VIBRATION ANALYSIS OF CURVED SINGLE-WALLED CARBON NANOTUBES EMBEDDED IN AN ELASTIC MEDIUM BASED ON NONLOCAL ELASTICITY

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In this paper, the flexural vibration of the curved single-walled carbon nanotube (SWCNT) is investigated, based on the nonlocal Euler-Bernoulli beam model. The SWCNT is assumed to be embedded in an elastic medium. Both Winkler-type and Pasternak-type foundation models are utilized to simulate the interaction of the SWCNT with the surrounding elastic medium. Three typical boundary conditions, namely clamped-clamped, clamped-pinned and pinned-pinned, are used to investigate the effect of the supported end conditions. Based on the Galerkin method, a solution for natural frequency is obtained. According to this study, the influences of the amplitude of waviness, the nonlocal effects, the Winkler modulus parameter, the Pasternak shear modulus parameter, the boundary condition and the aspect ratio are analyzed and discussed. It is shown that waviness in the curved SWCNT causes an obvious increase in the natural frequency in comparison with the straight SWCNT, especially for a compliant medium, pinned-pinned boundary condition and short SWCNT.

1. Introduction

After the invention of Carbon nanotubes (CNTs) by Iijima\(^1\) in 1991 a considerable amount of research has been conducted to investigate the mechanical, thermal and electrical properties of CNTs and directed toward understanding the static and dynamical behaviours of carbon nanotubes due to their enormous applications which in nanotechnology, electronics and other fields of materials science\(^2\).

Therefore these properties are a good deal attempt has so far been devoted to the study of the various aspects of nanotubes. Since controlled experiments at nanoscale are difficult and molecular dynamics simulations remain expensive and time-consuming, continuum mechanics models, such as Euler elastic-beam model and Timoshenko beam model, have been widely used to study the vibrational behaviour of CNTs. For instance, Lu et al\(^3\) studied the investigations of wave and vibra-
tion properties of single- or multi-walled carbon nanotubes based on nonlocal Euler and Timoshenko beam models. Rasekh and Khadem obtained the amplitude–frequency response curves of the nonlinear vibration and the effects of the surrounding elastic medium, also influence of internal moving fluid and compressive axial load on the nonlinear vibration and stability of embedded carbon nanotubes is investigated. In the other paper, the small-scale effects on vibration characteristics of CNTs have been studied using a nonlocal continuum mechanics model by Wang and Vardan. A qualitative validation study shows that the results based on nonlocal continuum mechanics are in agreement with the published experimental reports in this field. Murmu and Pradhan implemented the nonlocal Timoshenko beam theory to investigate the stability response of SWCNTs embedded in an elastic medium. This paper demonstrates that critical buckling loads of SWCNT are significantly dependent on the nonlocal constant and on the properties of surrounding medium.

In majority of previous studies are limited to classical beam theory for straight beams, while some recent experimental results show that these tiny structures are not usually straight and rather have certain degree of curvature or waviness along them. Joshi et al. investigated in the vibration response analysis of carbon nanotubes with waviness treated as thin shell. Also, Mayoof and Hawwa studied the dynamics of the CNT when it acts as first-mode resonator with a focus on the chaotic behaviour of a curved carbon nanotube under harmonic excitation.

In this study, a nonlocal Euler-Bernoulli beam model has been employed to investigate the transverse vibration of the wavy SWCNT embedded in an elastic medium. The natural frequency for the curved SWCNT is expressed using the Galerkin method. Also, the effect of amplitude of curvature on the fundamental frequency is discussed. Moreover, the variation of frequency has been considered based on the various parameters such as the surrounding elastic medium, the boundary conditions, the aspect ratio of SWCNT and the nonlocal coefficient.

2. Modeling

A curved SWCNT with length $L$ embedded in an elastic medium with two clamped ends which is described as a hollow tube as shown in Fig. 1. The elastic medium is simulated by Winkler-type and Pasternak-type models.

![Figure 1. A curved SWCNT embedded in an elastic medium with two fixed ends.](image)

Based on nonlocal Euler-Bernoulli beam theory and using the Hamilton principle, the governing equation of motion for the curved SWCNT can be expressed as
\[
EI \frac{\partial^4 W}{\partial x^4} + m \left( \frac{\partial^2 W}{\partial t^2} - (e_0 a)^2 \frac{\partial^2 W}{\partial x^2 \partial t^2} \right) - P + (e_0 a)^2 \frac{\partial^2 P}{\partial x^2} = 0
\]

Where \( x \) is the axial coordinate, \( t \) is the time, \( W(x,t) \) is the transverse displacement component and \( Y(x) \) represents the curvature of the SWCNT. \( EI, A \) and \( m \) are the bending rigidity, the cross-section area and the mass per unit length of the SWCNT. \( P(x,t) \) shows the external distributed load and \( e_0 a \) is a nonlocal parameter illuminating the nano-scale effect on the response of the structure. This equation can be reduced to the same equation in Ref. 3 when the SWCNT is assumed without curvature. Furthermore, if \( K_w=K_G=e_0 a=0 \), Eq. (1) will be changed to the same equation that suggested by Mayoof and Hawwa.\(^{8}\)

The external force \( P(x,t) \) is related to interaction between the SWCNT and the surrounding medium, which can be described by the Winkler-type and Pasternak-type models, and

\[
P = -K_w W + K_G \frac{\partial^2 W}{\partial x^2}
\]

\( K_w \) and \( K_G \) indicate the Winkler constant and Pasternak shear constant of the elastic medium, in that order. Substitution of Eq. (2) into Eq. (1) leads to the partial equation of motion (3) for the free vibration of the curved SWCNT

\[
EI \frac{\partial^4 W}{\partial x^4} + m \left( \frac{\partial^2 W}{\partial t^2} - (e_0 a)^2 \frac{\partial^2 W}{\partial x^2 \partial t^2} \right) + K_G \left( (e_0 a)^2 \frac{\partial^4 W}{\partial x^4} - \frac{\partial^2 W}{\partial x^2} \right) + K_w \left( W - (e_0 a)^2 \frac{\partial^2 W}{\partial x^2} \right) = 0
\]

In this paper, tree standard boundary conditions have been considered, a beam clamped at both ends, or the clamped–clamped boundary condition.

\[
\frac{\partial W(0,t)}{\partial x} = W(0,t) = \frac{\partial W(L,t)}{\partial x} = W(L,t) = 0
\]

A beam clamped at one end and simply supported at the other end, i.e. clamped–pinned condition

\[
\frac{\partial W(0,t)}{\partial x} = W(0,t) = \frac{\partial^3 W(L,t)}{\partial x^3} = W(L,t) = 0
\]

And a simply supported beam at both ends or pinned–pinned condition

\[
\frac{\partial^2 W(0,t)}{\partial x^2} = W(0,t) = \frac{\partial^2 W(L,t)}{\partial x^2} = W(L,t) = 0
\]

In addition, the sinusoidal small rise function \( Y \) is introduced by

\[
Y(x) = H \sin\left(\frac{\pi x}{L}\right)
\]

Where \( H \) is the amplitude of curvature.

3. Solution

Galerkin’s method is utilized in order to obtain an ordinary differential equation (ODE) from Eq. (3). Initially, Eq. (3) must become a dimensionless equation. For this purpose, introducing the dimensionless quantities as...
\[ \tau = t \omega, \quad \omega = \sqrt{\frac{EI}{mL^4}}, \quad \xi = \frac{x}{L}, \quad w = \frac{W}{L}, \]
\[ y = \frac{Y}{L}, \quad s = \frac{AL}{L}, \quad k_w = \frac{KWL}{EI} \]
\[ k_G = \frac{KGL^2}{EI}, \quad e_n = \frac{e_n a}{L}, \quad h = \frac{H}{L} \]

Eq. (3) can be written in the dimensionless form
\[
\frac{\partial^4 w}{\partial \xi^4} + \frac{\partial^2 w}{\partial \tau^2} - e_n^2 \frac{\partial^4 w}{\partial \xi^4} \frac{\partial^2 w}{\partial \tau^2} + k_G \left( \frac{\partial^4 w}{\partial \xi^4} - e_n^2 \frac{\partial^2 w}{\partial \xi^4} \right) + k_w \left( w - e_n^2 \frac{\partial^2 w}{\partial \xi^4} \right) = 0
\]
\[
\int_0^L \left( \frac{\partial^2 y}{\partial \xi^2} - e_n^2 \frac{\partial^4 y}{\partial \xi^4} \right) d\xi
\]

The process of Galerkin method is started by separating the dependences of the deflection of the beam, \( w(\xi, \tau) \), into temporal function \( q(\tau) \) and the fundamental mode shape \( \phi(\xi) \), as
\[ w(\xi, \tau) = \phi(\xi) q(\tau) \]

The spatially dependent mode shape satisfies the corresponding boundary conditions in Eqs. (4-6). Hence, the first mode of shape function for clamped-clamped beam can take from
\[ \phi(\xi) = \cos(4.73\xi) - 0.9825 \sinh(4.73\xi) - 0.9825 \sin(4.73\xi) \]

Basis function for clamped-pinned state will be
\[ \phi(\xi) = \cos(3.92\xi) - 1.0008 \sinh(3.92\xi) - 1.0008 \sin(3.92\xi) \]

And for pinned-pinned may be given as
\[ \phi(\xi) = \sin(3.14\xi) \]

Multiplying equation (9) by the mode shape and integrating over the length of the SWCNT, the governing ordinary differential equation (14) is obtained by assuming the dimensionless equation of curve \( y(\xi) = h \sin(n \pi \xi) \)
\[ \frac{\partial^2 q}{\partial \tau^2} + \omega_0^2 q = 0 \]

Where \( \omega_0 \) is the natural frequency and has the variable value for three boundary conditions.

For the clamped–clamped boundary condition may be expressed as
\[ \omega_0 = \sqrt{\frac{500.5e_n^2 k_G + 12.3e_n^2 k_w + 12.3k_G + k_w + 467.5sh^2 e_n^2 + 47.3sh^2 + 500.5}{12.3e_n^2 + 1}} \]
\[ \omega_0 = \sqrt{\frac{237.7e_n^2 k_G + 11.5e_n^2 k_w + 11.5k_G + k_w + 458.4sh^2 e_n^2 + 46.4sh^2 + 237.7}{11.5e_n^2 + 1}} \]

And for pinned-pinned we get
\[ \omega_0 = \sqrt{\frac{97.4e_n^2 k_G + 9.8e_n^2 k_w + 9.8k_G + k_w + 480.6sh^2 e_n^2 + 48.7sh^2 + 97.4}{9.8e_n^2 + 1}} \]

4. Results

In this study, the free vibration equation of the curved SWCNT has been derived by using the nonlocal Euler-Bernoulli theory. The outer diameter, thicknesses and Young’s modulus of the nano-
tube are assumed to be \( d_0 = 3.19 \text{ nm}, t_c = 0.137 \text{ nm}, \) and \( E = 2.407 \text{ TPa} \), respectively\(^\text{2b}\). The mass density of SWCNT is 2300 kg/m\(^3\) with nonlocal parameter \( e_0 \) of 2 nm and aspect ratio \( L/d_0 = 20 \). Also the Winkler-constant and Pasternak-constant are estimated the values of \( K_W = 1 \text{ MPa} \) and \( K_G = 5 \text{nN} \), in that order\(^\text{6}\).

Fig. 2 shows the fundamental frequency \( \omega_0 \) against the curvature amplitude \( H \) for three standard boundary conditions. It demonstrates that with increasing the amplitude of waviness, the frequency increases. Also, the results are completely dependent on the boundary conditions and the natural frequency is increased while the bending stiffness of the SWCNT rises from pinned-pinned to clamped-clamped, especially for the low curvature amplitude.

Moreover, to see the effects of curvature clearly, the “difference percent” is defined as a parameter that shows the percent increment of frequency for the curved SWCNT \((H\neq 0 \text{ nm})\) compared with the straight nanotube \((H=0 \text{ nm})\).

\[
\text{Difference percent} = \left( \frac{\omega_0^{H=0 \text{ nm}} - \omega_0^{H=0 \text{ nm}}}{\omega_0^{H=0 \text{ nm}}} \right) \times 100 \tag{18}
\]

Certainly, difference percent gives a better illustration for the pure effects of the amplitude of curvature. Figs. 3-7 represent the difference percent as a function of the waviness amplitude \( H \), while the effects of a certain parameter such as the stiffness of model, the aspect ratio and the nonlocal parameter have been evaluated in each figure. Obviously, the variation of fundamental frequency is increased when the waviness goes up, in all the figures.
The difference percent is highly sensitive to the stiffness of model, due to the foundation and boundary conditions. Figs. 4 and 5 depict the influence of medium on the vibration of curved SWCNT. The results show that, as the medium stiffness caused by Winkler-constant $K_W$ and Pasternak-constant $K_G$ increase, the difference percent decreases. In addition, Fig. 5 indicates that the boundary conditions have the considerable effects on the difference percent. The illustration demonstrates that with decreasing the stiffness of SWCNT from clamped-clamped to pinned-pinned, the effects of waviness on vibrational frequency increase.

![Figure 4](image1.png)

**Figure 4.** The difference percent against the curvature amplitude $H$ for clamped-clamped with different values of the Pasternak modulus $K_G$.

![Figure 5](image2.png)

**Figure 5.** The difference percent against the curvature amplitude $H$ with different types of boundary conditions.

Fig. 6 shows the importance of the aspect ratio $L/d_0$ in the natural frequency and associated difference percent. According to this figure, the rise of frequency with curvature amplitude is found to be significantly dependent on the aspect ratio, i.e., for $L/d_0=20$, the difference percent increases steeper than $L/d_0=80$. It means for long nanotube, the difference of frequency between the straight SWCNT and curved SWCNT is moderately reduced.

Finally, to investigate the effect of nonlocal theory, the difference percent is plotted as a function of the curvature amplitude $H$ and the nonlocal parameter $\epsilon_p\alpha$, in Fig. 7. As it shows, by increas-
ing the nonlocal effect, the difference percent increases for every fixed amplitude $H$, although, this effect is negligible

![Figure 6](image)

**Figure 6.** The difference percent against the curvature amplitude $H$ for clamped-clamped with different values of the aspect ratio $L/d_0$.

![Figure 7](image)

**Figure 7.** The difference percent against the curvature amplitude $H$ for clamped-clamped with different values of the nonlocal parameter $\epsilon_0a$.

5. **Conclusion**

A nonlocal continuum model has been developed to analyze the effect of waviness on the curved single-walled carbon nanotube. The surrounding elastic medium is simulated as the Winkler and Pasternak models. The Galerkin method is employed to solve the governing equation of motion. The results show the influence of the curvature, the stiffness of medium around the SWCNT, the boundary conditions, the aspect ratio of SWCNT and the nonlocal parameter on the natural frequency. Detailed results demonstrate that increasing of the amplitude of curvature, causes the fundamental frequency to increase. Furthermore, with an increase in the nonlocal constant, as the stiffness of model, due to the boundary conditions or to the foundation and length of SWCNT decrease, the frequency is obtained at higher values for a curved SWCNT in comparison a straight SWCNT.
REFERENCES