JIRSS (2011)
Vol. 10, No. 1, pp 1-12

Statistical Evidences in Type-II Censored Data

A. Habibi Rad[1], M. Emadi[1], M. Arashi[2], N. R. Arghami[1]

[1]Department of Statistics, School of Mathematical Sciences, Ferdowsi University of Mashhad, Iran.
[2]Faculty of Mathematics, Shahrood University of Technology, Iran.

Abstract. In this article, we use a measure of expected true evidence for determine the required sample size in type-II censored experiments for obtaining statistical evidence in favor of one hypothesis about the exponential mean against another.

Keywords. Exponential distribution, likelihood ratio, statistical evidence, type-II censored data.

MSC: 62A01, 62D05, 62N01.

1 Introduction

In a life-testing experiment, n items are placed on the test. The failure times observed from such a life-test, $X_{(1)} \leq ... \leq X_{(n)}$, are the order statistics from a random sample of size n from a parametric distribution with probability density function (pdf) $f(x;\theta)$ and cumulative distribution function (cdf) $F(x;\theta)$, where $\theta \epsilon \Re$. However, one may not continue the experiment until the last failure since the waiting time for the final failure is unbounded (Muenz and Green, 1977). For this reason, in some cases, the life-testing experiment is usually terminated when the rth failure $X_{(r)}$ is observed, which is referred to as a type-II censoring scheme. This censoring model saves time and cost, but some information about

A. Habibi Rad (✉)(ahabibi@um.ac.ir), M. Emadi(emadi@um.ac.ir),
M. Arashi(m_arashi_stat@yahoo.com), N. R. Arghami(arghami_nr@yahoo.com)
Received: August 23, 2010

the underlying parameters is lost in the censored data (Zheng and Park, 2004). So, the inference based on Type-II censored data will naturally be less efficient than that based on the complete data of n observations. More than the above specified scheme, there exist some other different sorts of censoring schemes such as random censoring, hybrid censoring (Epstein, 1954) and progressively Type-II censoring (Balakrishnan and Aggarwala, 2000).

Let $X_{(1)}, \cdots, X_{(n)}$ denote the ordered values of the random sample $X_1, \cdots, X_n$ (failure times). In Type-II plan, observations terminate after the rth failure occurs. So we only observe the r smallest observations in a random sample of n items. The likelihood function based on $X_{(1)}, \cdots, X_{(r)}$ is given by (Arnold et al. 1992)

$$\mathcal{L}_{cen.} = \frac{n!}{(n-r)!} \prod_{i=1}^{r} f\left(x_{(i)}\right) \left[1 - F(x_{(r)})\right]^{n-r}.$$

In Type-II censoring, the number of failure times r is fixed whereas the endpoint $X_{(r)}$ is a random observation.

An important role of the statistical analysis in science is interpreting observed data as evidence, that is assessing "What do the data say?". Although standard statistical methods (hypothesis testing, estimation, confidence intervals) are routinely used for this purpose, the theory behind those methods contains no defined concept of evidence and no answer to the basic question "when is it correct to say that a given body of data represent evidence supporting one statistical hypothesis against another?" (Royall, 1997, 2000). Emadi and Arghami (2003) and Emadi et al. (2007) have studied some measures of support of statistical hypotheses. Doostparast and Emadi (2006), Arashi and Emadi (2008) have studied some measures of support of statistical hypotheses based on independent and identically distribution (iid) observations and record statistics.

Habibi et al. (2006) generalized the concept of "expected true statistical evidence" based on the law of likelihood. So, when the objective of the study is to produce statistical evidence for one hypothesis against another, it is desirable to have a measure of performance of the experiments E_1 and E_2. This can be defined as

$$S_{\varphi}(E) = E_{\theta_1}\varphi(\lambda) + E_{\theta_2}\varphi(1/\lambda),$$

where $\varphi(.)$ is a non-decreasing function and

$$\lambda = \frac{f_1(\mathbf{X})}{f_2(\mathbf{X})},$$

where $f_1(.)$ and $f_2(.)$ are densities of $\mathbf{X}$ under H_1 and H_2, respectively, and when $\mathbf{X} = \mathbf{x}$, λ is the likelihood ratio.
By definition (Royall, 1997) when H_1 is true and we observe $\lambda > K$ or when H_2 is true and we observe $\lambda < 1/K$, we have strong true evidence under H_1 or H_2, respectively, where K is arbitrary and is usually between 8 and 32. Now if we take

$$\varphi(t) = \begin{cases} 1, & t \geq K \\ 0, & t < K, \end{cases}$$

$S_\varphi(E)$ is the sum of the probabilities of observing strong true evidence under H_1 and H_2.

If $\varphi(t) = t/(1+t)$, then $S_\varphi(E) = \text{abc}(E) =$ the area between the cumulative distribution function (cdf) curves (under H_1 and H_2) of $\eta = \lambda/(1+\lambda)$ (Emadi and Arghami, 2003).

If $\varphi(t) = \log(t)$, then

$$\begin{aligned} S_\varphi(E) &= E_{\theta_1}\left[\log \frac{f(\mathbf{X};\theta_1)}{f(\mathbf{X};\theta_2)}\right] + E_{\theta_2}\left[\log \frac{f(\mathbf{X};\theta_2)}{f(\mathbf{X};\theta_1)}\right] \\ &= D(f_{\theta_1}, f_{\theta_2}) + D(f_{\theta_2}, f_{\theta_1}) \\ &= J(f_{\theta_1}, f_{\theta_2}), \end{aligned}$$

where $D(p_{\theta_1}, p_{\theta_2})$ and $J(p_{\theta_1}, p_{\theta_2})$ are, respectively, asymmetric and symmetric Kullback-Leibler (K-L) distance (information) of p_{θ_1} and p_{θ_2}, (Kullback, 1959).
Also, other measures like as Fisher information (Park, 1996, Zheng and Gastwirth, 2000, Zheng and Gastwirth, 2001, and Zheng and Park, 2005) are used for this reason.

If the experimenter's object is obtaining statistical evidence about some competing hypotheses, then she/he would like to know the potential true evidence in the available data.

It turns out that "per failure expected true evidence" is a decreasing function of r, the number of failure times (Fig. 3), and thus the optimum value of r is one. The purpose of this paper, however, is to explain the process of choosing the value of r, when we are content with a percentage of the information contained in a non-censored sample.

The outline of this paper is as follows. In Section 2, we specify the model and the likelihood functions based on type-II censored data and iid observations. Section 3 contains some numerical studies and conclusions appear in Section 4. Finally the Maple program of the simulation study in section 3 is included in the Appendix.

2 Measuring Statistical Evidence

The model under study is the exponential model with the following density function

$$f_\theta(x) = \frac{1}{\theta}\ \exp\left(-\frac{x}{\theta}\right), \quad x > 0, \quad \theta > 0. \tag{1}$$

Then, the likelihood functions associated with iid observations and Type-II censored data are, respectively, given by

$$\mathcal{L}_{iid}(\theta) \quad = \quad \left(\frac{1}{\theta}\right)^n \exp\left\{-\frac{1}{\theta}\sum_{i=1}^{n} x_i\right\}, \tag{2}$$

and

$$\mathcal{L}_{cen.}(\theta) \quad = \quad \frac{n!}{(n-r)!}\left(\frac{1}{\theta}\right)^r \exp\left\{-\frac{1}{\theta}\left[\sum_{i=1}^{r} x_{(i)} + (n-r)x_{(r)}\right]\right\}, \tag{3}$$

where r is the number of failures.

Statistical evidence is represented and interpreted by the law of likelihood and its strength is measured by the likelihood ratio. The law of likelihood explains that the strength of statistical evidence for one hypothesis over another is measured by their likelihood ratio, (Blume, 2002)

Let random vector $\mathbf{X}$ have densities $f_1(\mathbf{x})$ and $f_2(\mathbf{x})$ (both completely known) under H_1 and H_2 respectively. We are going to use the one-to-one function $\eta = \lambda/(1+\lambda)$ of λ as a measure of statistical evidence in favor of H_1 against H_2. Note that the transformed measure η is a strictly increasing and continuous function in λ and takes values in the unit interval $(0, 1)$.

As a measure of expected true statistical evidence, we use $abc(\eta)$, defined by Emadi and Arghami (2003), as

$$abc(\eta) = E_1(\eta) - E_2(\eta), \tag{4}$$

where $E_i(\eta)$ is the expected value of η under H_i, $i = 1, 2$.

3 Optimal Stopping Point

In this section we find the value r (stopping point) in Type-II censored data which is optimum in the sense that it maximizes expected true

evidence per observed failure.
Consider the following hypotheses.

$$\left\{ \begin{array}{c} H_1 : \theta = \theta_1 \\ H_2 : \theta = \rho.\theta_1 \end{array} \right. , \tag{5}$$

where $\rho =\in (0,1)$ is known. Since H_1 and H_2 can be interchanged without loss of generality, we can not assume $\rho > 1$.

Theorem 3.1. *For the hypotheses in (5), γ depends only on ρ and not on θ_1, where*

$$\gamma = \frac{abc_{cen.}(\eta)}{abc_{iid}(\eta)}, \tag{6}$$

where $abc_{cen.}(\eta)$ and $abc_{iid}(\eta)$ are respectively expected true evidence provided by censored data and the complete sample.

Proof. The likelihood for the hypotheses in (5) is

$$\lambda = \frac{f_{\theta_1}(\mathbf{x})}{f_{\rho.\theta_1}(\mathbf{x})}.$$

Based on equations (1)-(3) we have

$$\lambda = \left(\frac{\rho.\theta_1}{\theta_1}\right)^r \exp\left[\left(\frac{1}{\rho.\theta_1} - \frac{1}{\theta_1}\right) Q\right], \qquad Q = \sum_{i=1}^{r} x_{(i)} + (n-r)x_{(r)}.$$

Now $\frac{2Q}{\theta} \sim \chi^2_{2r}$, where χ^2_{2r} is the central chi-square distribution with $2r$ degrees of freedom (see Cohen, 1991, 1995). By using (4) we have

$$\begin{aligned} abc(\eta)_{cen.} \quad = \quad & E\left\{ \frac{\rho^r \exp\left[\frac{1}{2}\left(\frac{1}{\rho} - 1\right)\chi^2_{2r}\right]}{1 + \rho^r \exp\left[\frac{1}{2}\left(\frac{1}{\rho} - 1\right)\chi^2_{2r}\right]} \right\} \\ & - E\left\{ \frac{\rho^r \exp\left[\frac{1}{2}(1-\rho)\chi^2_{2r}\right]}{1 + \rho^r \exp\left[\frac{1}{2}(1-\rho)\chi^2_{2r}\right]} \right\}. \end{aligned} \tag{7}$$

For the special case of $r = n$ (complete sample) we denote $abc(\eta)_{cen.}$ by $abc(\eta)_{iid}$.

As it can be seen, $abc(\eta)_{cen.}$ and $abc(\eta)_{iid}$ are independent of θ_1, and thus so is γ. Therefore γ depends only on ρ, r and n.
Since no explicit expression for γ exists we use the approximations of the next section.

3.1 Taylor Expansion

As one way to evaluate γ approximately, we use Taylor Expansion method and the following lemma (Lehmann and Casella, 1998).

Lemma 3.1. *Let $X_1, \cdots, X_n$ be iid with $E(X_1) = \mu$, $Var(X_1) = \sigma^2$ and finite forth moments. Suppose h is a function of a real variable whose first four derivatives $h'(t)$, $h''(t)$, $h'''(t)$, and $h^{(iv)}(t)$ exist for all $t \in I$ where I is an interval with $P(X_1 \in I) = 1$, and such that $|h^{(iv)}(X)| \leq M$ for all $X \in I$, for some $M < \infty$, then*

$$E(h(\bar{X})) = h(\mu) + \frac{\sigma^2}{2n} h''(\mu) + R_n,$$

where the reminder R_n is $O\left(\frac{1}{n^2}\right)$, that is there exist n_o and $A < \infty$ such that $R_n(\mu) < \frac{A}{n^2}$ for $n > n_o$ and for all μ.

To utilize the above lemma we define

$$h(X) = \frac{\rho^k e^{bX}}{1 + \rho^k e^{bX}}, \tag{8}$$

where, $X \sim \chi^2_{2r}$, $k \in \{r, n\}$, $b \in \{\frac{1}{2}\left(\frac{1}{\rho} - 1\right), \frac{1}{2}(1 - \rho)\}$. It is easy to see that

$$h^{(iv)}(2r) < \frac{b^4}{16[1 + \rho^r e^{rb}]^5} \times (\rho^r e^{rb} - 11\rho^{2r} e^{2rb} + 11\rho^{3r} e^{3rb} - \rho^{4r} e^{4rb}).$$

By using triangle inequality and Taylor Expansion of e^{rb}, it is easy to show that $|h^{(iv)}(2r)|$ is bounded.
So we can apply Lemma 3.1 to compute $abc(\eta)_{cen.}$, $abc(\eta)_{iid}$ and γ, up to the term of $O\left(\frac{1}{n^2}\right)$.

We have computed γ, for the value $\rho = 0.4$, $n \in \{10, 20, \cdots, 50\}$ and $\frac{r}{n} \in \{0.1, 0.2, \cdots, 1\}$ by using Lemma 3.1. The results are shown Figure 2.

3.2 Simulation

In this part we conducted a simulation study to compute γ and compare the results with the results of the Taylor expansion.

In the Simulation procedure, we derived $n \in \{10, \cdots, 50\}$ random sample from the central chi-square distribution with $2r$ and $2n$ degrees of freedom separately, and computed the expression given by (7) for

ρ	n	$\frac{r}{n}=$	0.1	0.2	0.3	0.4	0.5
0.4	10		0.202523	0.369415	0.508015	0.613638	0.704896
	20		0.305585	0.512608	0.652619	0.757141	0.837543
	30		0.399047	0.624931	0.75342	0.845239	0.898941
	40		0.482428	0.701458	0.830364	0.896938	0.940164
	50		0.544538	0.776487	0.883131	0.935503	0.966505
0.6	10		0.137523	0.265294	0.386540	0.492638	0.594627
	20		0.175876	0.325705	0.445917	0.557875	0.658970
	30		0.211849	0.375825	0.504908	0.616494	0.703653
	40		0.241618	0.419826	0.553572	0.666040	0.743990
	50		0.275646	0.462601	0.601401	0.705270	0.785825
0.8	10		0.109640	0.2130640	0.322066	0.426016	0.523189
	20		0.118806	0.234408	0.340034	0.447215	0.547017
	30		0.126604	0.245715	0.354872	0.462078	0.565139
	40		0.135231	0.258125	0.374564	0.477805	0.581519
	50		0.142545	0.269641	0.387160	0.496448	0.598006

ρ	n	$\frac{r}{n}=$	0.6	0.7	0.8	0.9
0.4	10		0.783827	0.85137	0.913219	0.954743
	20		0.883949	0.925347	0.956161	0.983479
	30		0.937293	0.962504	0.978512	0.992087
	40		0.967337	0.982124	0.991386	0.997352
	50		0.981916	0.991673	0.996402	0.998255
0.6	10		0.686114	0.771759	0.845768	0.927873
	20		0.741411	0.815743	0.896107	0.943600
	30		0.786855	0.853672	0.902141	0.958327
	40		0.822868	0.887156	0.928710	0.965933
	50		0.846718	0.906286	0.939001	0.974190
0.8	10		0.625049	0.724149	0.817888	0.917647
	20		0.642239	0.738213	0.826884	0.910137
	30		0.657518	0.752686	0.83903	0.919547
	40		0.672988	0.760869	0.852922	0.927635
	50		0.689780	0.778254	0.849786	0.936027

Table 1: **Values of γ**

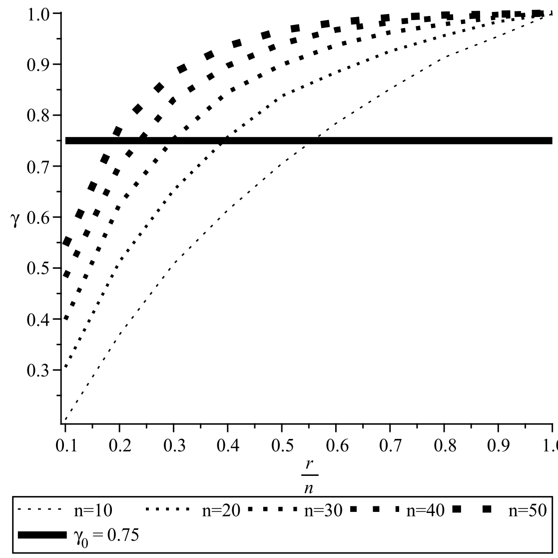

Figure 1: γ **computed by simulation for n=10**, $\cdots$, **50 and** $\rho = 0.4$**.**

different ratios $\frac{r}{n}$, when $\rho = 0.4$. The whole process was repeated 10^4 times. Averaging all, the support measure γ was evaluated through equation (6). The obtained results are shown in Table 1 and Figure 1.

Figure 3, by the way, shows that the optimum value of r that maximizes "per failure expected true evidence" is one. This means that it is better, in terms of statistical evidence generated, to set up n experiment each containing n items and in each observe only the first failure.

3.2.1 How to use Table 1 and figure 1

Suppose we are satisfied with $(100)\gamma_0$ percent of the expected true evidence of the complete sample, Table 1 and Figure 1 can help us find the ratio $\frac{r}{n}$ that would result in the desired percentage. For example if we choose $\gamma_0 = 0.75$, then the horizontal line in Figure 1 would tell us the required ratio $\frac{r}{n}$ for each n. For values of ρ other than those are given in Table 1, one can use the Maple12 program given in the Appendix.

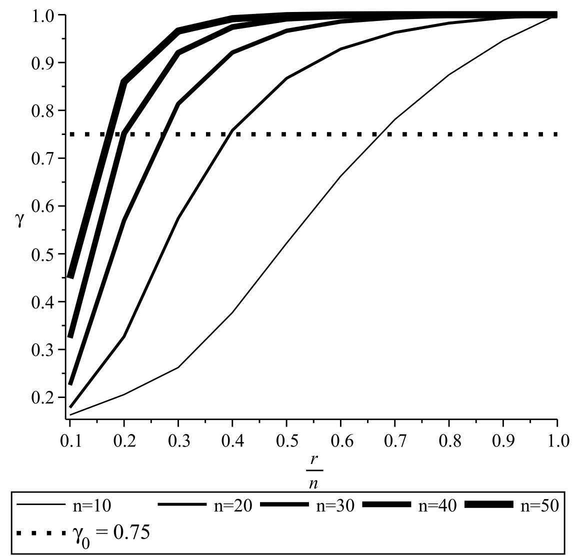

Figure 2: γ **computed from Taylor expansion for n=10**, $\cdots$, **50 and** $\rho = 0.4$**.**

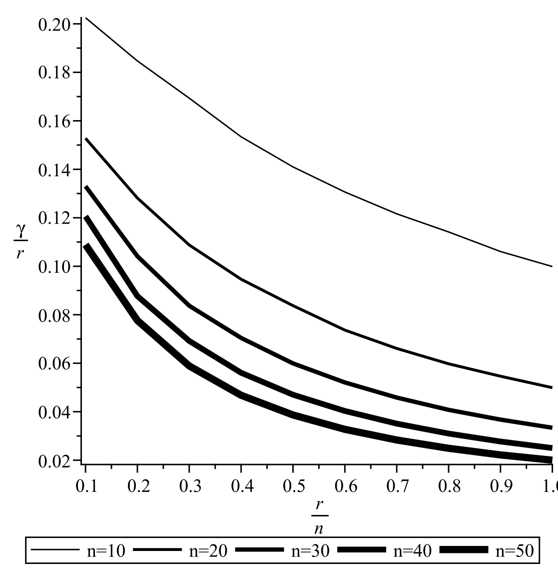

Figure 3: **Optimum value of** r **for n=10**, $\cdots$, **50 and** $\rho = 0.4$**.**

4 Conclusions

The problem of deciding about the stopping point (r) in Type-II censored sampling was considered and found the optimum in the sense that it maximize expected true evidence per observed failure.

For further research, it would be of much interest that one can include the cost factor in the model and propose a method for striking a balance between cost and accuracy.

Acknowledgements

The authors would like to thank the referees and the associate editor for constructive suggestions. We also wish to acknowledge support from the Ordered and Spatial Data Center of Excellence of Ferdowsi University of Mashhad.

References

Arashi, M. and Emadi, M. (2008), Evidential inference based on record data and inter-record times. Statist. Papers, **49**, 291-301.

Arnold, B. C., Balakrishnan, N., and Nagaraja, H. N. (1992), A First Course in Order Statistics. New York: John Wiley.

Balakrishnan, N. and Aggarwala, R. (2000), Progressive Censoring: Theory, Methods and Applications. Boston: Birkhauser.

Blume, J. D. (2002), Likelihood methods for measuring statistical evidence. Statistics in Medicine, **21**, 2563-2599.

Cohen, A. C. (1991), Truncated and Censored Samples: Theory and Applications. New York: Marcel Dekker.

Cohen, A. C. (1995), MLEs under censoring and truncation and inference. The Exponential Distribution: Theory, Methods and Applications, (eds. N. Balakrishnan and A. P. Basu), 33-51, Gordon and Breach, Amsterdam.

Doostparast, M. and Emadi, M. (2006), Statistical evidence methodology for model acceptance based on record values. J. Korean Statist. Soc., **35**, 167-177.

Emadi, M., Ahmadi, J. and Arghami, N. R., (2007), Comparing of record data and random observation based on statistical evidence. Statist. Papers, **48**, 1-21.

Emadi, M. and Arghami, N. R. (2003), Some measures of support for statistical hypotheses. J. Stat. Theory Appl., **2**, 165-176.

Epstein, B. (1954), Truncated life tests in the exponential case. Annals of Mathematical Statistics, **25**, 555-564.

Habibi, A., Arghami, N. R. and Ahmadi, J. (2006), Statistical evidence in experiments and in record values. Comm. Statist.-Theory Meth., **35**, 1971-1983.

Lehmann, E. L. and Casella, G. (1998), Theory of Point Estimation. 2nd Ed., Springer.

Muenz, L. R. and Green, S. B. (1977), Time saving in censored life testing. J. R. Statist. Soc., Ser. B, **39**, 269-275.

Park, S. (1996), Fisher information in order statistics. J. Amer. Statist. Assoc., **91**, 385-390.

Royall, R. (1997), Statistical Evidence: A Likelihood Paradigm. London: Chapman & Hall, New York.

Royall, R. (2000), On the probability of observing misleading statistical evidence. J. Amer. Statist. Assoc., **95**, 760-780.

Zheng, G. and Gastwirth, J. L. (2000), Where is the fisher information in an ordered sample? Statist. Sinica., **10**, 1267-1280.

Zheng, G. and Gastwirth, J. L. (2001), On the Fisher information in randomly censored data. Statist. Probab. Lett., **52**, 421-426.

Zheng, G. and Park, S. (2004), A note on time saving in censored life testing. J. Statist. Plan. Inf., **124**, 289-300.

Zheng, G. and Park, S. (2005), Another look at life testing. J. Statist. Plan. Inf., **127**, 103-117.

Appendix

(Maple Program)
with(Statistics):
randomize();
g:=fopen("D:\ \ $\cdots$ \ \data1.txt",APPEND)
Y:=RandomVariable(Chisquare(2·n));
$\rho := 0.8$;
n:=50;
fprintf(g,"%g%g \n",n,ρ);
for r from $\frac{n}{10}$ by $\frac{n}{10}$ to n do
λ:=unapply$\left(\rho^k e^{\frac{bq}{2}}, k, b, q\right)$;
X:=RandomVariable(Chisquare(2·r));
$\eta c1$:=0;
$\eta c2$:=0;
$\eta i1$:=0;
$\eta i2$:=0;
nn:=10000;
for i to nn do
x:=Sample(X,1);
y:=Sample(Y,1);
$\eta i1 := \eta i1 + \frac{\lambda\left(n,\frac{1}{\rho}-1,y_1\right)}{1+\lambda\left(n,\frac{1}{\rho}-1,y_1\right)}$;
$\eta i2 := \eta i2 + \frac{\lambda(n,1-\rho,y_1)}{1+\lambda(n,1-\rho,y_1)}$;
$\eta c1 := \eta c1 + \frac{\lambda\left(r,\frac{1}{\rho}-1,x_1\right)}{1+\lambda\left(r,\frac{1}{\rho}-1,x_1\right)}$;
$\eta c2 := \eta c2 + \frac{\lambda(r,1-\rho,x_1)}{1+\lambda(r,1-\rho,x_1)}$;
end do;
$\gamma := \frac{\frac{\eta c1}{nn}-\frac{\eta c2}{nn}}{\frac{\eta i1}{nn}-\frac{\eta i2}{nn}}$
fprintf(g,"%g,",γ);
end do;
fclose(g)