STABILITY OF EULER-LAGRANGE QUADRATIC FUNCTIONAL EQUATIONS IN NON-ARCHIMEDEAN NORMED SPACES

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Abstract
In this paper, we prove the stability of Euler-Lagrange quadratic mappings in the framework of non-Archimedean normed spaces. Our results in the setting of non-Archimedean normed spaces are different from the results in the setting of normed spaces.

Keywords: Generalized Hyers-Ulam stability, Euler-Lagrange functional equation, Non-Archimedean normed space, p-adic field.


1. Introduction and preliminaries

A classical question in the theory of functional equations is the following: “When is it true that a function which approximately satisfies a functional equation \( E \) must be close to an exact solution of \( E \)?” If there exists an affirmative answer, we say that the equation \( E \) is stable [4]. During the last decades several stability problems for various functional equations have been investigated by numerous mathematicians. We refer the reader to the survey articles [4, 5, 8, 19] and monographs [3, 6, 9, 10, 20], and references therein.

By a non-Archimedean field we mean a field \( K \) equipped with a function (valuation) \( | \cdot | \) from \( K \) into \([0, \infty)\) such that \( |r| = 0 \) if and only if \( r = 0 \), \( |rs| = |r| |s| \), and \( |r + s| \leq \max\{|r|, |s|\} \) for all \( r, s \in K \). Clearly \( |1| = |1| = 1 \) and \( |n| \leq 1 \) for all \( n \in \mathbb{N} \). By the trivial valuation we mean the mapping \( | \cdot | \) taking everything but 0 into 1 and \( |0| = 0 \).

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Let $X$ be a vector space over a field $K$ with a non-Archimedean non-trivial valuation $|\cdot|$. A function $\|\cdot\| : X \to [0, \infty)$ is called a non-Archimedean norm if it satisfies the following conditions:

(i) $\|x\| = 0$ if and only if $x = 0$;
(ii) For any $r \in K, x \in X, \|rx\| = |r|\|x\|;
(iii) The strong triangle inequality (ultrametric); namely,
$$\|x + y\| \leq \max\{\|x\|, \|y\|\} \quad (x, y \in X).$$

Then $(X, \|\cdot\|)$ is called a non-Archimedean normed space. Due to the fact that
$$\|x_n - x_m\| \leq \max\{\|x_{j+1} - x_j\| : m \leq j \leq n - 1\}, \quad (n > m)$$
holds, a sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean normed space. By a complete non-Archimedean normed space we mean one in which every Cauchy sequence is convergent.

For any nonzero rational number $x$, there exists a unique integer $n_x \in \mathbb{Z}$ such that $x = \frac{a}{b}p^{n_x}$, where $a$ and $b$ are integers not divisible by $p$. Then $|x|_p := p^{-n_x}$ defines a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric $d(x, y) = |x - y|_p$ is denoted by $\mathbb{Q}_p$, which is called the $p$-adic number field.

In [11], the authors investigated stability of approximate additive mappings $f : \mathbb{Q}_p \to \mathbb{R}$. In [11, 12, 13], the stability of Cauchy, quadratic and cubic functional equations were investigated in the context of non-Archimedean normed spaces.

In this paper, by following some ideas from [2, 12, 13, 16, 17, 18], we establish the stability of Euler-Lagrange equations in the setting of non-Archimedean normed spaces.

Throughout the paper, we assume that $X$ is a vector space and $Y$ is a complete non-Archimedean normed space.

2. Stability results

The functional equation $f(x + y) + f(x - y) = 2f(x) + 2f(y)$ is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a quadratic mapping. J. M. Rassias introduced the Euler-Lagrange quadratic mapping,

$$f(a_1x + a_2y) + f(a_2x - a_1y) = (a_1^2 + a_2^2)\{f(x) + f(y)\},$$

see [14, 15].

J. M. Rassias introduced the generalized pertinent Euler-Lagrange quadratic mappings via his paper [16] and investigated the stability problem for the following generalized functional equation

$$m_1m_2Q(a_1x + a_2y) + Q(m_2a_2x - m_1a_1y) = (m_1a_1^2 + m_2a_2^2)\{m_2Q(x) + m_1Q(y)\},$$

for all vectors $x, y \in X$, any fixed pair $(a_1, a_2)$ of nonzero reals and any fixed pair $(m_1, m_2)$ of positive reals. Consider a nonlinear mapping $Q : X \to Y$ satisfying the fundamental Euler-Lagrange functional equation

$$m_1^2m_2Q(a_1x) + m_1Q(m_2a_2x) = m_0^2m_2Q\left(\frac{m_1}{m_0}a_1x\right) + m_1^2m_1Q\left(\frac{m_2}{m_0}a_2x\right),$$

with $m_0 = \frac{m_1m_2+1}{m_1m_2}$ and $m = \frac{m_1+m_2}{m_1m_2+1}$ for all $x \in X$, any fixed nonzero reals $a_1, a_2$ and any fixed positive reals $m_1, m_2$.

A nonlinear mapping $Q : X \to Y$ is a called generalized Euler-Lagrange quadratic if it satisfies (2.2) and (2.3). It is said that the nonlinear mappings $\mathcal{Q} : X \to Y$, and
\( Q : X \to Y \) are 2-dimensional Euler-Lagrange quadratic weights of the first form if we have
\[
\overline{Q}(x) = \frac{m_0^2 m_2 Q(\frac{m_0}{m_2} a_1 x) + m_0^2 m_1 Q(\frac{m_0}{m_2} a_2 x)}{m_1 m_2 (m_1 a_1^2 + m_2 a_2^2)},
\]
and
\[
\overline{Q}(x) = \frac{m_1 m_2 Q(a_1 x) + Q(m_2 a_2 x)}{m_2 (m_1 a_1^2 + m_2 a_2^2)}
\]
for all \( x \in X \).

2.1. Lemma. ([16]) Let \( Q : X \to Y \) be a generalized Euler-Lagrange quadratic mapping satisfying (2.2). If \( m \neq 1 \), then we have
\[
Q(0) = 0, \; Q(m^n x) = m^{2n} Q(x)
\]
for all \( x \in X \) and all integers \( n \in \mathbb{Z} \).

Suppose that \( f : X \to Y \) is a mapping. We define a generalized Euler-Lagrange difference operator \( D^a_{m_1, a_2} \) of equation (2.2) as
\[
D^a_{m_1, a_2} f(x, y) := m_1 m_2 f(a_1 x + a_2 y) + f(m_2 a_2 x - m_1 a_1 y) - (m_1 a_1^2 + m_2 a_2^2) [m_2 f(x) + m_1 f(y)].
\]
In this section, we prove the stability of the generalized Euler-Lagrange quadratic functional equation in non-Archimedean normed spaces.

2.2. Theorem. Let \( \varphi : X \times X \to [0, \infty) \) and \( \psi : X \to [0, \infty) \) be functions such that
\[
\lim_{n \to \infty} \frac{\varphi[m^n x, m^n y]}{|m|^{2n}} = 0 \quad (x, y \in X)
\]
and
\[
\lim_{n \to \infty} \frac{\psi(m^n x)}{|m|^{2n}} = 0 \quad (x \in X),
\]
where, \( m = \frac{(m_1 + m_2)(m_1 a_1^2 + m_2 a_2^2)}{m_1 m_2 + 1} \) and \( |m| > 1 \) for any fixed pair \((a_1; a_2)\) of nonzero reals and any fixed pair \((m_1; m_2)\) of positive reals. Suppose that \( f : X \to Y \) is a mapping satisfying
\[
\|D^a_{m_1, a_2} f(x, y)\| \leq \varphi(x, y),
\]
for all \( x, y \in X \), and
\[
\|m_1 m_2 f(a_1 x) + m_1 f(m_2 a_2 x) - m_0^2 m_2 f(\frac{m_2}{m_0} a_1 x) - m_0^2 m_1 f(\frac{m_1}{m_0} a_2 x)\| \leq \psi(x).
\]
Then there exists a unique generalized Euler-Lagrange mapping \( Q : X \to Y \) such that
\[
\|f(x) - Q(x)\| \leq \sup \left\{ \frac{\varphi(m^n x, 0)}{|m|^{2n} |m_0 m_1 m_2|}, \frac{\varphi(m^n x)}{|m|^{2n} |m_0 m_1 m_2|} : n \in \mathbb{N} \cup \{0\} \right\},
\]
(2.8)

Proof. Observe that the functional inequality (2.7) can be written as follows:
\[
\|f(x) - \overline{Q}(x)\| \leq \frac{\psi(x)}{|m_1 m_2 (m_1 a_1^2 + m_2 a_2^2)|} = \frac{\psi(x)}{|m_0 m_1 m_2|} \quad (x \in X).
\]
Replacing \( x \) and \( y \) by \( 0 \) in (2.6) we have
\[
\|m_1 m_2 f(0) + f(0) - m_0 m(m_1 + m_2) f(0)\| \leq \varphi(0, 0),
\]
(2.9)
or
\[
\|f(0)\| \leq \frac{\varphi(0, 0)}{|(m_1 + m_2 + 1)(m - 1)|}.
\]

Moreover substituting \(y = 0\) in (2.6), one concludes the functional inequality
\[
\|m_1 m_2 f(a_1 x) + f(m_2 a_2 x) - m_0 m [m_2 f(x) + m_1 f(0)]\| \leq \varphi(x, 0)
\]
or
\[
\|f(x) - f(x) - \frac{m_1}{m_2} f(0)\| \leq \frac{\varphi(x, 0)}{|m_2(m_1 a_1^2 + m_2 a_2^2)|} = \frac{\varphi(x, 0)}{|m_0 m_2^2|}.
\]

Hence
\[
\|f(x) - f(x)\| \leq \max \left\{ \left\| \frac{f(x) - f(x) - \frac{m_1}{m_2} f(0)}{m_2 f(0)} \right\|, \frac{m_1}{m_2} f(0) \right\}
\]
(2.11)
\[
\leq \max \left\{ \frac{\varphi(x, 0)}{|m_0 m_2^2|}, \frac{m_1}{m_2} f(0) \right\}.
\]

In addition, replacing \(x, y\) in (2.6) by \(\frac{m_1 a_1 x}{m_0}\) and \(\frac{m_2 a_2 x}{m_0}\) respectively, one gets the functional inequality
\[
\left\|m_1 m_2 f(mx) + f(0) - m_0 m \left[ m_2 f\left(\frac{m_1 a_1 x}{m_0}\right) + m_1 f\left(\frac{m_2 a_2 x}{m_0}\right)\right]\right\| 
\leq \varphi\left(\frac{m_1 a_1}{m_0}, \frac{m_2 a_2}{m_0}\right).
\]

or
\[
\left\| \frac{f(mx)}{m^2} + \frac{f(0)}{m^2 m_1 m_2} - f(x) \right\| \leq \frac{1}{|m^2 m_1 m_2^2|} \varphi\left(\frac{m_1 a_1}{m_0}, \frac{m_2 a_2}{m_0}\right).
\]

So
\[
\|f(mx)\| - f(x)\| \leq \max \left\{ \frac{1}{|m^2 m_1 m_2^2|} \varphi\left(\frac{m_1 a_1}{m_0}, \frac{m_2 a_2}{m_0}\right), \|f(0)\| \right\}.
\]

Using the functional inequalities (2.9), (2.11) and (2.12), and the triangle inequality, we have the basic inequality
\[
\left\| \frac{f(mx)}{m^2} - f(x) \right\| \leq \max \left\{ \|f(x) - f(x)\|, \|f(x) - f(x)\|, \|f(x) - f(x)\| \right\}
\]
(2.13)
\[
\leq \max \left\{ \frac{\varphi(x, 0)}{|m_0 m_2^2|}, \|f(0)\|, \frac{\psi(x)}{|m_0 m_2^2|}, \frac{\varphi\left(\frac{m_1 a_1}{m_0}, \frac{m_2 a_2}{m_0}\right)}{|m^2 m_1 m_2^2|}, \|f(0)\| \right\}.
\]

Replacing \(x\) by \(m^n x\) in (2.13) we obtain
\[
\left\| \frac{f(m^n x)}{m^{2n}} - \frac{f(m^{n+1} x)}{m^{2(n+1)}} \right\|
\leq \max \left\{ \frac{\varphi\left(\frac{m^n x}{m_0 m_2 m_0 m_2}, \frac{m_1}{m_0 m_2^2}, \|f(0)\|, \frac{\varphi\left(\frac{m_n x}{m_0 m_2 m_0 m_2}, \frac{m_1 a_1}{m_0}, \frac{m_2 a_2}{m_0}\right)}{|m^2 m_1 m_2^2|}, \|f(0)\| \right\}.
\]
(2.14)
It follows from (2.4), (2.5) and \(|m| > 1\) that the sequence \(\left\{ \frac{f(m^n x)}{m^{2n}} \right\}\) is Cauchy. Since \(Y\) is complete, we conclude that \(\left\{ \frac{f(m^n x)}{m^{2n}} \right\}\) is convergent. Set \(Q(x) = \lim_{n \to \infty} \left\{ \frac{f(m^n x)}{m^{2n}} \right\}\). Now, from the inequalities (2.13) and (2.14), one gets the inequalities

\[
\left\| \frac{f(m^n x)}{m^{2n}} - f(x) \right\| \leq \max \left\{ \left\| \frac{f(m^j x)}{m^{2j}} - \frac{f(m^{j+1} x)}{m^{2(j+1)}} \right\| : 0 \leq j \leq n - 1 \right\}
\]

(2.15)

\[
\leq \max \left\{ \frac{\varphi(m^j x, 0)}{|m^{2j}|} \frac{|m_1|}{|m^{2j}m_0 m m_2|}, \frac{\psi(m^j x)}{|m^{2j}m_0 m m_1 m_2|}, \frac{\varphi(m_{11} a_{1} m^j x, m_{22} a_{2} m^j x)}{|m^{2j}m_0 m m_1 m_2|} \right\}.
\]

Taking the limit as \(n \to \infty\) in (2.15) we find that the mapping \(Q\) satisfies the inequality (2.8).

Besides, we claim that the mapping \(Q\) satisfies the generalized Euler-Lagrange equation. In fact, it is clear from (2.6) that the following inequality

\[
\|D_{m_{11},m_{22}}^{a_{1},a_{2}} f(m^n x, m^n y)\| \leq \frac{1}{|m^{2n}|} c(m^n x, m^n y)
\]

(2.16)

holds for all \(x, y \in X\) and \(n \in \mathbb{N}\). Taking the limit \(n \to \infty\) we obtain from (2.4) that \(D_{m_{11},m_{22}}^{a_{1},a_{2}} Q(x, y) = 0\).

Now let \(\tilde{Q} : X \to X\) be another generalized Euler-Lagrange mapping satisfying the equation \(D_{m_{11},m_{22}}^{a_{1},a_{2}} \tilde{Q}(x, y) = 0\) and the inequality

\[
\|f(x) - \tilde{Q}(x)\| \leq \max \left\{ \frac{\varphi(m^n x, 0)}{|m^{2n}|} \frac{|m_1|}{|m^{2n}m_0 m m_2|} \frac{\psi(m^n x)}{|m^{2n}m_0 m m_1 m_2|}, \frac{\varphi(m_{11} a_{1} m^n x, m_{22} a_{2} m^n x)}{|m^{2n}m_0 m m_1 m_2|} \right\}.
\]

Since \(Q(x) = \frac{Q(m^n x)}{m^{2n}}\), \(\tilde{Q}(x) = \frac{\tilde{Q}(m^n x)}{m^{2n}}\) for all \(x \in G\) and all \(n \in \mathbb{N}\). Thus we have

\[
\|Q(x) - \tilde{Q}(x)\| = \left\| \frac{Q(m^n x)}{m^{2n}} - \frac{\tilde{Q}(m^n x)}{m^{2n}} \right\|
\]

\[
\leq \max \left\{ \frac{1}{|m^{2n}|} \|Q(m^n x) - f(m^n x)\| \frac{1}{|m^{2n}|} \left\| \frac{\tilde{Q}(m^n x) - f(m^n x)}{m^{2n}} \right\| \right\}
\]

\[
\leq \max \left\{ \frac{\varphi(m^{n+k} x, 0)}{|m^{2(n+k)}|} \frac{|m_1|}{|m^{2(n+k)m_0 m m_2}|} \frac{\psi(m^{n+k} x)}{|m^{2(n+k)m_0 m m_1 m_2}|}, \frac{\varphi(m_{11} a_{1} m^{n+k} x, m_{22} a_{2} m^{n+k} x)}{|m^{2(n+k)m_0 m m_1 m_2}|} \right\}.
\]

\[
= \max \left\{ \frac{\varphi(m^j x, 0)}{|m^{2j}|} \frac{|m_1|}{|m^{2j}m_0 m m_2|} \frac{\psi(m^j x)}{|m^{2j}m_0 m m_1 m_2|}, \frac{\varphi(m_{11} a_{1} m^j x, m_{22} a_{2} m^j x)}{|m^{2j}m_0 m m_1 m_2|} \right\}.
\]

If \(k \to \infty\) we have \(Q = \tilde{Q}\).
2.3. **Theorem.** Let \( \varphi : X \times X \to [0, \infty) \) and \( \psi : X \to [0, \infty) \) be two functions such that
\[
(2.17) \quad \lim_{n \to \infty} |m|^{2n} \varphi \left( \frac{x}{m^n}, \frac{y}{m^n} \right) = 0 \quad (x, y \in X)
\]and
\[
(2.18) \quad \lim_{n \to \infty} |m|^{2n} \varphi \left( \frac{x}{m^n} \right) = 0 \quad (x \in X),
\]where, \( m = \frac{(m_1 + m_2)(m_1^2 + m_2^2 \lambda^2)}{m_1^2 m_2 + 1} \) and \( 0 < |m| < 1 \) for any fixed pair \((a_1; a_2)\) of nonzero reals and any fixed pair \((m_1; m_2)\) of positive reals, and \( f : X \to Y \) is a mapping satisfying
\[
(2.19) \quad \|D^{a_1, a_2}_{m_1, m_2} f(x, y)\| \leq \varphi(x, y)
\]and
\[
(2.20) \quad \left\| m_1^2 m_2 f(a_1 x) + m_1 f(m_2 a_2 x) - m_1^2 m_2 f \left( \frac{m_1}{m_0} a_1 x \right) - m_0^2 m_1 f \left( \frac{m_2}{m_0} a_2 x \right) \right\| \leq \psi(x)
\]for all \( x, y \in X \). Then there exists a unique generalized Euler-Lagrange mapping \( Q : X \to Y \) such that
\[
|f(x) - Q(x)| \leq \sup \left\{ \left| \frac{|m|^{2n+1} \varphi \left( \frac{x}{m^n}, 0 \right)}{|m_0 m_2 m_2^n} \right|, \left| \frac{|m|^{2n} |m_1| \|f(0)\|}{|m_2^n|} \right|, \left| \frac{|m|^{2n+1} \varphi \left( \frac{m_1}{m_0} a_1 x, 0 \right)}{|m_0 m_2 m_2^{n+1}} \right|, \left| \frac{|m|^{2n+2} \|f(0)\|}{|m_1^{n+1}|} : n \in \mathbb{N} \right\}.
\]

**Proof.** Using the same method as in Theorem 2.2, we conclude that
\[
Q(x) = \lim_{n \to \infty} \left\{ m^{2n} f \left( \frac{x}{m^n} \right) \right\}
\]is the unique Euler-Lagrange mapping satisfying (2.21). \( \square \)

In the next theorem we consider the case that \( m = 1 \).

2.4. **Theorem.** Assume that \( f : X \to Y \) and \( \phi : X \times X \to [0, \infty) \) are two mappings for which
\[
(2.22) \quad \|D^{a_1, a_2}_{m_1, m_2} f(x, y)\| \leq \phi(x, y)
\]holds for all \( x, y \in X \). Suppose that \( m := \frac{(m_1 + m_2)(m_1^2 + m_2^2 \lambda^2)}{m_1^2 m_2 + 1} = 1 \), \( m_2 a_2 = m_1 a_1 \), and if \( |l| > 1 \) then
\[
\lim_{n \to \infty} \phi \left( \frac{m^n x}{l^n}, 0 \right) = 0
\](if \( |l| < 1 \), then \( \lim_{n \to \infty} |l|^{2n} \phi \left( \frac{x}{l^n}, 0 \right) = 0 \)), where \( l := a_1 + a_2 \) is given with \( |l| \neq 0, 1 \). Then there exists a unique generalized Euler-Lagrange quadratic mapping \( Q : X \to Y \) satisfying \( D^{a_1, a_2}_{m_1, m_2} Q(x, y) = 0 \) and
\[
|f(x) - Q(x)| \leq \sup \left\{ \left| \frac{\phi \left( \frac{m^n x}{l^n}, \frac{x}{l^n} \right)}{|l|^{2n+2}|m_1 m_2^n|} \right|, \left| \frac{\|f(0)\|}{|l|^{2n+2}|m_1 m_2^n|} : n \in \mathbb{N} \right\} \quad \text{if } |l| > 1,
\]
\[
\sup \left\{ \left| \frac{|l|^{2n+2} \phi \left( \frac{x}{l^n}, \frac{x}{l^n} \right)}{|m_1 m_2^n|} \right|, \left| \frac{|l|^{2n+2} \|f(0)\|}{m_1 m_2^n} : n \in \mathbb{N} \right\} \quad \text{if } |l| < 1.
\]
Moreover, if there exists a mapping \( \psi : X \to [0, \infty) \), then the function \( f \) satisfies approximately the following fundamental functional equation
\[
\left\| m_1^2 m_2 f(a_1 x) + m_1 f(m_2 a_2 x) - m_0^2 m_1 f \left( \frac{m_1}{m_0} a_1 x \right) - m_0^2 m_1 f \left( \frac{m_2}{m_0} a_2 x \right) \right\| \leq \psi(x),
\]
and if $|l| > 1$, then

\[
(2.23) \quad \lim_{n \to \infty} \frac{\phi(l^n x)}{|l|^{2n}} = 0
\]

(if $|l| < 1$, then $\lim_{n \to \infty} |l|^{2n} \phi\left(\frac{x}{|l|^n}\right) = 0$) holds for all $x \in X$.

**Proof.** From the fact that $m := \frac{(m_1^2 + m_2^2)(m_1^2 + m_2^2)^2}{m_1 m_2 + 1} = 1$ and $m_2 a_2 = m_1 a_1$, we have

\[
\frac{m_1 m_2 + 1}{m_1 m_2} = \frac{m_1^2 a_1 + a_2}{m_1^2} = (a_1 + a_2)^2.
\]

Replacing $y$ by $x$ in (2.22), we obtain

\[
\|f(tx) - t^2 f(x)\| \leq \max \left\{ \frac{\phi(x, x)}{|m_1 m_2|}, \frac{\|f(0)\|}{|m_1 m_2|} \right\}
\]

and so

\[
\left\| \frac{f(l^{n+1} x)}{|l|^{2n+2}} - \frac{f(l^n x)}{|l|^{2n}} \right\| \leq \max \left\{ \frac{\phi(l^{2n} x, l^{2n} x)}{|l|^{2n+2} m_1 m_2}, \frac{\|f(0)\|}{|l|^{2n+2} m_1 m_2} \right\} \text{ if } |l| > 1
\]

and

\[
\left\| l^{2n+2} f\left(\frac{x}{l^{n+1}}\right) - t^{2n} f\left(\frac{x}{l^n}\right) \right\| \leq \max \left\{ |l|^{2n+2} \frac{\phi\left(\frac{x}{l^{n+1}}, \frac{x}{l^n}\right)}{|m_1 m_2|}, |l|^{2n+2} \frac{\|f(0)\|}{|m_1 m_2|} \right\} \text{ if } |l| < 1
\]

for all $x \in X$ and nonnegative integers $n$.

Now by a similar process to the proof of our previous theorems we may find that the sequences $\{\frac{f(l^n x)}{|l|^{2n}}\}$, when $|l| > 1$, and $\{\frac{f(l^n x)}{|l|^{2n}}\}$, when $|l| > 1$, are Cauchy and so are convergent, and

\[
Q(x) := \begin{cases} 
\lim_{n \to \infty} \frac{f(l^n x)}{|l|^{2n}} & \text{if } |l| > 1, \\
\lim_{n \to \infty} l^{2n} f\left(\frac{x}{l^n}\right) & \text{if } |l| < 1,
\end{cases}
\]

has our desired properties.

**2.5. Corollary.** Let $|m| \neq 1$, and $\rho : [0, \infty) \to [0, \infty)$ be defined by

\[
\rho(t) = \begin{cases} 
\frac{|m|^{2n}}{n+1} & \text{if } t = |m|^n r, \ |m| > 1, \ r < |m|, \ n \in \mathbb{N} \cup \{0\}, \ r > 0, \\
\frac{|m|^{2n}}{n+1} t & \text{if } t = \frac{r}{|m|^n}, \ 0 < |m| < 1, \ r < |m|, \ n \in \mathbb{N} \cup \{0\}, \ r > 0, \\
otherwise.
\end{cases}
\]

Suppose that $\delta_1, \delta_2 > 0$, $X$ is a normed space and $f : X \to Y$ fulfills the inequalities

\[
\|D_{m_1, m_2} f(x, y)\| \leq \delta_1 (\rho(||x||) + \rho(||y||)) \quad (x, y \in X)
\]

and

\[
(2.24) \quad \left\| m_1^2 m_2 f(a_1 x) + m_1 f(m_2 a_2 x) - m_0^2 m_2 f\left(\frac{m_1}{m_0} a_1 x\right) - m_0^2 m_1 f\left(\frac{m_2}{m_0} a_2 x\right) \right\|
\]

\[
\leq \delta_2 \rho(||x||).
\]

Then there exists a unique generalized Euler-Lagrange mapping $Q : G \to X$ such that if $|m| > 1$, then

\[
\|f(x) - Q(x)\| \leq \sup \left\{ \frac{\phi(x, 0)}{|m_0 m m_2|}, \frac{|m_1||f(0)||}{|m_2|}, \frac{\psi(x)}{|m_0 m m_1 m_2|}, \frac{\varphi(m_1 a_1 x_{m_0} m_2 a_2 x_{m_0})}{|m^2 m_1 m_2|}, \frac{\|f(0)\|}{|m^2 m_1 m_2|} \right\},
\]

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and if $0 < |m| < 1$, then

$$\|f(x) - Q(x)\| \leq \sup \left\{ \frac{|m| \varphi(\frac{m_1 x}{m_2}, 0)}{|m_0 m m_2|}, \frac{|m_1| \|f(0)\|}{|m_2|}, \frac{|m| \psi(x)}{|m_0 m m_1 m_2|}, \frac{|\varphi\left(\frac{m_1 x}{m_2}, \frac{m_2 x}{m_0}\right)}{|m_0 m_2|} \right\}.$$  

2.6. Remark. The hypotheses in Corollary 2.5 give us an example for which the crucial assumption $\sum_{i=0}^{\infty} \varphi\left(m_i x, m_i y\right) < \infty$ in the main theorem of [17] does not hold on balls of $X$ of the radius $r > 0$. Hence our results in the setting of non-Archimedean normed spaces are different from those of [17].

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