THE NON-ABELIAN TENSOR SQUARE AND SCHUR MULTIPLIER OF GROUPS OF ORDERS $p^2 q$ AND $p^2 qr$

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Abstract. The aim of this paper is to determine the non-abelian tensor square and Schur multiplier of groups of square free order and of groups of orders $p^2 q$, $pq^2$ and $p^2 qr$, where $p$, $q$ and $r$ are primes and $p < q < r$.

1. Introduction

The notion of non-abelian tensor product $G \otimes H$ of groups $G$ and $H$, which was introduced by R. Brown and J.-L. Loday [2, 3], is a generalization of the usual tensor product of the abelian groups. Let there be actions

$$G \times H \rightarrow G, (g, h) \mapsto h^g; \quad H \times G \rightarrow H, (h, g) \mapsto g^h$$

in such a way that for all $g, g_1 \in G$ and $h, h_1 \in H$,

$$g_1^h g = g_1^{-1}(h(g_1 g)) \quad \text{and} \quad h_1^h h = h_1^{-1}(g(h_1 h)),$$  \hfill (\star)

where $G$ acts on itself by conjugation $(g, g_1) \mapsto g_1 g = g g_1^{-1}$, and $H$ acts on itself similarly. Then the non-abelian tensor product $G \otimes H$ is defined to be the group generated by symbols $g \otimes h$, for all $g \in G, h \in H$, subject to the relations

$$g_1 g \otimes h = (g_1 \otimes g)(g \otimes h), \quad g \otimes h_1 h = (g \otimes h_1)(h_1 \otimes h)$$

for all $g, g_1 \in G, h, h_1 \in H$. In the case $G = H$ and $G$ acts on itself by conjugation, $G \otimes G$ is called the non-abelian tensor square of $G$.

Following the publications of Brown and Loday’s work, a number of purely group theoretic papers have appeared on this topic. Some of them investigate structural properties of the
tensor square, while the others are devoted to explicit descriptions for particular groups, for instance dihedral, quaternionic, symmetric and all groups of order at most 30 in [1].

Later, Hannebauer [7] determined the structure of tensor square of the linear groups SL(2, q), PSL(2, q), GL(2, q) and PGL(2, q) for all q \geq 5 and q \neq 9.

Ellis and Leonard [4] devised a computer algorithm for the computation of tensor square of finite groups which can handle much larger groups than those given in [1]. Using the CAYLEY-program, they compute the tensor square of B(2, 4), the 2-generator Burnside group of exponent 4, where |B(2, 4)| = 2^{12}. Recently, Hannebauer’s result is improved for the linear groups SL(n, q), PSL(n, q), GL(n, q) and PGL(n, q) for all n, q \geq 2 in [5]. These results extremely depend on knowing the order of M(G), the Schur multiplier of a group G. Also, some computations of G \otimes G for polycyclic groups have been done in [9].

In the present paper, we determine the non-abelian tensor square of the groups of square free order and groups of orders p^2 q, pq^2 and p^2 qr, where p, q and r are primes and p < q < r. Here we give some notations which will be used throughout the paper.

\( G^{ab} \) Abelianisation of G,
\( c(G) \) exponent of G,
\( Q_2 \) quaternion group of order 8,
\( A_4 \) alternating group of order 12,
\( (\mathbb{Z}_{p'})^k \) direct product of k copies of the cyclic group of order p'.

**Theorem A.** Let G be a group of order n, where n is a square free number. Then

\[ G \otimes G \cong \mathbb{Z}_n. \]

**Theorem B.** Let G be a group of order p^2 q, where p and q are prime numbers and p < q. The structure of G \otimes G is one of the following

(i) If \( G^{ab} = \mathbb{Z}_{p^2} \), then \( G \otimes G \cong \mathbb{Z}_{p^2 q} \).

(ii) If \( G^{ab} = (\mathbb{Z}_p)^2 \), then \( G \otimes G \cong (\mathbb{Z}_p)^4 \times \mathbb{Z}_q \).

(iii) If \( G^{ab} = \mathbb{Z}_3 \), then \( G \otimes G \cong \mathbb{Z}_3 \times Q_2 \).

**Theorem C.** Let G be a group of order pq^2, where p and q are prime numbers and p < q.
The structure of $G \otimes G$ is one of the following

(i) If $G' = \mathbb{Z}_q^2$, then $G \otimes G \cong \mathbb{Z}_{pq^2}$.

(ii) If $G' = (\mathbb{Z}_q)^2$ and $M(G) = 0$ or if $G' = \mathbb{Z}_q$, then $G \otimes G \cong \mathbb{Z}_p \times (\mathbb{Z}_q)^2$.

(iii) If $G' = (\mathbb{Z}_q)^2$ and $M(G) = \mathbb{Z}_q$, then $G \otimes G \cong \mathbb{Z}_p \times H$, where $H$ is an extra-special $q$-group of order $q^3$.

**Theorem D.** Let $G$ be a group of order $p^2qr$, where $p$, $q$, and $r$ are prime numbers $p < q < r$, $pq \neq 6$. The structure of $G \otimes G$ is one of the following

(i) If $|G'| = q$, $r$ or $qr$ where $q \not\equiv 1 \pmod{r}$, then $G \otimes G \cong \mathbb{Z}_{p^2qr}$ when $G^{ab}$ is cyclic, otherwise $G \otimes G \cong (\mathbb{Z}_p)^4 \times \mathbb{Z}_{qr}$.

(ii) If $|G'| = qr$ where $q \equiv 1 \pmod{r}$, then $G \otimes G \cong \mathbb{Z}_{p^2} \times G'$ when $G^{ab}$ is cyclic, otherwise $G \otimes G \cong (\mathbb{Z}_p)^4 \times G'$.

2. Basic Results

In this section we recall some definitions and basic results on the tensor square which are necessary for our main theorems.

Let $G$ be a group and $G \otimes G$ be the tensor square of $G$. The exterior square $G \wedge G$ is obtained by imposing the additional relation $g \otimes g = 1 \otimes (g \in G)$ on $G \otimes G$. Moreover, we denote by $\nabla(G)$ the subgroup of $G \otimes G$ generated by all elements $g \otimes g$ for all $g \in G$. The commutator map induces homomorphisms $\kappa : G \otimes G \to G$ and $\kappa' : G \wedge G \to G$ sending $g \otimes h$ and $g \wedge h$ to $[g, h] = ghg^{-1}h^{-1}$. The kernel of $\kappa$ is denoted by $J_2(G)$.

Results in [2, 3] give the following commutative diagram with exact rows and central extensions as columns

\[
\begin{array}{cccccc}
0 & 0 & & & & \\
\| & \| & \downarrow & \downarrow & \downarrow & \\
\Gamma(G^{ab}) & \longrightarrow & J_2(G) & \longrightarrow & M(G) & \longrightarrow & 0 \\
\| & \| & \downarrow & \downarrow & \downarrow & \\
\Gamma(G^{ab}) & \longrightarrow & G \otimes G & \longrightarrow & G \wedge G & \longrightarrow & 1 \\
\| & \| & \downarrow & \downarrow & \downarrow & \\
G' & \longrightarrow & G' & \longrightarrow & 1 & \longrightarrow & 1 \\
\end{array}
\]

where $\Gamma$ is the Whitehead’s quadratic functor (see Whitehead [10]).

For the sake of convenience of the reader we state some known results which are used in the proof of the main theorems.
Theorem 2.1. [1, Proposition 8]. If $G$ is a group in which $G'$ has a cyclic complement $C$, then $G \otimes G \cong (G \wedge G) \times G^{ab}$ and hence $|G \otimes G| = |G||M(G)|$.

Theorem 2.2. [6, Theorem A]. Let $G$ be a group such that $G^{ab} = \prod_{i=1}^{n} \prod_{j=1}^{k_i} \mathbb{Z}_{p_i}^{e_{ij}}$ where $1 \leq e_{i1} \leq e_{i2} \leq \ldots \leq e_{ik_i}$ for all $1 \leq i \leq n$, $k_i \in \mathbb{N}$ and $p_i \neq 2$. Then

$$|G \otimes G| = \prod_{i=1}^{n} p_i^{d_i}|G||M(G)|$$

in which $d_i = \sum_{j=1}^{k_i} (k_i - j)e_{ij}$.

3. Proof of Main Theorems

In this section we prove the main theorem as we mentioned earlier in section one.

Lemma 3.1. Let $G$ be a finite non-abelian group.

(i) If $G$ is a square free order group, then $G'$ is cyclic.

(ii) If $G$ is a group of order $p^2q$, then $G' = \mathbb{Z}_q$ or $G' = (\mathbb{Z}_2)^2$.

(iii) If $G$ is a group of order $pq^2$, then $G' = \mathbb{Z}_q$, $G' = \mathbb{Z}_{q^2}$ or $G' = (\mathbb{Z}_q)^2$.

(iv) If $G$ is a group of order $p^2qr$ and $pq \neq 6$, then $|G'| = q, r$ or $qr$.

(v) If $G$ is a group of order $p^2qr$ and $pq = 6$, then $|G'| = 3, r, 3r, 4$ or $4r$.

Proof. One may use Sylow theorems for examples in case (ii) to show the number of Sylow $p$-subgroups of $G$ is 1 or $q$ and the number of Sylow $q$-subgroups of $G$ is 1 or $p^2$. If $G$ has one Sylow $q$-subgroup $Q$, then $G/Q$ is abelian and so $G' = \mathbb{Z}_q$. Otherwise $p = 2$ and $q = 3$, hence $|G| = 12$ and we should have $G \cong A_4$.

The proof of other cases is similar and we omit it. \hfill $\Box$

Lemma 3.2. Let $G$ be a finite non-abelian group.

(i) If $G$ is a square free order group, then $M(G) = 0$.

(ii) If $G$ is a group of order $p^2q$, then

$$M(G) = \begin{cases} 
0 & \text{if } G' = \mathbb{Z}_q \text{ and } G^{ab} = \mathbb{Z}_{p^2} \\
\mathbb{Z}_p & \text{if } G' = \mathbb{Z}_q \text{ and } G^{ab} = (\mathbb{Z}_p)^2 \\
\mathbb{Z}_2 & \text{if } G' = (\mathbb{Z}_2)^2
\end{cases}$$
(iii) If $G$ is a group of order $pq^2$, then

$$M(G) = \begin{cases} 
0 & \text{if } G' = \mathbb{Z}_q \\
0 & \text{if } G' = \mathbb{Z}_q^2 \\
0 \text{ or } \mathbb{Z}_q & \text{if } G' = (\mathbb{Z}_q)^2 
\end{cases}$$

(iv) If $G$ is a group of order $p^2qr$ and $pq \neq 6$, then

$$M(G) = \begin{cases} 
0 & \text{if } G^{ab} \text{ is cyclic} \\
\mathbb{Z}_p & \text{otherwise} 
\end{cases}$$

(v) If $G$ is a group of order $p^2qr$ and $pq = 6$, then $M(G)$ is as the same as part (iv). Moreover if $|G'| = 4$ or $4r$, then $M(G) = \mathbb{Z}_2$.

Proof. (i) Since all Sylow subgroups of $G$ are cyclic, so $M(G) = 0$. For (ii), if $G' = \mathbb{Z}_q$ and $G^{ab} = \mathbb{Z}_p^2$, then again all Sylow subgroups of $G$ are cyclic and therefore $M(G) = 0$.

In the case $G' = \mathbb{Z}_q$ and $G^{ab} = (\mathbb{Z}_p)^2$, we can see that $|G'|$ and $|G^{ab}|$ are coprime, so by Schur-Zassenhaus lemma, $G'$ has a complement. The result follows from [8, Corollary 2.2.6].

The proof of the other parts is similar. $\square$

**Lemma 3.3.** Let $G$ be a finite non-abelian group.

(i) If $G$ is a square free order group of order $n$, then $|G \otimes G| = n$.

(ii) If $G$ is a group of order $p^2q$, then

$$|G \otimes G| = \begin{cases} 
p^2q & \text{if } G^{ab} = \mathbb{Z}_p^2 \\
p^4q & \text{if } G^{ab} = (\mathbb{Z}_p)^2 \\
24 & \text{if } G^{ab} = \mathbb{Z}_3
\end{cases}$$

(iii) If $G$ is a group of order $pq^2$, then

$$|G \otimes G| = \begin{cases} 
pq^2 & \text{if } G' = \mathbb{Z}_q^2 \\
pq^2 & \text{if } G' = \mathbb{Z}_q \text{ or } G' = (\mathbb{Z}_q)^2 \text{ and } M(G) = 0 \\
pq^3 & \text{if } G' = (\mathbb{Z}_q)^2 \text{ and } M(G) = \mathbb{Z}_q
\end{cases}$$
(iv) If $G$ is a group of order $p^2qr$ and $pq \neq 6$, then

$$|G \otimes G| = \begin{cases} p^2qr & \text{if } G^{ab} \text{ is cyclic} \\ p^4r & \text{otherwise} \end{cases}$$

(v) If $G$ is a group of order $p^2qr$ and $pq = 6$, then the order of $G \otimes G$ is similar to the part (iv). Moreover if $|G'| = 4$ or $4r$, then $|G \otimes G| = 24r$.

Proof. (i) Since $|G'|$ and $|G^{ab}|$ are coprime, $G'$ has a cyclic complement and $|G \otimes G| = |G|$ by Schur-Zassenhaus lemma, Theorem 2.1 and Lemma 3.2.

(ii) Suppose that $G^{ab} = Z_{p^2}$. In this case $|G'|$ and $|G^{ab}|$ are coprime and $|G \otimes G| = |G| = p^2q$ by Lemma 3.2.

Now assume that $G^{ab} = (Z_p)^2$, if $p \neq 2$, then $|G \otimes G| = p^4q$ by Theorem 2.2 and Lemma 3.2. Otherwise $e(\nabla(G))$ divides $e(G) = 2q$ and $e(\Gamma(G^{ab})) = 4$. Hence $e(\nabla(G)) = 2$ and $\nabla(G) = (Z_2)^3$ and it implies that $|G \otimes G| = 2^4q$.

We note that if $G^{ab} = Z_3$, then $G = A_4$ and this case is computed in [1].

(iii) Suppose that $G'$ is cyclic of order $q^2$, so we have $|G \otimes G| = pq^2$ by Theorem 2.1 and Lemma 3.2. The case that $G' = (Z_q)^2$ and $M(G) = 0$ holds similarly.

Now if $|G'| = q$ and $p \neq 2$, then $|G \otimes G| = pq^2$ by Theorem 2.2 and Lemma 3.2. Also if $p = 2$, then it is easy to see that $|G \otimes G| = 2q^2$.

(iv) Suppose that $G^{ab}$ is cyclic, so $|G \otimes G| = |G|$ by Theorems 2.1 and Lemma 3.2.

The other case is similar to the case (ii).

(v) It is straight forward. \(\Box\)

We are ready to prove main theorems. Notice that if $G'$ is cyclic, then $G \otimes G$ is abelian.

Proof of Theorem A. Since $G$ is a group of square free order, then $G \otimes G$ is abelian of order $n$ by Lemmas 3.1 and 3.3.

Proof of Theorem B. (i) It is clear that $G \otimes G$ is an abelian group of order $p^2q$ by Lemma 3.3. On the other hand, $e(G \otimes G)$ divides $|G \otimes G|$ and the epimorphism $\pi : G \otimes G \to G^{ab} \otimes G^{ab}$ implies that $e(G \otimes G) = p^2q$. Hence $G \otimes G \cong Z_{p^2q}$, as required.

(ii) It is as same as the case (i).
(iii) If $G' = (\mathbb{Z}_2)^2$, then $G = A_4$ and one may refer to the Table 1 given in [1].

**Proof of Theorem C.** (i) The exponent of $G \otimes G$ is equal to $pq^2$, so the proof follows by Lemma 3.3.

(ii) Since $G \otimes G/J_2(G) = G'$ is abelian, we have $(G \otimes G)' \subseteq J_2(G)$, where $|J_2(G)| = p$. In addition the epimorphism $\pi$ implies that $(G \otimes G)'$ is in it’s kernel which is of order $q^2$, so $(G \otimes G)' = 1$. The result holds by Lemma 3.3 and the fact that $e(G \otimes G) = pq$.

(iii) It is clear that $\nabla(G)$ and the Sylow $q$-subgroup of $G \otimes G$, say $H$, are normal. Thus by Lemma 3.3,

$$G \otimes G \cong \nabla(G) \times H,$$

in which $\nabla(G) = \mathbb{Z}_p$ and $H$ is of order $q^3$.

**Proof of Theorem D.** (i) If $G^{ab}$ is cyclic, then the epimorphism $\pi$ implies that $e(G \otimes G) = p^2qr$, so the result follows by Lemma 3.3.

If $G^{ab}$ is not cyclic, the proof is similar.

(ii) Assume that $G^{ab} = \mathbb{Z}_{p^2}$, then $|\nabla(G)| = |J_2(G)| = p^2$ and it can be easily seen that Ker $\pi$ is isomorphic to $G'$ and of order $qr$. Moreover since $e(G \otimes G) = p^2qr$, Lemma 3.3 implies that

$$G \otimes G \cong \nabla(G) \times \text{Ker } \pi \cong \nabla(G) \times G' \cong \mathbb{Z}_{p^2} \times G'.$$

If $G^{ab} = (\mathbb{Z}_p)^2$, then $e(G \otimes G) = pqr$, $|J_2(G)| = p^4$ and Ker $\pi \cong G'$. Therefore it follows from Lemma 3.3 that

$$G \otimes G \cong J_2(G) \times \text{Ker } \pi \cong (\mathbb{Z}_p)^4 \times G'.$$

Finally, we note that in Theorem D, if $pq = 6$, i.e. $|G| = 12r$, then $|G'| = 3, r, 3r, 4$ or $4r$. In the cases that $|G'| = 3, r$ or $3r$, the structure of $G \otimes G$ is similar to Theorem D. In other cases one may show that $G \otimes G \cong \mathbb{Z}_{3r} \times Q_2$.

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