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Inversion formula for the non-uniformly attenuated x-ray transform for emission imaging in $\mathbb{R}^3$ using quaternionic analysis

S M Saberi Fathi

Université de Cergy-Pontoise, Laboratoire de Physique Théorique et Modélisation,
95302 Cergy-Pontoise, France

E-mail: majid.saberi@u-cergy.fr

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Abstract

In this paper, we present a new derivation of the inverse of the non-uniformly attenuated x-ray transform in three dimensions, based on quaternionic analysis. An explicit formula is obtained using a set of three-dimensional x-ray projection data. The result without attenuation is recovered as a special case.

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1. Introduction

When a radiopharmaceutical emits radiation of photon energy $E_0$, an ideal SPECT camera records only emitted photons, which arrive perpendicularly to its surface. We are dealing uniquely with photons of energy $E_0$; thus, we have to solve a simplified photon transport equation, which may be expressed as

$$\mathbf{n} \cdot (\nabla u_0)(\mathbf{r}, \mathbf{n}, E_0) = -a_0(\mathbf{r}, E_0)u_0(\mathbf{r}, \mathbf{n}, E_0) - f_0(\mathbf{r}, \mathbf{n}, E_0).$$

Here $u_0(\mathbf{r}, \mathbf{n}, E_0)$ represents the photon flux density in the direction $\mathbf{n}$ of energy $E_0$, i.e. number of photons per unit surface perpendicular to $\mathbf{n}$ per second. Recall that $a_0(\mathbf{r}, E_0)$ is the linear attenuation coefficient or rate of depletion per unit length traversed and finally $f_0(\mathbf{r}, \mathbf{n}, E_0)$ is the number of photons emitted in the direction $\mathbf{n}$ per unit volume matter (of the extended radiation source). For simplicity, the energy label $E_0$ will be omitted hereafter.

The aim is to solve this partial differential equation with an isotropic source term $f_0(\mathbf{r})$:

$$\mathbf{n} \cdot (\nabla u_0)(\mathbf{r}, \mathbf{n}) = -a_0(\mathbf{r})u_0(\mathbf{r}, \mathbf{n}) - f_0(\mathbf{r}),$$

where the unknown photon flux density is $u_0(\mathbf{r}, \mathbf{n})$. Reconstructing $f_0$ from the data $u_0(\mathbf{x}, \mathbf{n})$ is the main problem posed here.

In three dimensions without attenuation, the solution is represented by the ‘x-ray cone beam’, without restriction on the set of source points $\mathbf{x}$. This has been worked out
mathematically in [1–4]. The reconstruction formula contains the average of the x-ray data on the unit sphere of $\mathbb{R}^3$. The case of point sources lying on a space curve is given by [5–8]. Finally, among the large amount of indirect inversion procedures, the most well known for efficiency and appeal are those by Smith, who developed a technique that converts divergent beam data into parallel beam data and used its known inversion procedure [9] and by Grangeat, who made a conversion of x-ray data into three-dimensional Radon data before using Radon inversion [10].

Reconstructing $f_0$ from equation (2) in two dimensions has been worked out by Novikov [11]. In this paper, we show that the use of quaternion analysis leads to a new inversion formula for the non-uniformly attenuated x-ray transform in $\mathbb{R}^3$. Quaternions are higher dimensional generalization of complex numbers. Although not widely used, they provide elegant compact local formulation for electromagnetism, solid mechanics and some other fields in engineering [12]. Recently, quaternions have been used in integral transforms, for example, in geophysical processes [13] or in signal processing [14]. In imaging science, [15] gets an inversion formula for the x-ray transform without attenuation. In another work [16], the inversion of exponential x-ray transform is given. The generalization of these works for the non-uniform attenuation is the subject of this paper. As we see later, this generalization is not trivial, because the fundamental solution of the Dirac operator with the non-uniform function $(D + a(x))$ in quaternion analysis has been studied only for an approximate vector potential of the form [17]

$$\left\{ \frac{x - x^{(i)}}{x - x^{(i)}_3}, i = 1, 2, \ldots \right\}. \quad (3)$$

This is not realistic in practical applications. However, the case of constant ‘$a = \text{constant}$’ has been studied in [18, 19].

In the next section, we introduce some useful notions on the algebra of real quaternions $\mathbb{H}$ and collect the main results of quaternion analysis needed for our problem. Section 3 describes the derivation of the inversion formula giving the reconstructed function in terms of the x-ray data, and we give an interpretation of this new result. This paper ends with a conclusion and some perspectives to invert the x-ray transform in the presence of other effects.

2. Quaternions

Let $x = (x_1, x_2, x_3)$ be an element of $\mathbb{R}^3$, expressed in an orthonormal basis formed by three unit vectors $t_1$, $t_2$ and $t_3$ by $x = \sum_{m=1}^{3} x_m t_m$. The conventional vector space structure is given by a scalar (inner) product rule for the basis unit vectors, i.e. $(t_i \cdot t_m) = \delta_{im}$ and by a vector (cross) product, i.e. $t_1 \times t_2 = t_3$ with its cyclic permutations and the non-commutativity $t_m \times t_n = -t_n \times t_m$.

To this structure, one can add a new one

- by promoting the unit vectors to be imaginary units, i.e. $t_1^2 = t_2^2 = t_3^2 = -1$ and
- by introducing a non-commutative multiplication rule between them: $t_i t_j = -t_j t_i$ for $i \neq j$ and $t_i t_j t_k = t_k$ for all cyclic permutations of $(i, j, k)$.

Then to each $x = \sum_{m=1}^{3} x_m t_m$, as a three-dimensional vector, corresponds a new object $x$ (also called $\text{Vec} x$ by some authors), which has the same formal expression but with $t_m$ following the new multiplication rule. Consequently, the identification

$$x \in \mathbb{R}^3 \mapsto x = \sum_{m=1}^{3} x_m t_m \quad (4)$$
is an isomorphism of \( \mathbb{R}^3 \) onto the set of ‘vector parts’ \( \{ \text{Vec} \} \) of more general objects called quaternions by Hamilton [27].

In fact, a quaternion \( x \) has four components, i.e. besides its imaginary vector part, there is also a scalar part \( \text{Sc} \, x = x_0t_0 \), where \( t_0 \) is the real (or non-imaginary) unit part (usually identified with the real unit \( 1 = t_0 \in \mathbb{R} \)) and \( x_0 \in \mathbb{R} \), such that

\[
x = x_0t_0 + \sum_{m=1}^{3} x_m t_m = \text{Sc} \, x + \text{Vec} \, x = x_0t_0 + \mathbf{x}, \quad (x_0, x_1, x_2, x_3 \in \mathbb{R}). \tag{5}
\]

The set of quaternions with real components should be called \( \mathbb{H}(\mathbb{R}) \), but for simplicity, will be denoted by \( \mathbb{H} \).

Following [20], we give some of their properties:

- **Conjugate operation:**
  \[
x = x_0t_0 - \sum_{m=1}^{3} x_m t_m, \tag{6}
\]

- **Square norm:**
  \[
  |x|^2 = xx = x_0^2 + x_1^2 + x_2^2 + x_3^2, \tag{7}
\]

- **Inverse:**
  \[
  x^{-1} = \frac{\overline{x}}{|x|^2} \text{ if and only if } x \overline{x} \neq 0. \tag{8}
\]

Finally, the ordered product of two quaternions \( y = y_0t_0 + \mathbf{y} \) and \( x = x_0t_0 + \mathbf{x} \) is a quaternion \( w = yx = (\text{Sc} \, w + \text{Vec} \, w) \), where

\[
w_0 = \text{Sc} \, w = y_0x_0 - (\mathbf{y} \cdot \mathbf{x}) \quad \text{and} \quad w = \text{Vec} \, w = \mathbf{y}x_0 + y_0\mathbf{x} + \mathbf{y} \times \mathbf{x}. \tag{9}
\]

In particular, i.e. the ordered product of \( \mathbf{y} \) by \( \mathbf{x} \) is

\[
\mathbf{y} \mathbf{x} = -\mathbf{y} \cdot \mathbf{x} + \mathbf{y} \times \mathbf{x}. \tag{10}
\]

For our purposes, we do not require the full machinery of quaternionic analyticity as developed by Fueter and others [20, 21]. Here we are only concerned with analytic properties useful for imaging processes in \( \mathbb{R}^3 \) modeled by the x-ray transform. They are essentially extracted from [18, 22]:

\[
D = \sum_{j=1}^{3} \mathbf{i}_j \frac{\partial}{\partial x_j}. \tag{11}
\]

The quaternionic operator \( D \) has been given different names according to authors: Dirac operator for [18], three-dimensional Cauchy–Riemann operator for [12], Moisil–Teodorescu differential operator for [23], etc.

Inspection shows that it is related to the three-dimensional Laplace operator by \( \triangle = -D^2 \). The solutions of \( Df(x) = 0 \), called frequently left-monogenic \( \mathbb{H} \)-valued functions, satisfy many generalizations of classical theorems from complex analysis to higher dimensional context [22]. Given the elementary solution of the Laplace operator, \( \triangle E(x) = -D^2 E(x) = \delta(x) \), as

\[
E(x) = \frac{1}{4\pi |x|}, \tag{12}
\]

the elementary solution of \( D \) can be worked out as [18]

\[
K(x) = \sum_{j=1}^{3} K_j(x) t_j = -\frac{x}{4\pi |x|^3}, \quad x \neq 0, \tag{13}
\]

1 Quaternions with complex-valued components are called biquaternions and denoted by \( \mathbb{H}(\mathbb{C}) \).
where
\[
K_j(x) = -\frac{x_j}{4\pi |x|^3} \quad (j = 1, 2, 3).
\]
(14)

Note that \(K(x)\) is a \(\mathbb{H}\)-valued fundamental solution of \(D\) and therefore monogenic in \(G \setminus \{0\}\) where \(G \subset \mathbb{R}^3\).

Now, we write the generalized Leibniz formula in quaternions [18]:
\[
D(uw) = \pi Dw + (Du)w + 2SC(uD)w, \quad u, w \in \mathbb{H}(\mathbb{R}^4),
\]
where \(\mathbb{H}(\mathbb{R}^4)\) is the set of \(u\) and \(v\), which are \(\mathbb{H}\)-valued functions with the domain in \(\mathbb{R}^4\).

Consequently, there exists a three-dimensional Cauchy integral representation for continuous left-monogenic \(\mathbb{H}\)-valued functions on \(\overline{G}\) [22],
\[
(Ff)(x) := \int_{\Gamma} K(x - y)g(y)f(y) d\Gamma_y, \quad x \in G \setminus \Gamma,
\]
where \(g(y) = \sum_{j=1}^{3} \alpha_j(y) y_j\) is the quaternionic outward pointing unit vector at \(y\) on the boundary \(\partial G = \Gamma\), \(d\Gamma_y\) is the Lebesgue measure on \(\Gamma\). Moreover one has \(D(Fff)(x) = 0\).

The operator \(D\) has a right inverse, called the Teodorescu transform [24]. It is defined for all \(f(x) \in C(G, \mathbb{H})\) by
\[
(Tf)(x) := \int_{G} K(x - y) f(y) dy \quad x \in G \subset \mathbb{R}^3.
\]
(17)

Roughly speaking, \(D\) is a kind of directional derivative and \(T\) is just the integration, the right inverse of this directional derivative.

Conversely, for any \(f(x) \in C^1(G, \mathbb{H}) \cap C(\overline{G}, \mathbb{H})\), it can be shown that it satisfies the so-called Borel–Pompeiu formula [18]
\[
(Ff)(x) + (TD)f(x) = \begin{cases} f(x), & x \in G \\ 0, & x \in \mathbb{R}^3 \setminus \overline{G}. \end{cases}
\]
(18)

A generalization of the concept of Cauchy principal value for \((Ff)(x)\) can be introduced when the variable \(x\) is approaching the boundary \(\partial G = \Gamma\). For a given \(f\), at each regular point \(x' \in \Gamma\) [18], the non-tangential limit of the Cauchy integral representation can be written as
\[
\lim_{x \to x'} (Ff)(x) = \frac{1}{2} (\pm f(x') + (Sf)(x')),
\]
where
\[
(Sf)(x) = 2 \int_{\Gamma} K(x - y)g(y)f(y) d\Gamma_y
\]
(20)
is understood as a ‘quaternionic Cauchy principal value’ of the integral over the smooth boundary \(\Gamma\) because of the singularity of \(K(x)\) in the integrand.

A Plemelj–Sokhotskij-type formula for \(f\), relative to \(\Gamma\), [22, 24] can now be given as
\[
\begin{align*}
(i) \quad & \lim_{x \to x'} (Ff)(x) = (Pf)(x'), \\
(ii) \quad & \lim_{x \to x'} (Ff)(x) = -(Qf)(x'),
\end{align*}
\]
(21)
where \(P\) is the projection operator \((P^2 = P)\) onto \(\mathbb{H}\)-valued functions, which have a left-monogenic extension into the domain \(G\), and \(Q\) is the projection operator \((Q^2 = Q)\) onto \(\mathbb{H}\)-valued functions, which have a left-monogenic extension into the domain \(\mathbb{R}^3 \setminus \overline{G}\) and vanish at infinity.

\(P\) and \(Q\) can be given, in turn, an alternative form in terms of the quaternionic principal value operator \(S\) as
\[
P := \frac{1}{2} (I + S) \quad Q := \frac{1}{2} (I - S),
\]
(22)
with the following operator relations

\[ SP = P, \quad SQ = -Q, \quad S^2 = SS = I. \tag{23} \]

Finally, we define a trace operator \( \text{tr} \) as a restriction map for an \( \mathbb{H} \)-valued function \( f \) on \( \Gamma \), smooth boundary of \( G \in \mathbb{R}^3 \), by

\[ \text{tr} f = f|_\Gamma. \tag{24} \]

**Notation.** Here we review our notation in this paper. Only ‘bold’ letters are used for vectors or vector functions in \( \mathbb{R}^3 \), such as \( x \) or \( f(x) \). The index ‘zero’ indicates the scalar part of a quaternion or quaternion function, e.g. \( x_0 \) or \( a_0(x) \). Underlined bold letters are used for the vector part of the quaternions or quaternion functions, e.g. \( \underline{x} \) or \( \underline{f}(x) \). Operators with index ‘\( a \)’ are the operators with attenuation, e.g. \( T_a, X_a \).

**3. The x-ray transform and its inverse**

We are now in a position to tackle the inversion problem for the non-uniform attenuated x-ray transform of a physical density \( f_0(x) \). By definition, this transform consists of integrating \( f_0(x) \), assumed to be an integrable function with compact support in a convex set \( G \), along a straight line from the source point \( x \) to infinity in the direction of the unit vector \( n \), i.e.

\[ (X_a f_0)(x, n) = \int_0^{\infty} dt \, e^{-\mathcal{D}a_0(x)t} f_0(x + tn), \tag{25} \]

where

\[ \mathcal{D}a_0(x) = -\frac{1}{4\pi} \int_{\Omega_n} (Xa_0)(x, n) \, d\Omega_n, \tag{26} \]

where \( d\Omega_n \) is the area element of the unit sphere \( \Omega_n \) in \( \mathbb{R}^3 \) and \( (Xa_0) \) is the x-ray transform

\[ (Xa_0)(x, n) = \int_0^{\infty} dt \, a_0(x + tn). \tag{27} \]

In transmission modality, \( f_0 \) represents the attenuation map of the object under study, whereas in emission modality \( f_0 \) is its radiation activity density.

The next point is that if \( f_0(\infty) = 0 \), it can be verified that \((X_a f_0)(x, n)\) satisfies a very simple partial differential equation, namely

\[ (n \cdot \nabla x + a_0(x)) (X_a f_0)(x, n) = -f_0(x). \tag{28} \]

This can be checked if we let the \((n \cdot \nabla x + a_0(x))\) operator act under the integral sign. After a change of variables, the integrand just turns into the differential of \( f_0(x) \) under the integral sign. Equation (28) is in fact a simplified stationary photon transport equation with loss by attenuation function \( a_0(x) \) and without source or sink term [25]. Since \((n \cdot \nabla x + a_0(x))\) is a directional derivative plus the attenuated term, clearly its inverse is an integration\(^2\). The solution of this partial differential equation is subjected to the following boundary condition. For a given direction \( n \), because of the support hypothesis and because of the prescription on the direction of integration, \((X_a f_0)(x, n) = 0\), whenever \( x \) is on the boundary \( \Gamma = \partial G \) of \( G \) and \( n \) points outward of \( \Gamma \).

To obtain the solution of the above equation by using real analysis, we write the solution of the homogenous form of equation (28), i.e.

\[ (n \cdot \nabla x + a_0(x)) v_0(x, n) = 0 \tag{29} \]

\(^2\) This is not the second-order ultra-hyperbolic partial differential equation of John [26].
from which \( v_0(\mathbf{x}, \mathbf{n}) \) is obtained as

\[
v_0(\mathbf{x}, \mathbf{n}) = e^{-\int_{\mathbb{R}^3} G_0(\mathbf{x}-\mathbf{y}, \mathbf{n})v_0(\mathbf{y})dy}, \quad \mathbf{y} \in \mathbb{R}^3, \tag{30}
\]

where \( G_0(\mathbf{x} - \mathbf{y}, \mathbf{n}) \) is the Green’s function of the \((\mathbf{n} \cdot \nabla_x)\) operator.

At this point, we define \( u_0(\mathbf{x}, \mathbf{n}) \) in (28) as

\[
u_0(\mathbf{x}, \mathbf{n}) = C_0(\mathbf{x})v_0(\mathbf{x}, \mathbf{n}). \tag{31}\]

By substituting \( u_0(\mathbf{x}, \mathbf{n}) \) into equation (28), we have

\[
C_0(\mathbf{x}) = \int_{\mathbb{R}^3} \tilde{G}_0(\mathbf{x} - \mathbf{y}, \mathbf{n})v_0^{-1}(\mathbf{y}, \mathbf{n})f_0(\mathbf{y})dy, \tag{32}\]

and

\[
u_0(\mathbf{x}, \mathbf{n}) = -\int_{\mathbb{R}^3} R_0(\mathbf{x}, \mathbf{y}, \mathbf{n})f_0(\mathbf{y})dy, \tag{33}\]

where

\[
R_0(\mathbf{x}, \mathbf{y}, \mathbf{n}) = v_0(\mathbf{x}, \mathbf{n})\tilde{G}_0(\mathbf{x} - \mathbf{y}, \mathbf{n})v_0^{-1}(\mathbf{y}, \mathbf{n}). \tag{34}\]

We will use a similar method in the quaternion analysis to obtain an inversion solution for equation (28).

### 3.1. Quaternion solution

By considering \( \mathbf{n} \) independent of \( \mathbf{x} \), we can rewrite equation (28) in the following form:

\[
\mathbf{n} \cdot (\nabla + \mathbf{a})u_0(\mathbf{x}) = -f_0(\mathbf{x}), \tag{35}\]

where we define \( \mathbf{a} := a_0\mathbf{n} \).

We would like to use the machinery of quaternion analysis to obtain the inversion of the three-dimensional x-ray transform. The idea is to consider equation (28) as part of an inhomogeneous equation (11), with an \( \mathbb{H} \) valued ‘source’ function \( f = f_0(\mathbf{x}) + \mathbf{f}(\mathbf{x}) \) on its right-hand side for an unknown scalar function \( u_0(\mathbf{x}) \). As can be checked, the quaternionic product rule (9) yields

\[
\mathbf{n}D_\mathbf{a}u_0(\mathbf{x}) = \mathbf{f}(\mathbf{x}), \tag{36}\]

where \( D_\mathbf{a} \) is defined as follows:

\[
D_\mathbf{a} = D + \mathbf{a}. \tag{37}\]

Explicitly equation (36) has the following form:

\[
\mathbf{n}D_\mathbf{a}u_0(\mathbf{x}) = -\mathbf{n} \cdot (\nabla_x + \mathbf{a}(\mathbf{x}))u_0(\mathbf{x}) + \mathbf{n} \times (\nabla_x + \mathbf{a}(\mathbf{x}))u_0(\mathbf{x}) = f_0(\mathbf{x}) + \mathbf{f}(\mathbf{x}), \tag{38}\]

which leads to a set of two equations for \( u_0 \):

\[
(\mathbf{n} \cdot \nabla_x + a_0(\mathbf{x}))u_0(\mathbf{x}) = -f_0(\mathbf{x}) \tag{39}\]

\[
(\mathbf{n} \times \nabla_x)u_0(\mathbf{x}) = \mathbf{f}(\mathbf{x}), \tag{39}\]

the first one being exactly the one of the x-ray transform. By solving equation (39), we can obtain the solution of equation (39) as a by product, for given \( f_0(\mathbf{x}) \), first. Then, \( \mathbf{f}(\mathbf{x}) \) can be computed from the curl term and the gradient term of the solution.

From (39) the case \( \mathbf{f} = (\mathbf{n} \times \nabla_x)u_0 = 0 \) means that the areolar derivative of \( u_0 \) is equal to zero. In the other words, the derivative of \( u_0 \) on the plane perpendicular to \( \mathbf{n} \) is equal to zero, or \( u_0 \) is constant on the plane perpendicular to \( \mathbf{n} \).

Considering \( \mathbf{f} = 0 \), equation (36) becomes

\[
\mathbf{n}D_\mathbf{a}u_0(\mathbf{x}) = f_0(\mathbf{x}), \quad \mathbf{x} \in G, \tag{40}\]
in which we can easily see that the above equation is the transport equation (28) in the quaternion formalism.

Now, by multiplying equation (40) by \(-n\) from the left-hand side we have

\[
D_a u_0(x, n) = -n f_0(x),
\]

where we used ‘nn = -1’.

We choose the Dirichlet boundary condition for equation (41), i.e. the boundary value of \(u_0\) on the smooth boundary surface of \(\Gamma\) is equal to \(w_0\) which is a monogenic function. Hence, we have

\[
D_a u_0(x) = -n f_0(x), \quad x \in G
\]

where \(w_0\) is a scalar function.

Before obtaining the solution of the above equation, we derive an explicit form of \(T a\):

\[
(T a)(x) = \int_G K(x - y) a(y) \, dy
\]

\[
= \frac{1}{4\pi} \int_G \frac{x - y}{|x - y|^3} a(y) \, dy = \frac{1}{4\pi} \sum_{i,j=1}^3 \int_G \frac{x_i - y_i}{|x - y|^3} a_j(y) \, dy \, i \, j.
\]

The photon transport in the \(n\) direction of the x-ray source located at \(x\) required that \(y = x + nt\), where \(t \in \mathbb{R}^+\). Consequently, the volume element \(dy \) in spherical coordinates becomes \(d\gamma = t^2 \, dt \, d\Omega_n\), where \(d\Omega_n\) is the area element of the unit sphere \(\Omega_n\) in \(\mathbb{R}^3\). Then, we have

\[
(T a)(x) = \frac{1}{4\pi} \int_{\Omega_n} \int_{\mathbb{R}^+} \frac{nt}{|nt|^3} a_0(x + nt) t^2 \, d\Omega_n \, dt
\]

\[
= -\frac{1}{4\pi} \int_{\Omega_n} \left( \int_{\mathbb{R}^+} a_0(x + nt) \, dt \right) \, d\Omega_n,
\]

where we have \(a = na_0\) and \(nn = -1\). Now, we use the definition of the x-ray transform of component \(f_0\):

\[
Xa_0(x, n) := \int_{\mathbb{R}^+} a_0(x + nt) \, dt.
\]

Then, we have

\[
Da_0(x) := (Ta)(x) = -\frac{1}{4\pi} \int_{\Omega_n} (Xa_0)(x, n) \, d\Omega_n.
\]

Here we showed that the \((Ta) = (Ta_n)_0\) is a scalar function. Thus, the following theorem gives the solution \(u_0\) of the above equation when \((Ta)\) is a scalar function. This is sufficient to solve our problem.

**Theorem.** Assuming that \((Ta) = Sc((Ta)_0) = (Ta)_0\), and for \(f_0\) and \(u_0\), \(v_0\) differentiable or weakly differentiable functions in a normed space with domain in \(G \subset \mathbb{R}^3\), the solution of equation (42) is given by

\[
v_0 = e^{-Ta_0}
\]

\[
u_0 = -\frac{1}{4\pi} \int_{\Omega_n} (Xa_0)(x, n) \, d\Omega_n.
\]

Then, we have

\[
Qa_0(x) := (Ta)(x) = -\frac{1}{4\pi} \int_{\Omega_n} (Xa_0)(x, n) \, d\Omega_n.
\]

where \(Ta_0 f_0 = v_0 T v_0^{-1} f_0 \) and \(Faw_0 = v_0 F v_0^{-1} w_0\). With the condition

\[
Qa u_0 = -\operatorname{tr}(v_0 T v_0^{-1} f_0),
\]

\[
Qa u_0 = -\operatorname{tr}(v_0 T v_0^{-1} f_0),\]
where $Q_w v_0 = v_0 Q v_0^{-1}$ the above condition follows from Plemelj–Sokhotski’s formula. This means that there exists an extension onto the domain $\mathbb{R}^3 \setminus G$.

**Proof.** $v_0$ is the solution of the following homogenous equation:

$$D_a v_0 = (D + a) v_0 = D v_0 + a v_0 = 0,$$

where the solution of the above equation is equal to

$$v_0 = e^{-T a}.$$

In appendix A, we solve equation (51) using real analysis. We can verify the above solution by substituting expression (52) into equation (51):

$$D e^{-T a} + a v_0 = -(D T a) e^{-T a} + a v_0 = -(D T a) v_0 + a v_0 = -av_0 + a v_0 = 0,$$

where we used the fact that $DT v_0 = v_0$ in $G$, which means that $T$ is the right inverse of $D$ [22]. Now, we introduce the general solution of (42) as $u_0^{(p)} := v_0 C_0$, where $C_0$ is a function with domain in $G \subset \mathbb{R}^3$ and $\text{tr} C_0 = 0$. We replace it in equation (42). Thus, we have

$$D(v_0 C_0) + a v_0 C_0 = -n f_0.$$

Finally, (54) gives

$$D(v_0 C_0) = v_0 D C_0 + (D v_0) C_0 = v_0 D C_0 - a v_0 C_0 = -n f_0. $$

Consequently, (55) gives

$$v_0 D C_0 = -n f_0. $$

By acting $v_0^{-1}$ on the above equation we obtain

$$D C_0 = -v_0^{-1} n f_0. $$

Now, taking into account that ‘$\text{tr} C_0 = 0$’ (which means that ‘$FC_0 = 0$’), $C_0$ has the following form:

$$C_0 = -T v_0^{-1} n f_0. $$

In a similar way where it was shown before that $(T a) = (T n a_0)$ is a scalar function, one can show that $(T v_0 n f_0)$ is a scalar function. Finally, $u_0^{(p)}$ is equal to

$$u_0^{(p)} = v_0 C_0 = -v_0 T v_0^{-1} n f_0 = -T a n f_0. $$

The proper solution $u_0^{(p)}$ of equation (42) which takes the value $u_0$ on the boundary, i.e. equation (43) is

$$\text{tr} u_0 = \text{tr} u_0^{(p)} + \text{tr} u_0^{(p)} = -tr (v_0 T v_0^{-1} n f_0) + tr u_0^{(p)}. $$

Using condition (50), we obtain

$$\text{tr} u_0 = Q_a w_0 + tr u_0^{(p)} = w_0. $$

Consequently,

$$\text{tr} u_0^{(p)} = (I - Q_a) w_0 = v_0 (I - Q) v_0^{-1} w_0 = v_0 P v_0^{-1} w_0 = P_a w_0. $$

where $P_a = v_0 P v_0^{-1}$. Thus, from the definition of $P w_0 = tr (F w_0)$ [18], $u_0^{(p)}$ is equal to

$$u_0^{(p)} = v_0 F v_0^{-1} w_0 = F_a w_0. $$

Finally, by considering $u_0^{(p)}$ and $u_0^{(p)}$, $u_0 = u_0^{(p)} + u_0^{(p)}$ is obtained by equation (49). We verify our solution by acting $D_a$ on equation (42). Then, we have

$$D_w u_0 = -D_w v_0 T v_0^{-1} n f_0 + D_a (F_a w_0) = -v_0 D_a T v_0^{-1} n f_0 + (D_a v_0) T v_0^{-1} n f_0,$$

where we used the generalized Leibniz formula (15) and
\[ D(F_u w) = D(v_0 F_{v_0^{-1}} w_0) = (Dv_0) F_{v_0^{-1}} w_0 + v_0 D F_{v_0^{-1}} w_0 = -\mathbf{a} F_{v_0^{-1}} w_0 + 0, \quad (65) \]

where we use \( D F_{v_0^{-1}} w_0 = 0 \), which means that \( (F_{v_0^{-1}} w_0) \) is a monogenic function \([22]\). Thus, as a result, we can conclude that \( D_a F_a w_0 = 0 \).

Thus, equation (64) is obtained as
\[ D_a u_0 = -v_0 v_1^{-1} \mathbf{n} f_0 - v_0 a T v_0^{-1} \mathbf{n} f_0 + (v_0 \mathbf{a}) T v_0^{-1} \mathbf{n} f_0 = -\mathbf{n} f_0, \quad (66) \]
where in the first term on the right-hand side we used \( DTu_0 = u_0 \).

Now, we check the solution at the boundary condition (43). Thus, by substituting \( u_0 \) from (48) into (43), we obtain
\[ w_0 = -\operatorname{tr}(v_0 T v_0^{-1} \mathbf{n} f_0 + F_u w_0) = -\operatorname{tr}(v_0 T v_0^{-1} \mathbf{n} f_0) + \operatorname{tr}(F_a w_0) = -\operatorname{tr}(v_0 T v_0^{-1} \mathbf{n} f_0) + P_a w_0, \quad (67) \]
where in the last equation we use: \( \operatorname{tr}(F_u w_0) = (F_a w_0) \Gamma = P_a w_0 \). Then, (67) yields
\[ -\operatorname{tr}(v_0 T v_0^{-1} \mathbf{n} f_0) = (I - P_a) w_0 = Q_a w_0. \quad (68) \]

\[ \square \]

3.2. The x-ray representation

Now, we reconstruct \( f_0 \) by using equation (49). As shown in (66), \( D_a u_0 = D_a (-v_0 v_1^{-1} \mathbf{n} f_0 + F_a w_0) = -\mathbf{n} f_0 \); thus,
\[ \mathbf{n} f_0 = D_a (v_0 T v_0^{-1} \mathbf{n} f_0) \quad (69) \]
which gives \( \mathbf{n} f_0 \). Now, replacing \( \mathbf{n} f_0 \) by \( f_0 \), we can obtain \( f_0 \):
\[ f_0 = D_a (v_0 T v_0^{-1} f_0) \quad (70) \]
or by using \( D_a v_0 = 0 \) (equation (51)), we have
\[ f_0 = v_0 D_a (T v_0^{-1} f_0). \quad (71) \]

To get the explicit form of \( f_0(x) \) in terms of the imaging data set, we first compute the Teodorescu transform of \( v_0^{-1} f_0 \). Thus, by using equation (47), \( v_0 \) is written as
\[ v_0(x) = e^{-T (x)} = e^{\frac{\theta}{2}} \int_{\Omega_n}(X_{\mathbf{n}}(x, \mathbf{n})d\Omega_n = e^{\mathbf{D}_{\mathbf{n}}(x)}. \quad (72) \]

Then, we obtain \((Tv_0^{-1} f_0)(x)\) by using the same method with which we obtained \((T \mathbf{a})\) in equation (47):
\[ (Tv_0^{-1} f_0)(x) = \int_G K(x - y)(v_0^{-1} f_0)(y) \ dy \]
\[ = \frac{1}{4\pi} \int_{\Omega_n} \mathbf{n}[X(v_0^{-1} f_0)](x, \mathbf{n}) d\Omega_n. \quad (73) \]

Now, we define the attenuated x-ray transform as follows:
\[ (X_{\mathbf{a}} f)(x, \mathbf{n}) := \int_{\mathbb{R}^2} e^{-\mathbf{D}_{\mathbf{n}}(x + \mathbf{n})} f_0(x + \mathbf{n}) \ df. \quad (74) \]

Thus, equation (73) is rewritten as
\[ (Tv_0^{-1} f_0)(x) = \frac{1}{4\pi} \int_{\Omega_n} \mathbf{n}(X_{\mathbf{a}} f_0)(x, \mathbf{n}) d\Omega_n. \quad (75) \]

Hence, \( f_0 \) is obtained by
\[ f_0(x) = v_0 D_a (Tv_0^{-1} f_0)(x) = \frac{1}{4\pi} e^{\mathbf{D}_{\mathbf{n}}(x)} D_a \int_{\Omega_n} \mathbf{n}(X_{\mathbf{a}} f_0)(x, \mathbf{n}) d\Omega_n. \quad (76) \]
As earlier in this paper we have introduced $(X_{a_0}f_0)(x, n)$, this is a monogenic (analytic) function on $\Omega_1n$; thus, $[a_0(x)(X_{a_0}f_0)(x, n)]_{\Omega_1n} \to 0$ as $t \to 0$. Finally, the reconstruction formula for $f_0$ is obtained as

$$f_0(x) = -\frac{1}{4\pi} e^{\mathcal{D}_a(x)} \int_{\Omega_1n} (n \cdot \nabla_x) (X_{a_0}f_0)(x, n) \, d\Omega_n. \quad (77)$$

**Case $a_0 = 0$.** In the special case where $a_0 = 0$, $\mathcal{D}_a = 0$. Equation (77) is given as

$$f_0(x) = \frac{1}{4\pi} \int_{\Omega_n} (n \cdot \nabla_x) (Xf_0)(x, n), \quad (78)$$

where $(Xf_0)(x + nt) = \int_0^\infty dt f_0(x + nt)$ is the x-ray transform without attenuation. Here the result is the one obtained by [15]. A comparison of the above formula with other results given by [15] is presented in appendix B.

### 4. Conclusion

In this paper, by using quaternion analysis we have obtained a successful inverse formula for the non-uniform x-ray transform in three dimensions. As we have shown in equation (77) for the case without attenuation $a_0 = 0$ has a different form, but it is essentially equivalent to the result obtained many years ago in previous works.

### Appendix A.

In this appendix we compute a solution for equation (51) using real analysis. Equation (51) can be written as

$$\nabla_x v_0 + na_0 v_0 = 0. \quad (A.1)$$

Multiplying by $n$ the left-hand side yields

$$(n \cdot \nabla_x) v_0 + a_0 v_0 = 0, \quad (A.2)$$

or

$$(n \cdot \nabla_x) \ln v_0 = -a_0, \quad (A.3)$$

where by introducing $\phi_0 := \ln v_0$ and $\rho_0 := -a_0$, we have

$$(n \cdot \nabla_x) \phi_0 = \rho_0. \quad (A.4)$$

The above equation is a stationary transport equation with the source term $\rho_0$ and without attenuation. The solution of this equation is known to be given by a divergent x-ray transform of the data [15, 28], i.e.

$$\phi_0(x, n) = (X\rho_0)(x, n) := \int_{\mathbb{R}} \rho_0(x + nt) \, dt. \quad (A.5)$$

Now, by replacing $\rho_0$ and $\phi_0$ by $\ln v_0$ and $-a_0$, respectively, we obtain

$$v_0(x, n) = e^{-\int_{\mathbb{R}} a_0(x + nt) \, dt}, \quad (A.6)$$

which is the same result as obtained from quaternion analysis. This solution is obtained without restriction on $a_0$. Thus, equation (51) or (A.1) does not impose any restriction on $a_0$. 

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Appendix B. Comparison of formula (78) with known results [15]

In [3], the inverse formula of the x-ray transform in three dimensions is given by

\[ f_0(x) = -\frac{1}{2\pi^2} \Delta_x \mathcal{R}^1 \int_{\Omega_n} (Xf_0)(x, \mathbf{n}) \, d\Omega_n = -\frac{1}{2\pi^2} \Delta_x \mathcal{R}^1 \int_{\Omega_n} u_0(x, \mathbf{n}) \, d\Omega_n, \]  

(B.1)

where

\[ \mathcal{R}^1 f_0(x) = \frac{1}{2\pi^2} \int \frac{1}{|x - y|^2} f_0(y) \, dy. \]  

(B.2)

Setting \( y = x + \mathbf{n}t \) in the above equation, we find

\[ \mathcal{R}^1 f_0(x) = \frac{1}{2\pi^2} \int_{\Omega_n} \int_{\mathbb{R}^n} f_0(x + \mathbf{n}t) \, dt \, d\Omega_n = \frac{1}{2\pi^2} \int_{\Omega_n} (Xf_0)(x, \mathbf{n}) \, d\Omega_n. \]  

(B.3)

In equation (B.1) we may define \( \frac{1}{4\pi} \int_{\Omega_n} u_0(x, \mathbf{n}) \, d\Omega_n = (u_0)_n \) as the average of \( u_0 \) over a unit ball. Thus, by using the above relation, equation (B.3) can be written as

\[ f_0(x) = -\Delta_x \mathcal{R}^1 (u_0)_n = -\frac{2}{\pi} \Delta_x (X(u_0)_n)(x) = -\frac{2}{\pi} \int_{\Omega_n} \Delta_x (Xu_0)(x, \mathbf{n}) \, d\Omega_n. \]  

(B.4)

Here we obtain another form for \( \Delta_x (Xu_0)_n \). From equation (39), which expresses \( f \) as \( f = \mathbf{n} \times \nabla_x u_0 \), we deduce that \( \nabla_x \cdot f = 0 \). Thus,

\[ \nabla_x \cdot (\nabla_x \times \mathbf{n}u_0) = \nabla_x \cdot \nabla_x u_0 = \nabla_x (\mathbf{n} \cdot \nabla_x u_0) = \mathbf{n} \Delta_x u_0 + \nabla_x f_0 = 0, \]  

(B.5)

which yields

\[ \Delta_x u_0 = - (\mathbf{n} \cdot \nabla_x) f_0. \]  

(B.6)

Substitution of this expression into equation (B.4) gives an alternative form of the reconstructed \( f_0 \):

\[ f_0(x) = \frac{2}{\pi} \int_{\Omega_n} (\mathbf{n} \cdot \nabla_x) (Xf_0)(x, \mathbf{n}) \, d\Omega_n. \]  

(B.7)

which, up to a normalization factor, has the same form as equation (78).

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