ON A CONJECTURE OF A BOUND FOR THE
EXPONENT OF THE SCHUR MULTIPLIER OF A
FINITE $p$-GROUP

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Communicated by Jamshid Moori

Abstract. Let $G$ be a $p$-group of nilpotency class $k$ with finite exponent $\exp(G)$ and let $m = \lfloor \log_p k \rfloor$. We show that $\exp(M(c)(G))$ divides $\exp(G)p^{m(k-1)}$, for all $c \geq 1$, where $M(c)(G)$ denotes the $c$-nilpotent multiplier of $G$. This implies that $\exp(M(G))$ divides $\exp(G)$, for all finite $p$-groups of class at most $p - 1$. Moreover, we show that our result is an improvement of some previous bounds for the exponent of $M(c)(G)$ given by M. R. Jones, G. Ellis and P. Moravec in some cases.

1. Introduction and motivation

Let a group $G$ be presented as a quotient of a free group $F$ by a normal subgroup $R$. Then, the $c$-nilpotent multiplier of $G$ (the Baer invariant of $G$ with respect to the variety of nilpotent group of class at most $c$) is defined to be

$$M(c)(G) = \frac{R \cap \gamma_{c+1}(F)}{[R, cF]},$$


Keywords: Schur multiplier, nilpotent multiplier, exponent, finite $p$-groups.

Received: 31 March 2009, Accepted: 1 August 2010.

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where \([R, \underbrace{F, \ldots, F}_{c\text{-times}}]\) denotes the commutator subgroup \([R, F, \ldots, F]\) and \(c \geq 1\).

The case \(c = 1\) which has been much studied is the Schur multiplier of \(G\), denoted by \(M(G)\). When \(G\) is finite, \(M(G)\) is isomorphic to the second cohomology group \(H^2(G, \mathbb{C}^*)\) (see G. Karpilovsky [6] and C. R. Leedham-Green and S. McKay [8] for further details).

It has been of interest to find a relation between the exponent of \(M(G)\) and the exponent of \(G\). Let \(G\) be a finite \(p\)-group of nilpotency class \(k \geq 2\) with exponent \(\exp(G)\). M. R. Jones [5] proved that \(\exp(M(G))\) divides \(\exp(G)^{k-1}\). This has been improved by G. Ellis [3] who showed that \(\exp(M(G))\) divides \(\exp(G)^{\lceil k/2 \rceil}\), where \(\lceil k/2 \rceil\) denotes the smallest integer \(n\) such that \(n \geq k/2\). For \(c = 1\), P. Moravec [11] showed that \(\lceil k/2 \rceil\) can be replaced by \(2\lceil \log_2 k \rceil\) which is an improvement, if \(k \geq 11\).

In this paper, we will show that if \(G\) is a finite exponent \(p\)-group of class \(k \geq 1\), then \(\exp(M(G))\) divides \(\exp(G)^{p^m(k-1)}\), for all \(c \geq 1\), where \(m = \lceil \log_p k \rceil\). Note that this result is an improvement of the results of Jones, Ellis and Moravec, if \(\lceil \log_p k \rceil(k-1)/k < e, \lceil \log_p k \rceil(k - 1)/\lceil k/2 \rceil - 1 < e, \lceil \log_p k \rceil(k - 1)/2\lceil \log_2 k \rceil - 1 < e\), respectively, where \(\exp(G) = p^e\).

It was a longstanding open problem as to whether \(\exp(M(G))\) divides \(\exp(G)\), for every finite group \(G\). In fact, it was conjectured that the exponent of the Schur multiplier of a finite \(p\)-group is a divisor of the exponent of the group itself. I. D. Macdonald and J. W. Wamsley [1] constructed an example of a group of order \(2^{21}\) which has exponent 4, whereas its Schur multiplier has exponent 8, hence the conjecture is not true in general. Also, Moravec [12] gave an example of a group of order 2048 and nilpotency class 6 which has exponent 4 and multiplier of exponent 8. He also proved that if \(G\) is a group of exponent 4, then \(\exp(M(G))\) divides 8. Nevertheless, Jones [5] has shown that the conjecture is true for \(p\)-groups of class 2 and emphasized that it is true for some \(p\)-groups of class 3. S. Kayvanfar and M. A. Sanati [7] have proved the conjecture for \(p\)-groups of class 4 and 5, with some conditions. A. Lubotzky and A. Mann [9] showed that the conjecture is true for powerful \(p\)-groups. The first and the third authors [10] showed that the conjecture is true for nilpotent multipliers of powerful \(p\)-groups. Finally, Moravec [11, 12] showed that the conjecture is true for metabelian groups of exponent \(p\), \(p\)-groups with potent filtration and \(p\)-groups of maximal...
class. Note that a consequence of our result shows that the conjecture
is true for all finite $p$-groups of class at most $p - 1$.

2. Preliminaries

In this section, we are going to recall some notions we will use in the
next section.

**Definition 2.1.** (M. Hall [4]). Let $X$ be an independent subset of a free
group, and select an arbitrary total order for $X$. We define the basic
commutators on $X$, their weight $\text{wt}$, and the ordering among them as
follows:

1. The elements of $X$ are basic commutators of weight one, ordered
   according to the total order previously chosen.
2. Having defined the basic commutators of weight less than $n$, the
   basic commutators of weight $n$ are the $c_k = [c_i, c_j]$, where:
   1. $c_i$ and $c_j$ are basic commutators and $\text{wt}(c_i) + \text{wt}(c_j) = n$,
      and
   2. $c_i > c_j$, and if $c_i = [c_s, c_t]$, then $c_j \geq c_t$.
3. The basic commutators of weight $n$ follow those of weight less
   than $n$. The basic commutators of weight $n$ are ordered among
   themselves lexicographically; that is, if $[b_1, a_1]$ and $[b_2, a_2]$ are
   basic commutators of weight $n$, then $[b_1, a_1] \leq [b_2, a_2]$ if and only
   if $b_1 < b_2$ or $b_1 = b_2$ and $a_1 < a_2$.

**Lemma 2.2.** (R. R. Struik [13]). Let $x_1, x_2, ..., x_r$ be any elements of
a group and let $v_1, v_2, ...$ be the sequence of basic commutators of weight
at least two in the $x_i$’s, in ascending order. Then,

\[(x_1x_2...x_r)^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} ... x_r^{\alpha_r} v_1^{f_1(\alpha)} v_2^{f_2(\alpha)} ... v_i^{f_i(\alpha)} ... ,\]

where $\{i_1, i_2, ..., i_r\} = \{1, 2, ..., r\}$, $\alpha$ is a nonnegative integer and

\[f_i(\alpha) = a_1 \binom{\alpha}{1} + a_2 \binom{\alpha}{2} + ... + a_{w_i} \binom{\alpha}{w_i}, \quad (I)\]

with $a_1, ..., a_{w_i} \in \mathbb{Z}$ and $w_i$ is the weight of $v_i$ in the $x_i$’s.
Lemma 2.3. (Struik [13]). Let $\alpha$ be a fixed integer and $G$ be a nilpotent group of class at most $k$. If $b_1, \ldots, b_r \in G$ and $r < k$, then

$$[b_1, \ldots, b_{i-1}, b_i^\alpha, b_{i+1}, \ldots, b_r] = [b_1, \ldots, b_r]^\alpha v_1 f_1(\alpha) v_2 f_2(\alpha) \ldots,$$

where $v_i$'s are commutators in $b_1, \ldots, b_r$ of weight strictly greater than $r$, and every $b_j$, $1 \leq j \leq r$, appears in each commutator $v_i$, the $v_i$'s listed in ascending order. The $f_i(\alpha)$'s are of the form (I), with $a_1, \ldots, a_{w_i} \in \mathbb{Z}$ and $w_i$ is the weight of $v_i$ (in the $b_j$'s) minus $(r-1)$.

Remark 2.4. Outer commutators on the letters $x_1, x_2, \ldots, x_n, \ldots$ are defined inductively as follows:

The letter $x_i$ is an outer commutator word of weight one. If $u = u(x_1, \ldots, x_s)$ and $v = v(x_{s+1}, \ldots, x_{s+t})$ are outer commutator words of weights $s$ and $t$, then $w(x_1, \ldots, x_{s+t}) = [u(x_1, \ldots, x_s), v(x_{s+1}, \ldots, x_{s+t})]$ is an outer commutator word of weight $s+t$ and will be written $w = [u, v]$.

It is noted by Struik [13] that Lemma 2.3 can be proved by a similar method, if $[b_1, \ldots, b_{i-1}, b_i^\alpha, b_{i+1}, \ldots, b_r]$ and $[b_1, \ldots, b_r]$ are replaced with outer commutators.

By a routine calculation we have the following useful fact.

**Lemma 2.5.** Let $p$ be a prime number and $k$ be a nonnegative integer. If $m = \lfloor \log_p k \rfloor$, then $p^t$ divides $\binom{p^m+t}{k}$, for all integers $t \geq 1$.

3. Main results

In order to prove the main result we need the following lemma.

**Lemma 3.1.** Let $G$ be a $p$-group of class $k$ and exponent $p^e$ with a free presentation $F/R$. Then, for any $c \geq 1$, every outer commutator of weight $w > c$ in $F/[R, \gamma F]$ has an order dividing $p^{e+m(c+k-w)}$, where $m = \lfloor \log_p k \rfloor$.

**Proof.** Since $\gamma_{k+1}(F) \subseteq R$, we have $\gamma_{c+k+1}(F) \subseteq [R, \gamma F]$. Also, for all $x$ in $F$ and $t \geq 0$ we have $x^{p^{e+t}} \in R$ and hence every outer commutator of weight $w > c$ in $F$, in which $x^{p^{e+t}}$ appears, belongs to $[R, \gamma F]$. Now, we use inverse induction on $w$ to prove the lemma. For the first step, $w = c + k$, the result follows by the above argument and Lemma 2.3.
Now, assume that the result is true, for all $l > w$. Put $\alpha = p^{e+m(c^l-w)}$ and let $u = [x_1, \ldots, x_w]$ be an outer commutator of weight $w$. Then, by Lemma 2.3 and Remark 2.4 we have
\[ [x_1^\alpha, \ldots, x_w] = [x_1, \ldots, x_w]^\alpha v_1^{f_1^i(\alpha)} v_2^{f_2^i(\alpha)} \cdots , \]
where the $v_i^{f_i^i(\alpha)}$ are as in Lemma 2.3. Note that $w < w_i = wt(v_i) \leq c+k$ modulo $[R, \ cF]$ and hence $f_i(\alpha) = a_1(\alpha) + a_2(\alpha) + \ldots + a_w(\alpha)$, where $k_i = w_i - w + 1 \leq c + k - w + 1 \leq k,$ for all $i \geq 1$. Thus, Lemma 2.5 implies that $p^{e+m(c^k-w-1)}$ divides the $f_i(\alpha)$'s. Now, by induction hypothesis $v_i^{f_i^i(\alpha)} \in [R, \ cF]$, for all $i \geq 1$. On the other hand, since $x_i^\alpha \in R$ and $w > c$, $[x_1^\alpha, \ldots, x_w] \in [R, \ cF]$. Therefore, $u^\alpha \in [R, \ cF]$ and this completes the proof.

**Theorem 3.2.** Let $G$ be a $p$-group of class $k$ and exponent $p^e$. Let $G = F/R$ be any free presentation of $G$. Then, the exponent of $\gamma_{c+1}(F)/[R, \ cF]$ divides $p^{e+m(k-1)}$, where $m = [\log_p k]$, for all $c \geq 1$.

**Proof.** It is easy to see that every element $g$ of $\gamma_{c+1}(F)$ can be expressed as $g = y_1 y_2 \cdots y_n$, where $y_i$'s are commutators of weight at least $c+1$. Put $\alpha = p^{e+m(k-1)}$. Now, Lemma 2.2 implies the identity
\[ g^\alpha = y_1^\alpha y_2^\alpha \cdots y_n^\alpha v_1^{f_1^i(\alpha)} v_2^{f_2^i(\alpha)} \cdots , \]
where $\{i_1, i_2, \ldots, i_n\} = \{1, 2, \ldots, n\}$ and $v_i^{f_i^i(\alpha)}$'s are as in Lemma 2.2. Then, the $v_i$'s are basic commutators of weight at least two and at most $k$ in the $y_i$'s modulo $[R, \ cF]$ (note that $\gamma_{c+k+1}(F) \subseteq [R, \ cF]$). Thus, Lemma 2.5 yields that $p^{e+m(k-2)}$ divides the $f_i(\alpha)$'s. Hence, $v_i^{f_i^i(\alpha)} \in [R, \ cF]$, for all $i \geq 1$ and $y_j^\alpha \in [R, \ cF]$, for all $1 \leq j \leq n$, by Lemma 3.1. Therefore, we have $g^\alpha \in [R, \ cF]$ and the desired result now follows.

Now, we are in a position to state and prove the main result of the paper.

**Theorem 3.3.** Let $G$ be a $p$-group of class $k$ and exponent $p^e$. Then, $\exp(M^{(c)}(G))$ divides $\exp(G)p^{m(k-1)}$, where $m = [\log_p k]$, for all $c \geq 1$.

**Proof.** Let $G = F/R$ be any free presentation of $G$. Then, $M^{(c)}(G) \leq \gamma_{c+1}(F)/[R, \ cF]$. Therefore, $\exp(M^{(c)}(G))$ divides $\exp(\gamma_{c+1}(F)/[R, \ cF])$. Now, the result follows by Theorem 2.3.
Note that the above result improves some previous bounds for the exponent of $M(G)$ and $M^{(c)}(G)$ as follows. Let $G$ be a $p$-group of class $k$ and exponent $p^e$, then we have the following improvements.

(i) If $\lfloor \log_p k \rfloor (k - 1)/k < e$, then $\exp(G)p^{\lfloor \log_p k \rfloor (k - 1)} < \exp(G)^{k-1}$. Hence, in this case our result is an improvement of Jones’s result [5]. In particular, our result improves the Jones’s one for every $p$-group of exponent $p^e$ and of class at most $p^e - 1$.

(ii) If $\lfloor \log_p k \rfloor (k - 1)/\lceil k/2 \rceil - 1 < e$, then $\exp(G)p^{\lfloor \log_p k \rfloor (k - 1)} < \exp(G)^{\lfloor k/2 \rfloor}$ which shows that in this case our result is an improvement of Ellis’s result [3]. In particular, our result improves the Ellis’s one for every $p$-group of exponent $p^e$ and of class $k < p^e/3$, for all $k \geq 3$, or of class $k < p^e/4$, for all $k \geq 4$.

(iii) If $\lfloor \log_p k \rfloor (k - 1)/2\lfloor \log_2 k \rfloor - 1 < e$, then $\exp(G)p^{\lfloor \log_p k \rfloor (k - 1)} < \exp(G)^{2\lfloor \log_2 k \rfloor}$. Thus, in this case our result is an improvement of Moravec’s result [11]. In particular, our result improves the Moravec’s one for every $p$-group of exponent $p^e$ and of class $k < e$, for all $k \geq 2$.

Corollary 3.4. Let $G$ be a finite $p$-group of class at most $p - 1$, then $\exp(M^{(c)}(G))$ divides $\exp(G)$, for all $c \geq 1$. In particular, $\exp(M(G))$ divides $\exp(G)$.

Note that the above corollary shows that the mentioned conjecture on the exponent of the Schur multiplier of a finite $p$-group holds for all finite $p$-group of class at most $p - 1$.

Remark 3.5. Let $G$ be a finite nilpotent group of class $k$. Then, $G$ is the direct product of its Sylow subgroups, $G = S_{p_1} \times \cdots \times S_{p_n}$. Clearly,

$$\exp(G) = \prod_{i=1}^n \exp(S_{p_i}).$$

By a result of G. Ellis [2, Theorem 5] we have

$$M^{(c)}(G) = M^{(c)}(S_{p_1}) \times \cdots \times M^{(c)}(S_{p_n}).$$

For all $1 \leq i \leq n$, put $m_i = \lfloor \log_{p_i} k \rfloor$. Then, by Theorem 3.3 we have

$$\exp(M^{(c)}(G)) \mid \exp(G)^{\prod_{i=1}^n p_i^{m_i(k-1)}}.$$
Hence, the conjecture on the exponent of the Schur multiplier holds for all finite nilpotent group $G$ of class at most $\operatorname{Max}\{p_1 - 1, \ldots, p_n - 1\}$, where $p_1, \ldots, p_n$ are all the distinct prime divisors of the order of $G$.

Acknowledgments

The authors would like to thank the referee for useful comments. This research was supported by a grant from Ferdowsi University of Mashhad; (No. MP87150MSH).

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