

# Comparison of Liao's optimal HAM and Niu's one-step optimal HAM for solving integro-differential equations

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## Abstract

In this paper, the Liao's optimal homotopy analysis method is compared with the Niu's one-step optimal homotopy analysis method for solving one system of linear Volterra integro-differential equations and one integro-differential equation. The results reveal that the Liao's optimal HAM has more accuracy to determine the convergence-control parameter than the one-step optimal HAM suggested by Zhao Niu.

**Mathematics Subject Classification :** 34K28, 45J05

**Keywords:** Integro-Differential Equations, Optimal HAM

## 1 Introduction

In 1992, Liao [5], for the first time, used the concept of homotopy to obtain analytic approximations of a nonlinear equation

$$\mathcal{N}[u(t)] = 0, \quad (1)$$

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by constructing a *one-parameter* family of equations called the zeroth-order deformation equation,

$$(1 - q)\mathcal{L}[\phi(t, q) - u_0(t)] = q\mathcal{N}[\phi(t, q)] \quad (2)$$

where  $q \in [0, 1]$  is an embedding parameter,  $\mathcal{N}$  is a nonlinear operator,  $u(t)$  is an unknown function,  $u_0(t)$  is an initial guess and  $\mathcal{L}$  is a linear operator. At  $q = 0$  and  $q = 1$ , we have  $\phi(t, 0) = u_0(t)$  and  $\phi(t, 1) = u(t)$ , respectively. So, if the Taylor series

$$\phi(t, q) = u_0(t) + \sum_{m=1}^{+\infty} u_m(t)q^m, \quad (3)$$

where

$$u_m(t) = \frac{1}{m!} \left. \frac{\partial^m \phi(t, q)}{\partial q^m} \right|_{q=0}, \quad (4)$$

converges at  $q = 1$ , we have the so-called homotopy-series solution

$$u(t) = u_0(t) + \sum_{m=1}^{+\infty} u_m(t), \quad (5)$$

which must satisfy the governing equation(1). However, Liao [6, 3] found that the above approach breaks down if the Taylor series (3) diverges at  $q = 1$ . So, to overcome this restriction, in 1997, he introduced [6] a nonzerorth auxiliary parameter  $c_0$ , to construct a *two-parameter* family of equations, i.e. the zeroth-order deformation equation:

$$(1 - q)\mathcal{L}[\phi(t, q) - u_0(t)] = c_0q\mathcal{N}(t, q) \quad (6)$$

This auxiliary parameter is also known as a convergence-control parameter [7]. Now, the homotopy-series solution (5) is not only dependent upon the embedding parameter  $q$  but also the convergence-control parameter  $c_0$ . From[3, 6, 8] we find out that the convergence-control parameter  $c_0$ , can provide us a convenient way to ensure the convergence of homotopy series solution and also adjust and control it's convergence region. A simple way of selecting  $c_0$  is to plot the curve of homotopy-series solution's derivatives (" $c_0$ -curves") with respect to  $c_0$  in some points [3, 9, 1]. So, if the homotopy-series solution is unique, all of them converge to the same value and hence there exists a horizontal line segment in it's figure that corresponds to a region of  $c_0$  called the valid region of  $c_0$ . It is a pity that the so-called " $c_0$ -curves" approach can not

give the optimal value of  $c_0$ , to obtain the fastest convergent series. so, during these years, some approaches suggested in order to determine the optimal value of  $c_0$ . In 2008, Marinca et al. [11, 12] introduced the so-called “homotopy asymptotic method” which is similar to the homotopy analysis method. This approach is based on a homotopy equation

$$[1 - q]\mathcal{L}[\phi(t, q) + g(t)] = H(t, q)\mathcal{N}[\phi(t, q)], \quad (7)$$

where

$$H(t, q) = qc_1 + q^2c_2 + \dots + q^m c_m(t), \quad (8)$$

is a nonzero auxiliary function for  $q \neq 0$ , and  $H(0, t) = 0$ .  $c_i, i = 0..m - 1$  are auxiliary constants and  $c_m(t)$  is a function of  $t$  and  $u(t)$  is an unknown function. Choosing the control-parameter  $c_m(t)$  depends on the given problem. for example, in [10, 12] he used  $H(q) = qc_1 + q^2c_2 + q^3c_3$ , and in [11] he assumed that  $H(q, t) = qc_1 + q^2(c_2 + c_3e^{-2t})$ . In Marinca’s approach, at the  $m$ th order of approximation, a set of nonlinear algebraic equations about  $c_1, c_2, \dots, c_m$  must be solved in order to find their optimal values and this leads to have a time-consuming approach. However, too many unknowns parameters greatly increase the cpu times and thus make the approach time-consuming. In 2009, to overcome this disadvantage, Zhao Niu [15] and Shijun Liao [4] suggested two new approaches. In this paper, we apply these approaches to solve one system of linear Volterra integro-differential equations and one integro-differential equation, and then, compare the obtained solutions.

## 2 Preliminary Notes

### 2.1 A one-step optimal HAM

Consider the equation (1),

$$\mathcal{N}[u(t)] = 0,$$

Niu et al. [15] employed the Liao’s [8] zeroth-order deformation equation

$$(1 - B(q))\mathcal{L}[\phi(t, q) - u_0(t)] = c_0A(q)\mathcal{N}[\phi(t, q)] \quad (9)$$

where  $A(q)$  and  $B(q)$  are the deformation functions satisfying

$$A(0) = B(q) = 0, \quad A(1) = B(1) = 1.$$

The Taylor series of  $A(q)$  and  $B(q)$  read

$$A(q) = \sum_{m=1}^{+\infty} \mu_m q^m, \quad B(q) = \sum_{m=1}^{+\infty} \sigma_m q^m,$$

which are convergent for  $|q| \leq 1$ . He set  $H(q) = c_0 A(q)$  and  $B(q) = q$  in this (8) equation to construct the zeroth-order deformation equation

$$(1 - q)\mathcal{L}[\phi(t, q) - u_0(t)] = H(q)\mathcal{N}[\phi(t, q)] \quad (10)$$

where  $\mathcal{L}$  is an auxiliary linear operator,  $q \in [0, 1]$  is the embedding parameter,  $u_0(t)$  an initial approximation of  $u(t)$ , and  $H(q)$  is the convergence-control function satisfying  $H(0) = 0$  and  $H(1) \neq 0$ . Like Liao [6]-[8], defining the vector  $\vec{u}_m = \{u_0(t), u_1(t), \dots, u_m(t)\}$ , differentiating the equation (10)  $m$  times with respect to the embedding parameter  $q$ , then divide it by  $m!$  and finally set  $q = 0$ , we have the so-called  $m$ th-order deformation equation

$$\mathcal{L}[u_m(t) - \chi_m u_{m-1}(t)] = \sum_{k=1}^m c_k R_{m-k}(t) \quad (11)$$

where

$$R_n(t) = \frac{1}{n!} \frac{\partial^n \mathcal{N}[\phi(t, q)]}{\partial q^n} \quad (12)$$

and

$$\chi_m = \begin{cases} 0, & \text{if } m \leq 1, \\ 1, & \text{if } m > 1. \end{cases} \quad (13)$$

Let  $\Delta_n(c_n)$  denote the square residual error of the governing equation (1) and express as

$$\Delta_n(c_n) = \int_{\Omega} (\mathcal{N}[\tilde{u}_n(t)])^2 d\Omega, \quad (14)$$

where

$$\vec{c}_m = c_1, c_2, \dots, c_m,$$

and

$$\tilde{u}_m(t) = u_0(t) + \sum_{k=1}^m u_k(t).$$

At the 1st-order of approximation,  $\Delta_1$  is only dependent upon  $c_1$ , so, the optimal value of  $c_1$  is obtain by solving the nonlinear algebraic equation

$$\frac{d\Delta_1}{dc_1} = 0.$$

At the 2nd-order, since  $c_1$  is known, the square residual error  $\Delta_2$  is only dependent upon  $c_2$ , thus we can gain the optimal value of  $c_2$  by solving the nonlinear algebraic equation

$$\frac{d\Delta_2}{dc_2} = 0.$$

and so on.

## 2.2 Liao's optimal HAM

This approach is based on the zeroth-order deformation equation (9) and used the *one-parameter* deformation functions

$$A_1(q, c_1) = \sum_{m=1}^{+\infty} \mu_m(c_1)q^m, \quad B_1(q, c_2) = \sum_{m=1}^{+\infty} \sigma_m(c_2)q^m,$$

where  $|c_1| < 1$  and  $|c_2| < 1$  are constants, and

$$\begin{aligned} \mu_1 &= (1 - c_1), & \mu_m &= (1 - c_1)c_1^{m-1}, & m > 1, \\ \sigma_1 &= (1 - c_2), & \sigma_m &= (1 - c_2)c_2^{m-1}, & m > 1. \end{aligned} \quad (15)$$

So, the new zeroth-order deformation equation is

$$[1 - B_1(q, c_2)]\mathcal{L}[\phi(t, q) - u_0(t)] = c_0 A_1(q, c_1)\mathcal{N}[\phi(t, q)], \quad q \in [0, 1]. \quad (16)$$

Note that  $c_1$  and  $c_2$  are the convergence-control parameters. Like the previous approach, differentiating the equation (16)  $m$  times with respect to the embedding parameter  $q$ , then dividing it by  $m!$  and finally setting  $q = 0$ , we have the  $m$ th-order deformation equation

$$\mathcal{L} \left[ u_m(t) - \sum_{k=1}^{m-1} \sigma_{m-k}(c_2)u_k(t) \right] = c_0 \sum_{k=0}^{m-1} \mu_{m-k}(c_1)\delta_k(t).$$

Liao [4] found out that, when we use more than one unknown parameters, the cpu time increases exponentially so that the square residual error  $\Delta_n$  (14) is

often inefficient in practice. So, to decrease the cpu time, he introduced the so-called average residual error

$$E_m = \frac{1}{K} \sum_{j=0}^K \left[ \mathcal{N} \left( \sum_{k=0}^m u_k(j\Delta x) \right) \right]^2, \quad (17)$$

where  $\Delta x = \frac{1}{K}$ . Now we solve some examples and then we will compare the obtained results.

### 3 Applications

**Example 3.1.** Consider the following system of linear Voltra integro-differential equations [16]

$$\begin{cases} u_1' = 1 + t + t^2 - u_2(t) - \int_0^t (u_1(s) + u_2(s)) ds, & u_1(0) = 1, \\ u_2' = -1 - t + u_1(t) - \int_0^t (u_1(s) - u_2(s)) ds, & u_2(0) = -1, \end{cases} \quad (18)$$

with exact solutions  $u_1(t) = t + \exp(t)$  and  $u_2(t) = t - \exp(t)$ . We assume that the solution of above system can be expressed by a set of base function  $\{\exp(nt) | n \geq 0\}$  in the form

$$u(t) = \sum_{i=0}^{+\infty} a_i t^i \quad (19)$$

where  $a_i$  are coefficients to be determined. We choose auxiliary linear operator

$$\mathcal{L}[\phi_i(t, q)] = \frac{\partial \phi_i(t, q)}{\partial t},$$

with the property

$$\mathcal{L}[b_i] = 0,$$

where  $b_i$  are constant. From (18) we define the nonlinear operators

$$\begin{aligned} \mathcal{N}_1[\phi_1(t, q)] &= \frac{\partial \phi_1(t, q)}{\partial t} - (1 + t + t^2) + \phi_2(t, q) + \int_0^t (\phi_1(s, q) + \phi_2(s, q)) ds \\ \mathcal{N}_2[\phi_2(t, q)] &= \frac{\partial \phi_2(t, q)}{\partial t} + (1 + t) - \phi_1(t, q) + \int_0^t (\phi_1(s, q) - \phi_2(s, q)) ds \end{aligned}$$

according to the boundary conditions (18) and Eq. (19), the initial approximations should be in the form  $u_{1,0}(t) = \exp(t)$ ,  $u_{2,0}(t) = -\exp(t)$ . From (12) and (18), we have

$$\begin{aligned} \mathcal{R}_{1,n}(t) &= u'_{1,n-1} - (1 - \chi_n)(1 + t + t^2) + u_{2,n-1} + \int_0^t (u_{1,n-1} + u_{2,n-1}) ds \\ \mathcal{R}_{2,n}(t) &= u'_{2,n-1} + (1 - \chi_n)(1 + t) - u_{1,n-1} + \int_0^t (u_{1,n-1} - u_{2,n-1}) ds \end{aligned}$$

where the prime denotes differentiation with respect to the  $t$ . Now, the solution of the  $m$ th order deformation equation (11) becomes

$$u_{i,m}(t) = \chi_m u_{i,m-1}(t) + \int c_k \mathcal{R}_{i,m-k}(t) dt + b_i, \quad m \geq 1, i = 1, 2,$$

where the constants  $b_i$  are determined by the initial conditions

$$u_{1,m}(0) = 0, u_{2,m}(0) = 0.$$

So, the first several approximations can be obtained as follows :

$$\begin{aligned} \tilde{u}_{1,1}(t) &= \exp(t) - c_1(t + 0.5t^2 + 0.3333t^3), \\ \tilde{u}_{2,1}(t) &= -\exp(t) - c_1(t - 0.5t^2), \\ \tilde{u}_{1,2}(t) &= \exp(t) + 0.912522t + 0.208289t^2 + 0.221517t^3 - \\ &\quad 0.008266t^5 - c_2(t + 0.5t^2 + 0.3333t^3), \\ \tilde{u}_{1,2}(t) &= -\exp(t) + 0.912522t - 0.208289t^2 + 0.082667t^3 - 0.008266t^5 - c_2(t - 0.5t^2). \end{aligned}$$

It is found that

$$\begin{aligned} \Delta_{1,1}(t) &= \frac{37}{10} + \frac{3287}{315}c_1 + \frac{336059}{45360}c_1^2, \\ \Delta_{2,1}(t) &= \frac{1}{3} + \frac{23}{90}c_1 + \frac{4687}{9072}c_1^2, \\ \Delta_{1,2}(t) &= 0.104904 - 1.617464c_2 + 7.408708c_2^2, \\ \Delta_{2,2}(t) &= 0.078324 - 0.225431c_2 + 0.516645c_2^2. \end{aligned}$$

and so on.

At the 1th-order of approximation, in order to determine the optimal value of  $c_1$ , each of the equations in (18) is solved separately. So, the obtained values and corresponding square residual errors are,

$$c_1 = -0.704233, \quad \Delta_{1,1}(t) = 0.0256897, \quad \Delta_{2,1}(t) = 0.409589,$$

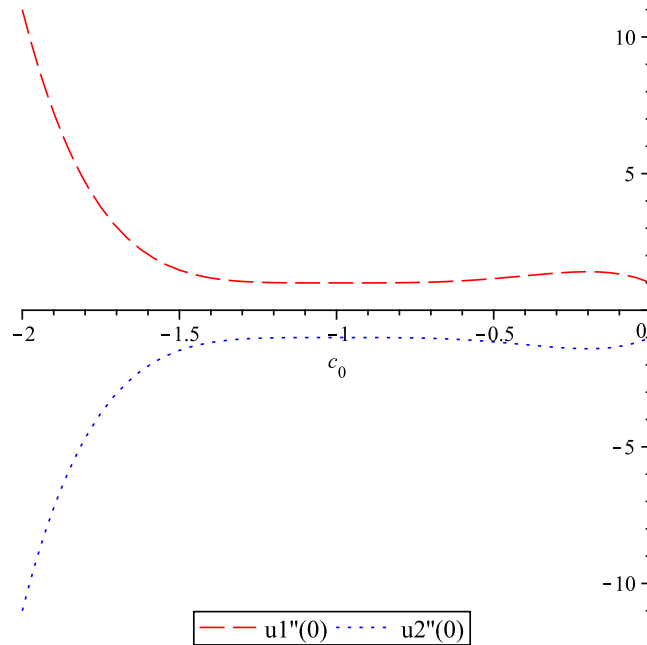


Figure 1:  $c_0$ -curve of 4th-order approximation for Example 3.1

for the first equation, and

$$c_1 = -0.247322, \quad \Delta_{1,2}(t) = 1.572389, \quad \Delta_{2,2}(t) = 0.301731,$$

for the second one. So, the minimum of the  $\Delta_{1,2}$  and  $\Delta_{2,2}$  is correspond to the optimal value of  $c_1$ . Thus,  $c_1 = -0.704233$  is chosen. This procedure lead to the best approximate solution of the system. The 4th-order of approximate solution is obtained as follows

$$\begin{aligned} \tilde{u}_{1,4}(t) &= \exp(t) + 0.907434t + 0.082059t^2 + 0.958009e - 1t^3 - 0.104494e - 1t^4 - \\ &\quad 0.103389e - 1t^5 + 0.6966315e - 3t^6 + 0.394245e - 3t^7 - 0.271121e - 5t^9, \\ \tilde{u}_{2,4}(t) &= -\exp(t) + 0.907434t - 0.082059t^2 + 0.4109446e - 1t^3 + 0.104494e - 1t^4 - \\ &\quad 0.615912e - 2t^5 - 0.6966315e - 3t^6 + 0.195207e - 3t^7 - 0.271121e - 5t^9, \end{aligned}$$

Now, we apply the Liao's optimal HAM. First consider Figure 1, the  $c_0$ -curve of the 4th-order approximation of Example 3.1. It is obvious that the best region of  $c_0$  is  $-1.5 \leq c_0 \leq -0.5$ . Note that in this approach we have at most three unknown convergence-control parameter  $c_0, c_1, c_2$ . Now we compare the different cases of these parameters.



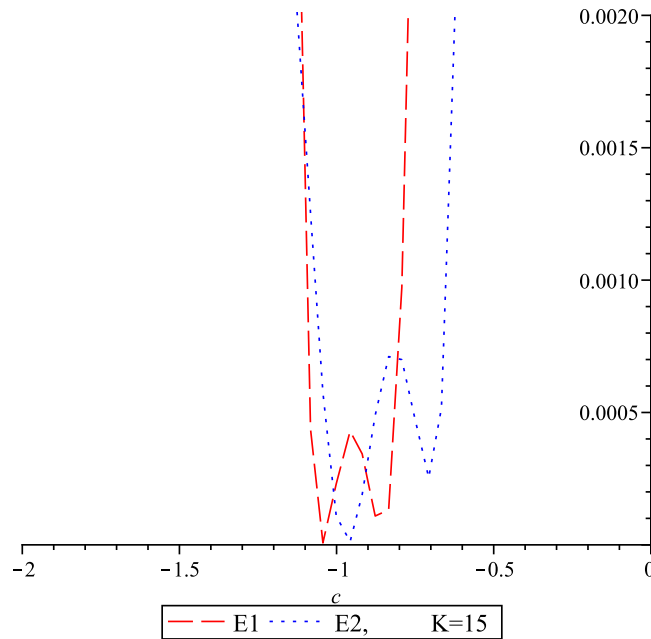


Figure 2: Residual of 4th-order approximation of Example 3.1

*Optimal  $c_0$  in case of  $c_1 = c_2 = 0$  :*

This case, is the traditional HAM which was used by Liao (5). In case of  $c_1 = c_2 = 0$  there is only one unknown convergence-control parameter  $c_0$ , thus, the optimal value of  $c_0$  is found by the minimum of  $E_4$ , with  $K = 15$ , as shown in Table 2. Figure 2 is shown the residual of the 4th-order approximation of this system of equations. The 4th-order of approximate solution through this approach is obtained as follows

$$\begin{aligned} \tilde{u}_{1,4}(t) &= \exp(t) + 0.999994t + 0.227256e - 3t^2 + 0.234468e - 2t^3 - 0.705523e - 2t^4 - \\ &\quad 0.985020e - 2t^5 + 0.470348e - 3t^6 + 0.782842e - 3t^7 - 0.900634e - 5t^9, \\ \tilde{u}_{2,4}(t) &= -\exp(t) + 0.999994t - 0.227256e - 3t^2 + 0.2193178e - 2t^3 + 0.104494e - 1t^4 - \\ &\quad 0.702811e - 2t^5 - 0.470348e - 3t^6 + 0.648456e - 3t^7 - 0.900634e - 5t^9, . \end{aligned}$$

Tables 3, 4 shows the fourth approximations of the solutions of system (18) via Liao’s optimal approach in two ways and it’s comparison with the one-step optimal HAM, HVIM [16] and the exact solutions of (18).

Table 1: Square residual error for Example 3.1

order M	$c_n$	$\Delta_1$	$\Delta_2$	cpu(s)
1	-0.7042	$0.2569e-1$	0.4096	0.094
2	0.1092	$0.1663e-1$	$0.5987e-1$	0.094
3	$0.3667e-1$	$0.5691e-2$	$0.2925e-1$	0.094
4	$0.2266e-1$	$0.3091e-2$	$0.1343e-1$	1.172

Table 2: Average residual error for Example 3.1

order M	$c_0$	$E_1$	$E_2$	cpu(s)
1	-0.7018	$0.2925e-1$	0.4357	0.125
2	-0.9180	0.1346	$0.2308e-3$	0.141
3	-0.9326	$0.1217e-3$	$0.5986e-2$	0.296
4	-0.9326	$0.5008e-3$	$0.9106e-6$	1.420

The first way of calculation of  $c_0$  is exactly the same as we determine  $c_1$  in one-step optimal approach. But here, we introduce a good way to determine  $c_0$ . In this way, at each order of approximation, we solve  $E_1 - E_2 = 0$ , then, put the solutions in both  $E_1$  and  $E_2$ . One that gives less residual, is the optimal one. The approximate solutions are denote by “\*” in Tables 3, 4.

This way, gives better approximations of the solutions than the first way as shown in Tables 3, 4. Besides, through these Tables we find that the one-step optimal HAM is not suitable approach to determine convergence-control parameters for system of equations.

*Optimal  $c_1 = c_2$  in case of  $c_0 = -1$  :*

Here, we assumed that  $c_0 = -1$  and investigate the optimal value of  $c_1 = c_2$ . In this case, at the 4th-order of approximation, we obtained  $E_1 = 0.621164e-3$ ,  $E_2 = 0.574729e-2$  in 3.869s which is not better than the corresponding order's residuals in Table 2. Using more than one convergence-control parameter is not suggested for solving system of equations, because, it becomes more and more complicated to determine the optimal values.

Table 3: Comparison of  $u_1$  given by OHAM, one-step optimal, HVIM, exact for Example 3.1

$t_i$	$u_{1,4}$ OHAM	$u_{1,4}$ one step optimal	$u_{1,4}$ HVIM	$u_{1,4}$ exact	$u_{1,4}^*$ OHAM*
0	1	1	1	1	1
0.2	1.421415	1.406918	1.421400	1.421402	1.421403
0.4	1.891730	1.873689	1.891734	1.891824	1.891793
0.6	2.421067	2.414699	2.421423	2.422119	2.421754
0.8	3.021051	3.045654	3.022583	3.025541	3.023801
1	3.705187	3.783876	3.709226	3.718281	3.712749

Table 4: Comparison of  $u_2$  given by OHAM, one-step optimal, HVIM, exact for Example 3.1

$t_i$	$u_{2,4}$ OHAM	$u_{2,4}$ one step	$u_{2,4}$ HVIM	$u_{2,4}$ exact	$u_{2,4}^*$ OHAM*
0	-1	-1	-1	-1	-1
0.2	-1.021386	-1.042854	-1.021400	-1.021402	-1.021407
0.4	-1.091615	-1.139148	-1.0917347	-1.091825	-1.091948
0.6	-1.221366	-1.297474	-1.221423	-1.222119	-1.222963
0.8	-1.423969	-1.528951	-1.422583	-1.425541	-1.428827
1	-1.716126	-1.848026	-1.709226	-1.718281	-1.727631

**Example 3.2.** Consider the nonlinear integro-differential equation [2]

$$u'(t) = -1 + \int_0^t u^2(s)ds, \tag{20}$$

for  $t \in [0, 1]$  with the boundary condition  $u(0) = 0$ .

Consider high-order deformation equation (11) subject to the  $u_m(0) = 0$ , where

$$\mathcal{R}_n(t) = u'_n + (1 - \chi_{n+1}) - \int_0^t \sum_{j=0}^n u_j(s)u_{n-j}(s)ds.$$

The initial approximation should be in the form  $u_0(t) = -t$ .

So, the 2th-order of approximate solution via one-step optimal HAM is

obtained as follows

$$\tilde{u}_2(t) = -t + 0.083234t^4 - 0.003699t^7.$$

and

$$\tilde{u}_2(t) = -t + 0.083039t^4 - 0.003548t^7,$$

is the 2th-order of approximation which is obtained by OHAM. The comparison between residuals via two approaches is shown in Table 5.

Table 5: Comparison between residuals, via two approaches for Example 3.2

order M	$c_n$	$\Delta_n$	cpu(s)	$c_0$	$E$	cpu(s)
1	-0.945677	$0.447006e - 5$	0.062	-0.942067	$0.588916e - 5$	0.032
2	$0.571303e - 3$	$0.718753e - 8$	0.062	-0.965568	$0.182630e - 8$	0.109
3	$0.260778e - 4$	$0.219308e - 10$	0.062	-0.971777	$0.247549e - 12$	0.281
4	0.147745	$0.872873e - 13$	0.062	-0.973225	$0.671736e - 16$	1.404

*Optimal  $c_0$  in case of  $c_1 = c_2 = 0$  :*

The optimal values of  $c_0$  are shown as Table 5. It is obvious that, in this example, the optimal HAM when  $c_1 = c_2 = 0$  has more accuracy than the one-step HAM.

Other cases of these convergence-control parameters is not suggested, because, the obtained residuals are not better than this case (Optimal  $c_0$  in case of  $c_1 = c_2 = 0$ ). For example, in case of  $c_0 = -1, c_1 = 0.002572, c_2 = 0$  we have  $E = 0.349442e - 7$ .

## 4 Conclusion

In this article, first, we have described Liao's optimal analysis method and Niu's one-step optimal analysis method, then we have applied these methods for solving one system of linear Volterra integro-differential equations and one

integro-differential equation. In order to illustrate the differences between these methods, we solve two examples. The results compared show that, in both examples, Liao's approach gives better approximations than the Niu's approach. In example 1, we have introduced a simple way to determine the optimal value of convergence-control parameter in system of equations.

**ACKNOWLEDGEMENTS.** The authors would like to thank the referees for their valuable suggestions, which greatly improved the paper.

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