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A Novel Computational Approach to Solve Nonlinear Boundary Value Problems Using Extended Modal Series Method

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Abstract: In this paper, a new approach is proposed to solve nonlinear boundary value problems (BVPs). In this approach, the original nonlinear BVP transforms into a sequence of linear BVPs. Solving the proposed linear BVP sequence in a recursive manner leads to the exact solution of original problem in the form of uniformly convergent series. Hence, to obtain the exact solution, only the techniques of solving linear ordinary differential equations are employed. This confirms that the proposed method is straightforward and easy to implement. Besides, we present an efficient algorithm with low computational complexity and fast convergence rate. Through the finite iterations of the algorithm, an approximate closed-form solution is obtained for the nonlinear BVP. Finally, a numerical example is employed to demonstrate efficiency, simplicity, and high accuracy of the proposed method.

Keywords: Nonlinear boundary value problem, extended Modal series method, approximate closed-form solution.

1. Introduction

Boundary value problems (BVPs) occur frequently in engineering, applied mathematics, physics, economy, etc. However, it is generally impossible to obtain the analytic closed-form solutions for BVPs, especially nonlinear ones. Therefore, these problems are attacked by various approximate methods. However, it is often difficult to obtain an approximate closed-form solution than a numerical one for a given nonlinear BVP. Research works are still carried out in this direction, and some new methods have been used for problem solver.

A well-known method for solving nonlinear BVPs is perturbation method [1]. However, this method depends on the existence of small/large physical parameters in system model. To handle nonlinear BVPs, there are some non-perturbation methods such as Adomian's decomposition method (ADM) [2,3], homotopy perturbation method (HPM) [4-7], homotopy analysis method (HAM) [8,9], etc. However, these methods have also their own limitations. For example, the ADM can not ensure the convergence of solution series. In addition, calculating the so-called Adomian's Polynomials is a very intricate problem. Also, the HPM and HAM usually require providing an initial guess for the solution of nonlinear BVP. Moreover, convergence rate in these methods depends on choosing the linear and nonlinear operators.

The Modal series method, which is recently developed in the nonlinear system analysis [10-15], was initially introduced in [10]. It provides the solution of autonomous nonlinear systems in terms of fundamental and interacting modes. This solution, which is called Modal series, yields a good deal of physical insight into the system behavior. In contrast with the perturbation method [1], the modal series method does not depend on the small/large physical parameters in system model. In addition, unlike the other non-perturbation methods such ADM [2,3], HPM [4-7], and HAM [8,9], the solution series obtained via the Modal series method converges uniformly to the exact solution.

In this paper, we extend the Modal series method for solving nonlinear BVPs. By this extension, the exact solution of nonlinear BVP is determined in the form of uniformly convergent series with easy-computable terms. The proposed method only requires solving a sequence of linear BVPs in a recursive manner. In addition, uniformly convergence of the obtained solution series is guaranteed. Therefore, in view of computational complexity, the proposed method is more practical than the other approximate methods. The rest of paper is organized as follows. The problem statement is described in section 2. Section 3 explains how to extend the Modal series method for solving nonlinear BVPs. Section 4 presents an efficient algorithm with low computational complexity and fast convergence rate. Through the finite iterations of the algorithm, an approximate closed-form solution is obtained for the nonlinear BVP. In section 5, effectiveness of the proposed method is verified using a numerical example, and a comparison is made with the existing results. Finally, conclusions are given in the last section.

2. Statement of Problem

Consider the following nonlinear ordinary differential equation (ODE):

$$M_D\left(y(t)\right)=N_D\left(f(y(t))\right),\ a<t<b \tag{1}$$

where $f(\cdot)$ is an analytic function on $[a,b]$ with $f(0)=0$, M_D and N_D are the following differential operators:

$$M_D=D^n+a_1(t)D^{n-1}+a_2(t)D^{n-2}+\cdots+a_n(t) \tag{2}$$

$$N_D=b_0(t)D^n+b_1(t)D^{n-1}+b_2(t)D^{n-2}+\cdots+b_n(t) \tag{3}$$

in which D denotes the derivative with respect to the time, and $a_i(t):i=1,2,\ldots,n$ and $b_j(t):j=0,1,\ldots,n$ are known continuous functions. In order to define the boundary conditions for the aforementioned equation, let n be an even natural number and consider the following boundary conditions for equation (1):

$$\begin{cases} y(a)=\alpha_0\ ,\ y(b)=\beta_0 \\ \left.\dfrac{d^2y(t)}{dt^2}\right|_{t=a}=\alpha_2\ ,\ \left.\dfrac{d^2y(t)}{dt^2}\right|_{t=b}=\beta_2 \\ \vdots \\ \left.\dfrac{d^{n-2}y(t)}{dt^{n-2}}\right|_{t=a}=\alpha_{n-2}\ ,\ \left.\dfrac{d^{n-2}y(t)}{dt^{n-2}}\right|_{t=b}=\beta_{n-2} \end{cases} \tag{4}$$

where α_i and β_j for $i,j=0,2,\ldots,n-2$ are real constants.

In general, it is extremely difficult to solve analytically the nonlinear BVP (1) with boundary conditions (4) except for a few simple cases. In order to overcome this difficulty, in the next section, we will extend the Modal series method.

3. Extending the Modal Series Method to Solve Nonlinear BVPs

In this section, we explain how to extend the Modal series method for solving nonlinear BVP (1) with boundary conditions (4). To do so, we present the following theorem:

Theorem 3.1. The solution of nonlinear BVP (1) with boundary conditions (4) can be expressed as:

$$y(t)=\sum_{i=1}^{\infty}y_i(t) \tag{5}$$

where the i^{th} order term $y_i(t)$ for $i\geq 1$ is achieved by solving recursively the following sequence of linear BVPs:

$$\begin{aligned} &M_D\left(y_1(t)\right)=N_D\left(\lambda_1 y_1(t)\right)\\ &M_D\left(y_2(t)\right)=N_D\left(\lambda_1 y_2(t)+\lambda_2 y_1^2(t)\right)\\ &M_D\left(y_3(t)\right)=N_D\left(\lambda_1 y_3(t)+2\lambda_2 y_1(t)y_2(t)+\lambda_3 y_1^3(t)\right)\\ &\quad\vdots \end{aligned} \tag{6}$$

with boundary conditions:

$$\begin{cases} y_1(a)=\alpha_0\ ,\ y_1(b)=\beta_0 \\ \left.\dfrac{d^2y_1(t)}{dt^2}\right|_{t=a}=\alpha_2\ ,\ \left.\dfrac{d^2y_1(t)}{dt^2}\right|_{t=b}=\beta_2 \\ \vdots \\ \left.\dfrac{d^{n-2}y_1(t)}{dt^{n-2}}\right|_{t=a}=\alpha_{n-2}\ ,\ \left.\dfrac{d^{n-2}y_1(t)}{dt^{n-2}}\right|_{t=b}=\beta_{n-2} \end{cases} \tag{7a}$$

$$\begin{cases} y_i(a)=y_i(b)=0 \\ \left.\dfrac{d^2y_i(t)}{dt^2}\right|_{t=a}=\left.\dfrac{d^2y_i(t)}{dt^2}\right|_{t=b}=0 \\ \vdots \\ \left.\dfrac{d^{n-2}y_i(t)}{dt^{n-2}}\right|_{t=a}=\left.\dfrac{d^{n-2}y_i(t)}{dt^{n-2}}\right|_{t=b}=0 \end{cases}\ ,\ i\geq 2 \tag{7b}$$

where $\lambda_j=\dfrac{1}{j!}\left.\dfrac{d^jf(y)}{dy^j}\right|_{y=0}$.

Proof. The solution of nonlinear BVP (1) with boundary conditions (4) for arbitrary $y_b=(\alpha_0,\alpha_2,...,\alpha_{n-2},\beta_0,\beta_2,...,\beta_{n-2})$ and for all $t\in[a,b]$ can be expressed as:

$$y(t)=\Lambda(y_b,t) \tag{8}$$

where $\Lambda:R^n\times[a,b]\to R$ is an analytic function since nonlinear function in (1) is assumed to be analytic [16]. In addition, it is easy to show that $\Lambda(0,t)=0,\forall t\in[a,b]$. Since Λ is an analytic function and y_b is assumed to be arbitrary, we can expand (8) as Maclaurin series with respect to y_b which yields:

$$\begin{aligned} y(t)=\Lambda(y_b,t)&=\underbrace{\sum_{i=1}^{n}\left.\frac{\partial\Lambda(y_b,t)}{\partial y_{b,i}}\right|_{y_b=0}y_{b,i}}_{y_1(t)}\\ &+\underbrace{\frac{1}{2!}\sum_{k=1}^{n}\sum_{l=1}^{n}\left.\frac{\partial^2\Lambda(y_b,t)}{\partial y_{b,k}\partial y_{b,l}}\right|_{y_b=0}y_{b,k}y_{b,l}}_{y_2(t)}\\ &+\underbrace{\frac{1}{3!}\sum_{p=1}^{n}\sum_{q=1}^{n}\sum_{r=1}^{n}\left.\frac{\partial^3\Lambda(y_b,t)}{\partial y_{b,p}\partial y_{b,q}\partial y_{b,r}}\right|_{y_b=0}y_{b,p}y_{b,q}y_{b,r}}_{y_3(t)}\\ &+\cdots\\ &=y_1(t)+y_2(t)+y_3(t)+\cdots \end{aligned} \tag{9}$$

where $y_{b,j}$ is the j^{th} element of vector y_b, and $y_i(t)$ is linear combination of all terms depending on the multiplication of i elements of vector y_b. For example, $y_2(t)$ contains linear combination of all terms of the form $y_{b,k}y_{b,l}$ for $k,l \in \{1,2,...,n\}$. Moreover, since Λ is an analytic function, existence and uniformly convergence of the Maclaurin series in (9) are guaranteed. Now, let the boundary conditions be εy_b where ε is an arbitrary scalar parameter. This parameter only simplifies the calculations and its value does not have any significance. Similarly to (9), we can write:

$$y_\varepsilon(t) = \Lambda(\varepsilon y_b, t) = \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \varepsilon^3 y_3(t) + \cdots \tag{10}$$

Satisfying (10) in (1) results in:

$$M_D(y_\varepsilon(t)) = N_D(f(y_\varepsilon(t))) \tag{11}$$

Substituting the Maclaurin series of f in (11), and noting that $f(0)=0$, we obtain:

$$M_D(y_\varepsilon(t)) = N_D\left(\lambda_1 y_\varepsilon(t) + \lambda_2 y_\varepsilon^2(t) + \lambda_3 y_\varepsilon^3(t) + \cdots\right) \tag{12}$$

where $\lambda_j = \frac{1}{j!}\frac{d^j f(y)}{dy^j}\bigg|_{y=0}$. Rearranging (12) with respect to the order of ε yields:

$$\begin{aligned}
&\varepsilon M_D(y_1(t)) + \varepsilon^2 M_D(y_2(t)) + \varepsilon^3 M_D(y_3(t)) + \cdots \\
&\quad = \varepsilon N_D(\lambda_1 y_1(t)) \\
&\quad + \varepsilon^2 N_D\left(\lambda_1 y_2(t) + \lambda_2 y_1^2(t)\right) \\
&\quad + \varepsilon^3 N_D\left(\lambda_1 y_3(t) + 2\lambda_2 y_1(t) y_2(t) + \lambda_3 y_1^3(t)\right) \\
&\quad + \cdots
\end{aligned} \tag{13}$$

Since (13) must hold for any ε, terms with the same order of ε on each side must be equal. This procedure yields:

$$\varepsilon^1 : M_D(y_1(t)) = N_D(\lambda_1 y_1(t)) \tag{14a}$$

$$\varepsilon^2 : M_D(y_2(t)) = N_D\left(\lambda_1 y_2(t) + \lambda_2 y_1^2(t)\right) \tag{14b}$$

$$\begin{aligned}
&\varepsilon^3 : \\
&M_D(y_3(t)) = N_D\left(\lambda_1 y_3(t) + 2\lambda_2 y_1(t) y_2(t) + \lambda_3 y_1^3(t)\right) \\
&\qquad \vdots
\end{aligned} \tag{14c}$$

and so on. Note that (14a) is a homogeneous linear ODE. From (14a), $y_1(t)$ can be easily obtained. Assume that $y_1(t)$ has been obtained from (14a) in the first step. Then, $y_2(t)$ can be easily obtained from (14b) in the second step since (14b) is a nonhomogeneous linear ODE. Notice that, the nonhomogeneous term of (14b) is calculated using the solution of (14a). Continuing as above, $y_i(t)$ for $i \ge 2$ can be easily obtained in the i^{th} step only by solving a nonhomogeneous linear ODE. In addition, at each step nonhomogeneous term is calculated using the information obtained from previous steps. Therefore, solving the presented sequence is a recursive process.

To obtain the boundary conditions for the above-mentioned sequence, set $t=a$ and $t=b$ in (10) and its derivatives with respect to the time. Equating the coefficients of ε^i on each side of the resulted equations yields:

$$\varepsilon : \begin{cases} y_1(a) = \alpha_0 \ , \ y_1(b) = \beta_0 \\ \left.\frac{d^2 y_1(t)}{dt^2}\right|_{t=a} = \alpha_2 \ , \ \left.\frac{d^2 y_1(t)}{dt^2}\right|_{t=b} = \beta_2 \\ \vdots \\ \left.\frac{d^{n-2} y_1(t)}{dt^{n-2}}\right|_{t=a} = \alpha_{n-2} \ , \ \left.\frac{d^{n-2} y_1(t)}{dt^{n-2}}\right|_{t=b} = \beta_{n-2} \end{cases} \tag{15a}$$

$$\varepsilon^i : \begin{cases} y_i(a) = y_i(b) = 0 \\ \left.\frac{d^2 y_i(t)}{dt^2}\right|_{t=a} = \left.\frac{d^2 y_i(t)}{dt^2}\right|_{t=b} = 0 \\ \vdots \\ \left.\frac{d^{n-2} y_i(t)}{dt^{n-2}}\right|_{t=a} = \left.\frac{d^{n-2} y_i(t)}{dt^{n-2}}\right|_{t=b} = 0 \end{cases} , \ i \ge 2 \tag{15b}$$

Thus, the proof is complete.

Remark 3.1. Let the solution of nonlinear BVP (1) with boundary condition y_b be available as in (5). To find the solution for any other boundary condition of the form εy_b, there is no need to repeat the recursive process. The solution can be easily obtained as:

$$y(t) = \sum_{i=1}^{\infty} \varepsilon^i y_i(t) \tag{16}$$

4. Approximate Closed-Form Solution for Nonlinear BVPs

In fact, it is almost impossible to obtain the solution of nonlinear BVPs as in (5) since (5) contains infinite series. Therefore, in practice, by replacing ∞ with a finite positive integer M in (5), the M^{th} order approximate closed-form solution is obtained as follows:

$$y^{(M)}(t) = \sum_{i=1}^{M} y_i(t) \tag{17}$$

In order to find an approximate closed-form solution with enough accuracy, we present an iterative algorithm with low computational complexity. According to previous discussions, the presented algorithm has also a relatively fast convergence rate. Therefore, only a few iterations of the algorithm are required to get a desirable accuracy. This fact reduces the size of computations, effectively.

Algorithm:
Step 1. Let $i=1$.
Step 2. Calculate the i^{th} order term $y_i(t)$ from the presented sequence of linear BVPs in Theorem 3.1.
Step 3. Let $M=i$ and obtain $y^{(M)}(t)$ from (17).

Step 4. Stop the algorithm if there is no significant difference between $y^{(M)}(t)$ and $y^{(M-1)}(t)$; else replace i by $i+1$ and go to step 2.

5. Numerical Example

In this section, effectiveness of the proposed method is verified using a numerical example.

Example 5.1. Consider the following nonlinear sixth order BVP [4]:

$$\begin{cases} D^6(y(t)) = e^{-t}y^2(t)\ ,\ 0<t<1 \\ y(0) = \left.\dfrac{d^2y(t)}{dt^2}\right|_{t=0} = \left.\dfrac{d^4y(t)}{dt^4}\right|_{t=0} = 1 \\ y(1) = \left.\dfrac{d^2y(t)}{dt^2}\right|_{t=1} = \left.\dfrac{d^4y(t)}{dt^4}\right|_{t=1} = e \end{cases} \tag{18}$$

The exact solution of problem (18) is $y(t)=e^t$.

Based on the proposed method in section 3, instead of directly solving the nonlinear BVP (18), we solve a sequence of linear BVPs as follows:

$$\varepsilon^1: \begin{cases} D^6(y_1(t)) = 0 \\ y_1(0) = \left.\dfrac{d^2y_1(t)}{dt^2}\right|_{t=0} = \left.\dfrac{d^4y_1(t)}{dt^4}\right|_{t=0} = 1 \\ y_1(1) = \left.\dfrac{d^2y_1(t)}{dt^2}\right|_{t=1} = \left.\dfrac{d^4y_1(t)}{dt^4}\right|_{t=1} = e \end{cases}$$

$$\Rightarrow y_1(t) = 1 + 1.006979226t + 0.5t^2 + 0.1553169205t^3 + 0.04166666667t^4 + 0.01431901524t^5. \tag{19a}$$

$$\varepsilon^2: \begin{cases} D^6(y_2(t)) = e^{-t}y_1^2(t) \\ y_2(0) = \left.\dfrac{d^2y_2(t)}{dt^2}\right|_{t=0} = \left.\dfrac{d^4y_2(t)}{dt^4}\right|_{t=0} = 0 \\ y_2(1) = \left.\dfrac{d^2y_2(t)}{dt^2}\right|_{t=1} = \left.\dfrac{d^4y_2(t)}{dt^4}\right|_{t=1} = 0 \end{cases}$$

$$\begin{aligned} \Rightarrow y_2(t) &= -3.572993076\times10^6 \\ &+1.242930455\times10^6 t - 1.869782688\times10^5 t^2 \\ &+15335.62449t^3 - 693.9053464t^4 \\ &+14.11491600t^5 + 3.572993076\times10^6 e^{-t} \\ &+2.330062613\times10^6 te^{-t} + 7.305443440\times10^5 t^2e^{-t} \\ &+1.456762703\times10^5 t^3e^{-t} + 20567.06102t^4e^{-t} \\ &+2161.195696t^5e^{-t} + 172.3953566t^6e^{-t} \\ &+10.39527278t^7e^{-t} + 0.4581342835t^8e^{-t} \\ &+0.01349530311t^9e^{-t} + 2.050341974\times10^{-4}t^{10}e^{-t}. \end{aligned} \tag{19b}$$

$$\vdots$$

and so on. Therefore, the second order approximate closed-form solution for nonlinear BVP (18) is obtained as follows:

$$\begin{aligned} & y(t) \cong y^{(2)}(t) = y_1(t) + y_2(t) \\ & \Rightarrow y(t) \cong -3.572992076\times10^6 + 1.242931462\times10^6 t \\ & -1.869777688\times10^5 t^2 + 15335.77981t^3 \\ & -693.8636797t^4 + 14.12923502t^5 \\ & +3.572993076\times10^6 e^{-t} + 2.330062613\times10^6 te^{-t} \\ & +7.305443440\times10^5 t^2e^{-t} + 1.456762703\times10^5 t^3e^{-t} \\ & +20567.06102t^4e^{-t} + 2161.195696t^5e^{-t} \\ & +172.3953566t^6e^{-t} + 10.39527278t^7e^{-t} \\ & +0.4581342835t^8e^{-t} + 0.01349530311t^9e^{-t} \\ & +2.050341974\times10^{-4}t^{10}e^{-t}. \end{aligned} \tag{20}$$

Problem (18) has also been solved in [4] by the homotopy perturbation method (HPM). Simulation results of both methods are shown in Fig. 1 as compared to the exact solution. From Fig. 1 it is observed that the solution obtained by 2 iterations of the proposed method is in good agreement with the exact solution. In addition, the result obtained by 2 iterations of the proposed method is much more accurate than that obtained by 2 iterations of the HPM. Therefore, the extended Modal series method is superior to the HPM for solving nonlinear BVP (18).

5. Conclusion

This main objective of this paper was to extend the Modal series method for solving nonlinear BVPs. Extending the Modal series method, the exact solution of nonlinear BVP is determined in the form of uniformly convergent series with easy-computable terms. The proposed method only requires solving a sequence of linear BVPs in a recursive manner. In addition, uniformly convergence of the obtained solution series is guaranteed. Therefore, in view of computational complexity, the proposed method is more practical than the other approximate methods. Besides, an iterative algorithm has been presented which has low computational complexity and fast convergence rate. Through the finite iteration of the algorithm, an approximate closed-form solution has been obtained for the nonlinear BVP. Numerical example verified the efficiency, simplicity, and high accuracy of the proposed method.

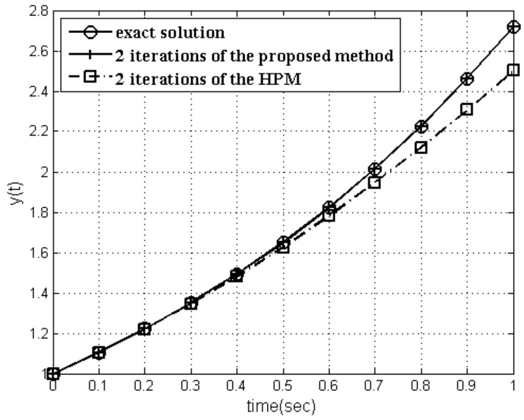

Figure 1. Simulation results of the proposed method and HPM as compared to the exact solution.

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