Fuzzy Optimal Control under Generalized Differentiability of Fuzzy-number-valued Functions

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Abstract: In this paper optimal control problem of fuzzy linear time invariant systems is considered. Determining of system optimal input with minimizing quadratic integral criterion by using Pontryagin Minimum Principle will make the Fuzzy Partial Differential Equations [FPDEs] system solve the problem. The quality of obtaining system optimal input by solving the FPDEs system has been studied based on new definition of fuzzy functions differentiability. Finally, some examples illustrate the proposed method.

Keywords: Optimal Control, Fuzzy Differential Equations, Generalized Differentiability, Pontryagin Minimum Principle.

1. Introduction

The investigation of the optimal control theory has been an important branch of modern control theory since the fifties of the last century. With more use of the methods and results on mathematics and computer science, optimal control theory has greatly achieved development and been applied to many fields such as production engineering, programming, economy, management etc.

Theory and application of optimal control for the phenomena that have been modeled with linear time invariant systems have been completely developed, but since description of many of the real phenomena is ambiguous and one cannot always withdraw from the existent uncertainty in real dynamical systems, determining optimal input based on models that uncertainty and ambiguity are not considered in them in fact cannot satisfy the demands which we are looking for them in optimal control problem, demands such as shortest path , minimum time, minimum energy, etc.

Fuzzy logic is one of the useful tools for describing and modeling of real systems that can consider well uncertainty and ambiguity existing in the real systems in their mathematical model. So, by expression of fuzzy optimal control problem for real systems that have been modeled by fuzzy math the optimal control input can be obtained that compared with previous state determination of optimal input based on definitive models is closer to reality and also has been improved, that's why in recent years several papers have focused on fuzzy optimal control problem. Wan developed an optimal fuzzy controller to stabilize a Linear Time Invariant [LTI] system via Pontryagin Minimum Principle [1], in [2] application of optimal fuzzy control will reduce the energy consumption of a Benzene-Toluene distillation column.

Authors in [1], [2] by using fuzzy IF-Then rules and then by defuzzification of fuzzy dependency functions and finally by using Pontryagin Minimum Principle have presented optimal fuzzy controller design. A special fuzzy optimal control model with a quadratic objective function subject to linear fuzzy differential equation is investigated in [3] and authors have used the expected value-based method to optimize the fuzzy objective function. In [4] the fuzzy logic controller to manage the power split between the fuel cell and battery has been parameterized and by using Direct Algorithm [5] the optimal values of parameterized fuzzy controller were found. In [6] the fuzzy optimal controller has been used practically by fuzzy IF-Then rules and genetic algorithm design. To reduce the current of battery effectively, and helpfully to prolong its service life, authors in [7] proposed the fuzzy optimal controller based on IF-Then fuzzy rules and Mamdani's fuzzy inference method. However, the optimality equation generally does not have analytic solutions except some special cases. In this paper we will study optimal input determination problem for fuzzy linear time invariant systems with minimizing the quadratic integral criteria. By using Pontryagin Minimum Principle and Hamilton functional form, optimal input fuzzy is determined by solving fuzzy partial differential equations system. Solving of fuzzy differential equations system requires overall concept of differentiability in fuzzy-number-valued functions. In several literatures different ways for solving fuzzy differential equations has been proposed [8-10]. The significant point is that these literatures have considered fuzzy differential equations description based on usual concept of differentiability of fuzzy-number-valued functions. Bede and Gal [11] have offered more comprehensive and complete concept of

fuzzy functions differentiability called Generalized Differentiability. They have shown that the conventional concept of fuzzy functions differentiability is not general and is defective. For example:
If c is a fuzzy number and $g:[a,b] \to \mathbb{R}$ is an usual real-valued function differentiable on $x_0 \in (a,b)$ with $g'(x_0) \le 0$, then based on the conventional concept of differentiability, $f(x) = c \odot g(x)$ is not differentiable on x_0. While based on Generalized Differentiability, $f(x)$ is derivative in x_0. In this paper, system optimal input by using Pontryagin Minimum Principle and based on the concept of generalized differentiability is determined.

2. The Pontryagin Minimum Principle

Consider the system

$$\dot{X}(t) = G\big(X(t), U(t)\big)$$
$$X(0) = X_0$$

Where $X \in \mathbb{R}^n$ is the state vector, $U \in \mathbb{R}^m$ is the control input vector, and $G = (g_1, \dots, g_n)$ is a vector of linear or nonlinear functions. The optimal control problem for the system (1) is as follows: determine the control $U(t) = (u_1, \dots, u_m)$ such that the performance criterion

$$J = \int_0^T f(X(t), U(t), t)\, dt$$

Is minimized, where f as the cost function is given and the final time T is known in this paper. The Pontryagin Minimum Principle for solving this optimal control problem proceeds as follows: defined the so-called Hamiltonian

$$H(X, U, \lambda) = f(X(t), U(t), t) + \lambda(t) G(X(t), U(t))$$

And find $U(t)$ such that $H(X, U, \lambda)$ is minimized with this $U(t)$. By solving these differential equations the optimal control is obtained

$$\frac{\partial H}{\partial u_i} = 0 \ ; \ \frac{\partial H}{\partial x_i} = -\dot{\lambda}_i(t) \ ; \ \frac{\partial H}{\partial \lambda_i} = g_i(x_i, u_i)$$

With boundary conditions $X(0) = X_0$, $\lambda_i(T) = 0$.

3. The Generalized Differentiability

The authors in [11] introduce a more general definition of derivative for fuzzy mappings by considering a lateral type of H-derivatives [12].

We will denote by E^n the space of all compact and convex fuzzy sets on $\mathbb{R}^n$.

Definition 3.1. Let $f:(a,b) \to E^1$ and $t_0 \in (a,b)$, then :

f is (1)-differentiable at t_0 if for $h > 0$ sufficiently near to 0, there exist the H-differences $f(t_0 + h) \ominus f(t_0)$, $f(t_0) \ominus f(t_0 - h)$ and the limits (in the metric D):

$$f'(t_0) = \lim_{h \to 0^+} \frac{f(t_0 + h) \ominus f(t_0)}{h} = \lim_{h \to 0^+} \frac{f(t_0) \ominus f(t_0 - h)}{h}$$

f is (2)-differentiable at t_0 if for $h > 0$ sufficiently near to 0, there exist the H-differences $f(t_0) \ominus f(t_0 + h)$, $f(t_0 - h) \ominus f(t_0)$ and the limits (in the metric D):

$$f'(t_0) = \lim_{h \to 0^+} \frac{f(t_0) \ominus f(t_0 + h)}{(-h)} = \lim_{h \to 0^+} \frac{f(t_0 - h) \ominus f(t_0)}{(-h)}$$

4. The Fuzzy Optimal Control Problem

In this work we consider the system under control the LTI system

$$\dot{\tilde{X}}(t) = \tilde{A}\tilde{X}(t) + \tilde{B}U(t)$$

And we have the following optimal control problem:

$$Minimize \int_0^T [\tilde{x}_1^2(t) + \cdots + \tilde{x}_n^2(t) + u_1^2(t) + \cdots + u_m^2(t)]\, dt$$
$$s.t.$$
$$\dot{\tilde{X}}(t) = \tilde{A}\tilde{X}(t) + \tilde{B}U(t)$$

By solving the above problem we will obtain system fuzzy optimal input.

5. References

[1] Wang, L. X. "Stable and Optimal Fuzzy Control of Linear Systems". *IEEE Trans. Fuzzy Sys.* **6**, pp. 137–143. 1998.

[2] A. Khelassi, C. Bouyahiaoui, A. Maidi, A. Benhalla, M. Djellal, "Optimal Fuzzy Control To Optimise Energy Consumption in Distillation Columns", *15th Triennial World Congress, Barcelona*, Spain, 2002.

[3] Y. Zhao, Y. Zhu, "Fuzzy optimal control of linear quadratic models", *Computers and Mathematics with Applications*, 60, pp. 67-73, 2010.

[4] Chun-Yan Li, Guo-Ping Liub, "Optimal fuzzy power control and management of fuel cell/battery hybrid vehicles", *Journal of Power Sources*, 192, pp. 525–533, 2009.

[5] D.R. Jones, C.D. Perttunen, B.E. Stuckman, "Lipschitzian optimization without the Lipchitz constant", *Journal of Optimization Theory and Applications*, 79, pp. 157–181, 1993.

[6] John W. Labadie, Yongshan Wan, " Fuzzy optimal control of reservoir-assisted stormwater treatment areas for aquatic ecosystem restoration", *Environmental Modelling & Software*, 25, pp. 1692-1701, 2010.

[7] Guiping Wang, Panpan Yang, Jinjin Zhang, " Fuzzy Optimal Control and Simulation of Battery-Ultracapacitor Dual-Energy Source Storage System for Pure Electric Vehicle", *International Conference on Intelligent Control and Information Processing*, August 13-15, Dalian, China, 2010.

[8] Y. Chalco-Cano, H. Roma´n-Flores, "On new solutions of fuzzy differential equations", *Chaos, Solitons and Fractals*, 38, pp. 112–119, 2008.

[9] Mehran Mazandarani, Ali Vahidian Kamyad, "Numerical Solution of Fuzzy Linear Non-Autonomous Systems by AVK Method", *4th International Conference of Fuzzy Information & Engineering*, Iran-Amol, 14-15 October, 2010.

[10] Y.Chalco-Cano, H.Román-Flores , "Comparation between some approaches to solve fuzzy differential equations", *Fuzzy Sets and Systems*, 160, pp. 1517–1527, 2009.

[11] Bede B, Gal SG, "Generalizations of the differentiability of fuzzy number valued functions with applications to fuzzy differential Equations", *Fuzzy Set Syst*,151, pp. 581–599, 2005.

[12] Puri M, Ralescu D, "Differential and fuzzy functions", *J Math Anal Appl*, 91, pp. 552–558, 1983.