^{(0)}$ are fulfilled conditions $\alpha^T(A^{(0)}(A^{(0)}e)) = \frac{1}{6}$ and $\alpha^T(A^{(0)}e)^2 = \frac{1}{3}$,then the SRK method is of order three in the case of $b^j \equiv 0$ for $1 \leq j \leq m$ in SDE (1.1).Therefore,let (p_D, p_S) with $p_D \geq p_S$ denote the order of convergence of the SRK method.
The SRK method RI2W1 with $p_D = 3$ and $p_S = 2$ presented in Table1(left).While the SRK scheme PL1W1 with $p_D = 2$ and $p_S = 2$ presented in Table1(right).

$\frac{2}{3}$			$\frac{1}{3}$					
$\frac{-1}{3}$	0		$\frac{4}{3}$	0				
1			1					
1	0		-1	0				
$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{2}$	$\frac{-1}{2}$

1			1					
0	0		0	0				
1			1					
1	0		-1	0				
$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{2}$	$\frac{-1}{2}$

Table1:SRK method RI2W1(left) and SRK method PL1W1(right)

3. Numerical results

In order to verify theoretical results,the SRK schemes RI2W1 and PL1W1 are compared.We approximate $E(f(X_T))$ by Monte Carlo simulation.Therefore,we estimate $E(f(Y_T))$ by the sample average of M independently simulated realizations of the approximations $f(Y_{T,k})$,$k = 1, ..., M$,with $Y_{T,k}$ calculated by the scheme under consideration.Then the error is denoted by

$$\hat{\mu} = E(f(X_T)) - \frac{1}{M}\sum_{k=1}^{M} f(Y_{T,k}) \tag{3.1}$$

References

1. A.Robler, *Runge-Kutta methods for itô stochastic differential equations with scalar noise* , BIT., **46** (2006) 97-110.
2. A.Robler, *Runge-Kutta methods for the Numerical Solution of Stochastic Differential Equations* , Ph.D.thesis.Darmstadt University of Technology.Shaker Verlag.Aachen,2003.
3. P.E.KLOEDEN and E.PLATEN, *Numerical Solution of Stochastic Differential Equations*, Springer-Veriag,Berlin, 1999.
4. A.Robler, *Rooted tree analysis for order conditions of stochastic Runge-Kutta methods for the weak approximations of stochastic differential equations* , Stochastic Anal.Appl., **24** (2006) 97-134.
5. K.Debrabant and A.Robler, *Classification of stochastic Runge-Kutta methods for the weak approximation of stochastic differential equations*, Math.Comput.Simulation., **77** (2008) 408-420.

[1] Department of Mathematics, Ferdowsi of University,Mashhad, Iran.
E-mail address: soheili@um.ac.ir

[2] Department of Mathematics, sistan and baluchestan of University , zahedan , Iran.
E-mail address: r.gharechahi_64@yahoo.com