ON THE COMMUTING GRAPH ASSOCIATED WITH
THE SYMMETRIC AND ALTERNATING GROUPS

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The commuting graph of a group G, denoted by \( \Gamma(G) \), is a simple undirected graph whose vertices are all non-central elements of G and two distinct vertices \( x, y \) are adjacent if \( xy = yx \). The commuting graph of a subset of a group is defined similarly. In this paper we investigate the properties of the commuting graph of the symmetric and alternating groups and subsets of transpositions and involutions in the symmetric groups.

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1. Notations

In this paper we consider simple graphs which are undirected, with no loops or multiple edges. For any graph \( \Gamma \), we denote the sets of vertices and the edges of \( \Gamma \) by \( V(\Gamma) \) and \( E(\Gamma) \), respectively. Two distinct vertices are adjacent if they are joined by an edge in \( \Gamma \). The degree \( d_\Gamma(v) \) of a vertex \( v \) in \( \Gamma \) is the number of edges incident to \( v \) and if the graph is understood, then we denote \( d_\Gamma(v) \) simply by \( d(v) \). The order of \( \Gamma \) is \( |V(\Gamma)| \) and its maximum and minimum degrees will be denoted by \( \Delta(\Gamma) \) and \( \delta(\Gamma) \), respectively. A graph \( \Gamma \) is regular if the degrees of all vertices of \( \Gamma \) are the same. A subset \( X \) of the vertices of \( \Gamma \) is called a clique if the induced subgraph on \( X \) is a complete graph. The maximum size of a clique in a graph \( \Gamma \) is called the clique number of \( \Gamma \) and is denoted by \( \omega(\Gamma) \). A subset \( X \) of the vertices of \( \Gamma \) is called an independent set if the induced subgraph on \( X \) has no edges. The maximum size of an independent set in a graph \( \Gamma \) is called the independence number of \( \Gamma \) and is denoted by \( \alpha(\Gamma) \). A path \( P \) is a sequence of distinct vertices \( v_0, v_1, \ldots, v_k \) whose terms are alternately adjacent vertices. In this case \( P \) is called a path between \( v_0 \) and \( v_k \).
and the number \(k\) is called the length of \(P\). In addition, if \(v_0\) and \(v_4\) are adjacent in \(\Gamma\), then \(v_0v_1 \cdots v_4v_0\) is called a cycle of the length \(k + 1\). The length of the shortest cycle in \(\Gamma\) is called girth of \(\Gamma\) and denoted by \(girth(\Gamma)\). A graph \(\Gamma\) is connected if there is a path between each pair of distinct vertices of \(\Gamma\), otherwise \(\Gamma\) is said to be disconnected. A component of a graph \(\Gamma\) is one of its maximal connected subgraphs.

If \(u\) and \(v\) are vertices in \(\Gamma\), then the distance between \(u\) and \(v\), denoted by \(d(u, v)\), is the length of the shortest path between \(u\) and \(v\). If such a path exists; otherwise, we define \(d(u, v) = \infty\). The diameter of a graph \(\Gamma\) is

\[
\text{diam}(\Gamma) = \max\{d(u, v) \neq \infty \mid u \text{ and } v \text{ are distinct vertices of } \Gamma\}.
\]

Let \(G\) be a finite group and \(S \subseteq G\) be any non-empty non-abelian subset of \(G\) and as usual define \(Z(S)\), the center of \(S\), to be the set \(\{x \in S \mid xa = ax, \text{ for all } a \in S\}\). For each \(x \in S\), the centralizer of \(x\) in \(S\), denoted by \(C_S(x)\), is defined as \(\{y \in S \mid yx = xy\}\). The commuting graph of \(G\), denoted by \(\Gamma(G)\), is a graph whose vertex set is the set \(G \setminus Z(G)\) and join two distinct vertices \(x\) and \(y\) whenever \(xy = yx\). The commuting graph of \(S\) is defined similarly. Obviously for a subset \(S\) of a group \(G\) and each \(v \in V(\Gamma(S))\), we have \(d(v) = |C_S(v)| - 1 - |Z(S)|\). The prime graph of \(G\), denoted by \(\Pi(G)\), is a graph whose vertex set is \(\pi(G)\), the set of all prime divisors of \(G\), and two distinct vertices \(p\) and \(q\) are adjacent if and only if \(G\) contains an element of order \(pq\) (see [22]).

We denote the symmetric group and the alternating group on \(n\) letters by \(S_n\) and \(A_n\), respectively. Also \(T_n\) and \(I_n\) denote the sets of all transpositions and involutions in \(S_n\), respectively. For an element \(\sigma \in S_n\), the support of \(\sigma\), denoted by \(\supp(\sigma)\), is the set of all letters \(k \in \{1, 2, \ldots, n\}\) for which \(\sigma(k) \neq k\). If \(\sigma\) is a cycle in \(S_n\) then the length of \(\sigma\), denoted by \(l(\sigma)\), is the number \(|\supp(\sigma)|\). Our notations are standard and you can find them for example in [8,17,20].

2. Introduction

We can associate a graph to a group or a subset of a group in different ways to investigate algebraic properties of the group using the properties of the graph.

In [12,22], the authors investigated the connectivity of the prime graph of finite simple groups. In [1,14], the authors discussed the non-commuting graph of groups, which is the complement of the commuting graph of the group and in [2], the authors introduced the non-cyclic graph of a group and investigated some of its properties. In [4-6], the authors investigated some properties of the commuting graph associated with the conjugacy classes of involutions in the symmetric groups. In [9], the authors characterized most of the finite simple groups by their commuting graphs and proved for those groups:

Conjecture 2.1. Let \(M\) be a finite simple group. If \(G\) is any finite group such that \(\Gamma(M) \cong \Gamma(G)\), then we have \(M \cong G\).

In [19], the authors investigated some conditions on the connectivity of the commuting graph of finite simple groups and proved that for all finite classical simple groups \(G\), \(\text{diam}(\Gamma(G)) \leq 10\), where \(\Gamma(G)\) is connected. Hence we are motivated to the following conjecture:

Conjecture 2.2. There is a natural number \(b\) such that if \(G\) is a finite non-abelian group with \(\Gamma(G)\) connected, then \(\text{diam}(\Gamma(G)) \leq b\).

In Sec. 3, we find the necessary and sufficient conditions for the connectivity of the commuting graph of the symmetric and alternating groups and subsets of transpositions and involutions in the symmetric groups and find their diameter, clique number, girth and a good bound for their independence number. Also we find the number of their components in the disconnected case and maximum and minimum degrees for these graphs. We will see that the diameter of the commuting graph of the symmetric and alternating groups can be at most 5 which proves the above conjecture for these groups. In Sec. 4, we find the relation between the commuting graph and the prime graph of finite centerless groups.

3. The Commuting Graph of the Symmetric and Alternating Groups

For a natural number \(m > 1\) which is not a prime, let \(g_{m, p}\) be the greatest proper divisor of \(m\). It is obvious that \(\sqrt{m} \leq g_{m, p} < m\) and if \(d > 1\) is a proper divisor of \(m\), then \(\frac{m}{d} \leq g_{m, p}\), because \(\frac{m}{d}\) is also a proper divisor of \(m\).

Theorem 3.1. For \(n \geq 3\), \(\Gamma(S_n)\) is connected if and only if \(n\) and \(n - 1\) are not primes and in this case \(\text{diam}(\Gamma(S_n)) \leq 5\) and this bound is sharp.

Proof. Let \(p \geq 3\) be a prime number. It is obvious that \(\Gamma(S_p)\) and \(\Gamma(S_{p+1})\) are not connected. For the converse, suppose that \(n\) and \(n - 1\) are not prime numbers. We prove that \(\Gamma(S_n)\) is connected. Obviously we have \(n \geq 9\). Suppose \(\sigma \in S_n\). We claim that there exists a path of length at most 2 in \(\Gamma(S_n)\) between \(\sigma\) and a cycle of length at most \(\frac{n}{2}\). The following cases can occur:

1. \(\sigma\) is a cycle. In this case, if \(\sigma\) is a cycle of length at most \(n - 2\), then there are distinct letters \(a, b\) such that \(a, b \not\in \supp(\sigma)\) and so \(\sigma(a, b) = (a, b)\sigma\) and \(\sigma - (a, b)\) is the claimed path in \(\Gamma(S_n)\). If \(\sigma\) is a cycle of length \(n - 1\), then let \(g = g_{n, 2} = g_{n-1}\). Then \(g \geq 3\) and \(\sigma^g\) is the product of \(g\) disjoint cycles \(\sigma_1, \sigma_2, \ldots, \sigma_g\) of length \(n - 1\) and \(\frac{n}{2} \leq \sigma^g\). We have \(\sigma_1^{k} = \sigma^{k}\) and \(\sigma_1 = \sigma_1\sigma^k\) and so we have the path \(\sigma - \sigma_1 - \sigma^k - \sigma_1\) in \(\Gamma(S_n)\) which is the claimed path because \(l(\sigma_1) \leq \frac{n}{2}\). If \(\sigma\) is a cycle of length \(n\), with the same proof as the case \(n - 1\), we obtain a path with the claimed property.

2. \(\sigma\) is the product of two or more cycles. In this case, suppose \(\sigma_1\) is the cycle with minimum length appearing in \(\sigma\). Thus \(l(\sigma_1) \leq \frac{n}{2}\) and we have \(\sigma_1\sigma_1 = \sigma_1\sigma_1\). Since \(n \geq 9\), there are distinct letters \(a, b\) such that \(a, b \not\in \supp(\sigma_1)\) and thus \(\sigma_1(a, b) = (a, b)\sigma_1\). Therefore \(\sigma - \sigma_1 - (a, b)\) is the claimed path in \(\Gamma(S_n)\).
Now let \( \sigma \) and \( \mu \) be two nontrivial elements in \( S_n \). If \( \sigma \) is a cycle of length at most \( n-2 \), then there are permutations \((a, b)\), \(\sigma_1\) and \(\mu_2\) in \( S_n \) with \( l(\sigma_2) \leq \frac{n}{2} \) such that \( \sigma = (a, b) \) and \( \mu = \mu_1 \mu_2 \) are paths in \( \Gamma(S_n) \). Hence there are distinct letters \( c, d \) such that \( \sigma = (a, b) - (c, d) \) and \( \mu_1 \mu_2 = (c, d) \mu \) is a path in \( \Gamma(S_n) \) and so \( d(\sigma, \mu) \leq 5 \).

Suppose \( \sigma \) and \( \mu \) are not cycles of length at most \( n-2 \). If \( C_1, C_2 \) and \( C_3 \) are sets of cycles of length \( n-1 \), cycles of length \( n \) and permutations in \( S_n \), which are products of two or more disjoint cycles, respectively, then \( \sigma, \mu \in C_1 \cup C_2 \cup C_3 \).

If \( \sigma, \mu \in C_1 \cup C_2 \), then without loss of generality we can assume that \( g_\sigma \leq g_\mu \). Let \( \sigma^\sigma = \sigma_1 \sigma_2 \cdots g_\sigma \) and \( \mu^\mu = \mu_1 \mu_2 \cdots g_\mu \) such that \( \sigma_1, \sigma_2, \ldots, \) are disjoint cycles of length \( \frac{n}{2} \) and \( \mu_1, \mu_2, \ldots, \) are disjoint cycles of length \( \frac{n}{2} \). We have \( l(\sigma_1) = \frac{n}{2} \leq g_\sigma \leq g_\mu \), because \( \frac{n}{2} \) is a proper divisor of \( l(\sigma) \). If \( l(\sigma_1) < g_\mu \), then according to the pigeonhole principle, there is some \( 1 \leq j \leq g_\mu \), such that \( \sigma_1 \mu_j = \mu_1 \mu_j \). Therefore \( \sigma = \sigma^\sigma - \sigma_1 - \mu_j - \mu^\mu - \mu \) is a path in \( \Gamma(S_n) \) and so \( d(\sigma, \mu) \leq 5 \).

If \( l(\sigma_1) = g_\mu \), then \( g_\sigma = g_\mu \) and \( l(\sigma) = (g_\mu)^2 \), so \( l(\sigma) = l(\mu) \). If \( \sigma_1 \mu_j = \mu_1 \mu_j \), then \( \sigma = \sigma^\sigma - \sigma_1 - \mu_j - \mu^\mu - \mu \) is a path in \( \Gamma(S_n) \) and so \( d(\sigma, \mu) \leq 5 \).

Otherwise, without loss of generality we can assume that \( \sigma_1 = (1 2 \cdots l) \) and \( \mu_1 \neq (j 1 \cdots l) \) for all \( 1 \leq j \leq l \). Let \( \alpha_\sigma = (a_1 a_2 \cdots a_l) \) and \( \alpha_\mu = (b_1 b_2 \cdots b_l) \). It is easily checked that \( \alpha_\sigma \alpha_\mu = \mu^\mu \alpha_\sigma \). Therefore \( \sigma = \sigma^\sigma - \sigma_1 - \alpha - \mu^\mu - \alpha \) is a path in \( \Gamma(S_n) \) and so \( d(\sigma, \mu) \leq 5 \).

If \( \sigma, \mu \in C_1 \) and \( \mu \in C_2 \), then put \( \sigma^{\sigma^\sigma} = \sigma_1 \sigma_2 \cdots g_\sigma \), and \( \mu = \mu_1 \mu_2 \cdots \mu_l \) such that \( \sigma_1, \sigma_2, \ldots \) are disjoint cycles of length \( \frac{n}{2} \) and \( \mu_1, \mu_2, \ldots \) are disjoint cycles with \( l(\mu_1) \leq \cdots \leq l(\mu_l) \). Hence there are distinct letters \( c, d \) such that \( \sigma = \sigma^\sigma - \sigma_1 - (c, d) - \mu_1 - \mu \) is a path in \( \Gamma(S_n) \) and so \( d(\sigma, \mu) \leq 5 \). The proof for the case \( \sigma \in C_2 \) and \( \mu \in C_3 \) is the same.

If \( \sigma, \mu \in C_3 \), then put \( \sigma = \sigma_1 \sigma_2 \cdots g_\sigma \), such that \( \sigma_1, \sigma_2, \ldots \) are disjoint cycles with \( l(\sigma_1) \leq \cdots \leq l(\sigma_\sigma) \) and \( \mu = \mu_1 \mu_2 \cdots \mu_l \) such that \( \mu_1, \mu_2, \ldots \) are disjoint cycles with \( l(\mu_1) \leq \cdots \leq l(\mu_\mu) \). Let \( a, b \) be disjoint letters such that \( \sigma_1 = (a b) \sigma_1 \). Hence there are distinct letters \( c, d \) such that \( \sigma = \sigma_1 - (a b) - (c d) - \mu_1 - \mu \) is a path in \( \Gamma(S_n) \) and so \( d(\sigma, \mu) \leq 5 \).

Therefore \( \Gamma(S_n) \) is connected and \( \text{diam}(\Gamma(S_n)) \leq 5 \). This bound is sharp, because if \( \sigma = (1 2 3 4 5 6 7 8) \) and \( \mu = (1 2 3 4 5 6 7 8) \), then by using GAP [18], we obtain \( d(\Gamma(S_n), \sigma, \mu) = 5 \) and \( \text{diam}(\Gamma(S_n)) = 5 \).

**Remark 3.2.** The above theorem shows that for each prime number \( p \geq 3 \), graphs \( \Gamma(S_p) \) and \( \Gamma(S_{p+1}) \) are disconnected. With a simple calculation, \( \Gamma(S_5) \) has four components \( \{(1, 2), (1, 3), (1, 4), (2, 3)\} \) and \( \Gamma(S_6) \) has five components with representatives \( \{(1, 2), (1, 3), (1, 4), (1, 5), (2, 3)\} \), and \( \{(1, 2), (1, 3), (1, 4), (2, 3)\} \) and these components have \( 15, 22, 22, 2 \) and \( 2 \) vertices, respectively.

In the next lemma, we obtain the number of components of \( \Gamma(S_p) \) and \( \Gamma(S_{p+1}) \).
Corollary 3.6. If \( n \geq 5 \), then we have \( \Delta(\Gamma(S_n)) = d((1 \, 2)) = 2(n - 2)! - 2 \) and \( \delta(\Gamma(S_n)) = d((1 \, 2 \ldots \, n - 1)) = n - 3 \).

Proof. Keep the notations of the previous lemma. We have

\[
\Delta(\Gamma(S_n)) = \max_{K \subseteq S_n} \{|C_{S_n}(x)| - 2 \mid x \in S_n, x \neq \text{id}\} = \max_{K \subseteq S_n} f_n(K) - 2 = 2(n - 2)! - 2
\]

With a similar proof we obtain the second statement.

If \( G \) is a centerless group, then \( A \) is an abelian subgroup of maximal order of \( G \) if and only if \( A \setminus \{1\} \) is a greatest clique in \( \Gamma(G) \) if and only if \( \omega(\Gamma(G)) = |A| - 1 \). In [7], the authors proved the following theorem:

Theorem 3.7. Let \( G \) be an abelian subgroup of maximal order of the symmetric group \( S_n \). Then

1. \( G \cong Z_3 \) if \( n = 3k \);
2. \( G \cong Z_2 \times Z_3^k \) if \( n = 3k + 2 \);
3. either \( G \cong Z_4 \times Z_3^{k-1} \) or \( G \cong Z_2 \times Z_3 \times Z_3^{k-1} \) if \( n = 3k + 1 \).

This implies that the following groups are abelian subgroups of \( S_n \) of maximal order:

1. \( \{(1 \, 2 \, 3), (4 \, 5 \, 6), \ldots, (3k - 2 \, 3k - 1 \, 3k)\} \) if \( n = 3k \);
2. \( \{(1 \, 2), (3 \, 4 \, 5), \ldots, (3k \, 3k + 1 \, 3k + 2)\} \) if \( n = 3k + 2 \);
3. \( \{(1 \, 2), (3 \, 4), (5 \, 6 \, 7), \ldots, (3k - 1 \, 3k + 1)\} \) if \( n = 3k + 1 \).

Then we have the following Lemma:

Lemma 3.8. For \( n \geq 4 \), we have girth\((\Gamma(S_n)) = 3\).

\[
\omega(\Gamma(S_n)) = \begin{cases} 3^{k-1} - 1, & n = 3k; \\ 4.3^{k-1} - 1, & n = 3k + 1; \\ 2.3^{k-1}, & n = 3k + 2. 
\end{cases}
\]

Proof. It is obvious that \( \{(1 \, 2), (3 \, 4), (1 \, 2) \times (3 \, 4)\} \) is a cycle in \( \Gamma(S_n) \) and so girth\((\Gamma(S_n)) = 3\).
In [15], the author proved that if \( n = n(G) \) is the maximum size of a non-commuting subset of the group \( G \), then \( |G : Z(G)| \leq \alpha(G) \) for some constant \( c \). Obviously \( n(G) = \alpha(G) \). Hence we have the following corollary:

**Corollary 3.9.** For \( n \geq 4 \), we have

\[
\alpha(\Gamma(S_n)) \geq \frac{\ln(n)}{\ln(c)}.
\]

Of course we can use GAP to obtain \( \alpha(\Gamma(S_n)) \) for small \( n \)’s. For example

\[
\begin{align*}
\alpha(S_n) &= 10, & n = 4; \\
&= 31, & n = 5; \\
&= 175, & n = 6.
\end{align*}
\]

**Theorem 3.10.** For \( n \geq 5 \), \( \Gamma(A_n) \) is connected if and only if \( n \), \( n - 1 \), and \( n - 2 \) are not primes and in this case \( \text{diam}(\Gamma(A_n)) \leq 5 \) and this bound is sharp.

**Proof.** If \( p \geq 5 \) is a prime number and \( G \) is one of the groups \( A_p, A_{p+1} \) or \( A_{p+2} \), then it is obvious that the commuting graph of \( G \) is disconnected. For the converse, suppose \( n \), \( n - 1 \), and \( n - 2 \) are not primes, then \( n \geq 10 \). If \( n = 10 \), then by using GAP, we can see that for all \( \sigma, \mu \in \Gamma(A_{10}) \) we have \( d(\sigma, \mu) \leq 5 \) and so \( \text{diam}(\Gamma(A_{10})) \leq 5 \). Hence assume that \( n > 10 \), therefore \( n \geq 16 \). Let \( \sigma, \mu \in A_n \), be a permutation. If \( |\text{supp}(\sigma)| \leq n - 3 \), then choose distinct letters \( a, b, c \notin \text{supp}(\sigma) \) and \( -a, -b, -c \) is a path in \( \Gamma(A_n) \). Otherwise, the following cases occur for \( \sigma \):

1. \( \sigma \) is a cycle of length more than \( n - 3 \). If \( \sigma \) is a cycle of length \( n - 2 \), then \( \sigma = (a b c) \) which is not a prime. Thus there are disjoint cycles \( a_2, \ldots, a_n \), where \( a_i = a_{n-i} \), \( \sigma = a_2 \cdots a_n \). Therefore \( \sigma = a_2 \cdots a_n \) is a path in \( \Gamma(A_n) \). In addition we have \( \frac{n-2}{2} \leq \frac{n}{2} \). For the cases where \( \sigma \) is a cycle of length \( n - 1 \) or \( n \) we use the same method.

2. \( \sigma \) is a cycle of length \( n \). Let \( i \geq 2 \) and \( a = a_1 \cdots a_i \) is the cycle decomposition of \( \sigma \) to disjoint cycles. The number of odd cycles in this decomposition must be even (probably zero). Two cases can occur:

   a. At least one even cycle, namely \( a_1 \), appears in the decomposition of \( \sigma \) and so \( \sigma a_1 = \sigma a_2 \sigma \) in \( A_n \). On the other hand, there must be at least one even cycle or two odd cycles in the decomposition of \( \sigma \). Therefore \( |\text{supp}(\sigma)| \leq n - 3 \) and there are distinct letters \( a, b, c \notin \text{supp}(\sigma) \) and hence \( \sigma = (a b c) \sigma \). Therefore \( \sigma = a_2 \cdots a_n \sigma \) is a path in \( \Gamma(A_n) \). Otherwise, \( \sigma = a_2 \cdots a_n \). Suppose \( l(\sigma) \leq 2l(\sigma_2) \leq \cdots \leq l(\sigma_4) \). If \( l(\sigma_1) = l(\sigma_2) = 2 \), then \( \sigma = a_1 \sigma_2 \) is a path in \( \Gamma(A_n) \). Otherwise, \( l(\sigma_2) = 2k \geq 4 \). Let \( \sigma = a_1 \cdots a_n \) be disjoint decomposition of \( \sigma_2 \) to transpositions. Therefore \( \sigma = (a_1 \cdots a_n) \) is a path in \( \Gamma(A_n) \). We have proved that there is a path of length at most two between \( \sigma \) and a permutation \( \sigma' \in A_n \) such that \( |\text{supp}(\sigma')| \leq \frac{n}{2} \).

Now let \( \sigma \) and \( \mu \) be two non-trivial elements in \( A_n \). If \( |\text{supp}(\sigma)| \leq n - 3 \), then there are permutations \( (a b c), \mu_1 \) and \( \mu_2 \) in \( A_n \) with \( |\text{supp}(\mu)| \leq \frac{n}{2} \) such that \( \sigma = (a b c) \mu_1 - \mu_2 \mu_1 \mu_2 \) are paths in \( \Gamma(A_n) \). Hence there are distinct letters \( d, e, f \) such that \( \sigma = (d e f) \mu_1 - \mu_2 \mu_1 \mu_2 \) is a path in \( \Gamma(A_n) \) and so \( d(\sigma, \mu) \leq 5 \). Suppose \( |\text{supp}(\sigma)| \leq n - 2 \) and \( |\text{supp}(\mu)| \geq 2 \). Let \( C_1, C_2, C_3, C_4, C_5 \) be subsets of the set of elements \( a \in A_n \) with \( |\text{supp}(\sigma)| \geq n - 2 \) including cycles of length \( n - 2 \), cycles of length \( n - 1 \), cycles of length \( n \), permutations which are products of two or more disjoint even cycles, permutations that their decomposition to disjoint cycles include at least two transpositions and permutations that their decomposition to disjoint cycles include some odd cycle, but at most one transposition, respectively.

Let \( \sigma, \mu \in C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5 \cup C_6 \). Then \( |\sigma' \mu| = 1 \).\( \sigma' \mu \) is disjoint cycles of length \( i \leq \frac{n}{2} \). If \( \mu \in C_1 \cup C_2 \cup C_3 \), then with a proof similar to the proof of Theorem 3.1.1, we can prove that \( d(\sigma, \mu) \leq 5 \). If \( \mu \in C_4 \), then \( \mu = \mu_1 \mu_2 \cdots \mu_k \) such that \( k \geq 2 \) and \( \mu_1, \mu_2, \ldots, \mu_k \) are disjoint even cycles with \( l(\mu_1) \leq l(\mu_2) \cdots \leq l(\mu_k) \). Therefore \( l(\mu_1) \leq \frac{n}{2} \)

\[
n - |\text{supp}(|\text{supp}(\sigma)| \cup \text{supp}(\mu))| \geq n - l(\sigma_1) - l(\mu) \geq 2.
\]

Hence \( n - |\text{supp}(\sigma)| \cup \text{supp}(\mu)| \geq 3 \) and there are distinct letters \( a, b, c \) such that \( (a b c) \) commutes with \( \mu_1 \) and \( \mu_2 \). Therefore \( d(\sigma, \mu) \leq 5 \). If \( \mu \in C_5 \), then \( \mu = \mu_1 \mu_2 \cdots \mu_k \) such that \( k \geq 2 \) and \( \mu_1, \mu_2, \ldots, \mu_k \) are disjoint cycles of length \( i \leq \frac{n}{2} \). Therefore \( l(\mu_1) \leq l(\mu_2) \cdots \leq l(\mu_k) \). Hence \( l(\mu_1) \leq \frac{n}{2} \).

According to the pigeonhole principle, there is some \( i \leq 4 \), such that \( \text{supp}(\sigma) \cap \text{supp}(\mu) = \emptyset \). Therefore \( d(\sigma, \mu) \leq 5 \). Let \( \sigma \in C_4 \), then \( \sigma = \sigma_1 \cdots \sigma_k \) such that \( k \geq 2 \) and \( \sigma_1, \sigma_2, \ldots, \sigma_k \) are disjoint even cycles with \( l(\mu_1) \leq l(\mu_2) \cdots \leq l(\mu_k) \). Therefore \( l(\mu_1) \leq \frac{n}{2} \). If \( \mu \in C_4 \), then \( \mu = \mu_1 \mu_2 \cdots \mu_k \) such that \( j \geq 2 \) and \( \mu_1, \mu_2, \ldots, \mu_k \) are disjoint even cycles with \( l(\mu_1) \leq l(\mu_2) \cdots \leq l(\mu_k) \). Therefore \( l(\mu_1) \leq \frac{n}{2} \). If \( \mu \in C_5 \), then \( \mu = \mu_1 \mu_2 \cdots \mu_k \) such that \( j \geq 2 \) and \( \mu_1, \mu_2, \ldots, \mu_k \) are disjoint cycles of length \( l(\mu_1) = l(\mu_2) = 2 \). Therefore there are distinct letters \( a, b, c \) such that \( \mu_1, \mu_2 \) commute with \( (a b c) \). Hence \( \sigma = (a b c) \mu_1 - \mu_2 \mu_1 \mu_2 \) is a path in \( \Gamma(A_n) \) and so \( d(\sigma, \mu) \leq 5 \). If \( \mu \in C_6 \), then
transpositions. We have \(|\supp(\mu)| = 2s + 2u \geq 14\) and so \(s + u \geq 7\). According to the pigeonhole principle, there are distinct numbers \(1 \leq a, b \leq s + u\) such that \(t_{a1} \cap t_{b1}\) and \(t_{a2} \cap t_{b2}\) commute. Hence \(\sigma - (a_2' - t_{a2} \cdot t_{a2}' - (a_2')^2 - \mu) - \mu = \) a path in \(\Gamma(A_m)\) and so \(d(\sigma, \mu) \leq 5\).

Thus \(\Gamma(A_m)\) is connected and \(\text{diam}(\Gamma(A_m)) \leq 5\). This bound is sharp, because if \(\sigma = (1 2 3 4 5 6 7 8 9)\) and \(\mu = (1 3 5 7 9 4 6 8 10)\), then by using GAP, we obtain \(d_{\Gamma(A_9)}(\sigma, \mu) = 5\) and so \(\text{diam}(\Gamma(A_{10})) = 5\).

**Remark 3.11.** The above theorem shows that for each prime number \(p \geq 5\), graphs \(\Gamma(A_p)\), \(\Gamma(A_{p+1})\) and \(\Gamma(A_{p+2})\) are disconnected. With a simple calculation, \(\Gamma(A_2)\) has five components and \(\Gamma(A_3)\) has 21 components. In the next Lemma, we obtain the number of components of \(\Gamma(A_m)\) for all natural numbers \(n \geq 9\), when it is disconnected:

**Lemma 3.12.** (a) If \(p \geq 11\) and \(p - 2\) is not prime, then \(\Gamma(A_p)\) has \((p-2)! + 1\) components.

(b) If \(p \geq 13\) and \(p - 2\) is prime, then \(\Gamma(A_p)\) has \((p-2)! + \frac{(p-2)!}{2!} + 1\) components.

(c) If \(p \geq 11\), then \(\Gamma(A_{p+1})\) has \((p+1)! - 1\) components.

(d) If \(p \geq 7\) and \(p + 2\) is not prime, then \(\Gamma(A_{p+2})\) has \(\frac{(p+2)!}{2!} + 1\) components.

**Proof.** Let \(m, n, l\) be natural numbers such that \(m \neq l\) and \(m \geq n \geq l\). Set \(A_m^m = A_n^m|\{\sigma \in A_n^m| \sigma^2 = 1\}\) and \(A_{n}^{m} = A_n^m|\{\sigma \in A_n^m| \sigma^2 = 1\text{ or } \sigma^2 = 1\}\) and suppose \(p\) is a prime number. Similar to the proof of Lemma 3.3, we have:

(a) \(\Gamma(A_p^p)\) is a connected graph and \(\text{diam}(\Gamma(A_p^p)) \leq 6\) with a simple calculation, the number of components of \(\Gamma(A_p^p)\) is \((p-2)!\). The result immediately follows.

(b) \(\Gamma(A_p^{p-2})\) is a connected graph and \(\text{diam}(\Gamma(A_p^{p-2})) \leq 6\) and the number of components of \(\Gamma(A_p\setminus A_p^{p-2})\) is \((p-2)! + \frac{(p-2)!}{2!} + 1\). We get the result immediately.

(c) \(\Gamma(A_{p+1}^{p+1})\) is a connected graph and \(\text{diam}(\Gamma(A_{p+1}^{p+1})) \leq 6\) and the number of components of \(\Gamma(A_{p+1}\setminus A_{p+1}^{p+1})\) is \((p+1)! - 1\). Hence the result follows immediately.

(d) \(\Gamma(A_{p+2}^{p+2})\) is a connected graph and \(\text{diam}(\Gamma(A_{p+2}^{p+2})) \leq 6\) and the number of components of \(\Gamma(A_{p+2}\setminus A_{p+2}^{p+2})\) is \(\frac{(p+2)!}{2!} + 1\). Therefore we have the result.

**Remark 3.13.** We can use GAP to check that:

\[
\Delta(\Gamma(A_m)) = \begin{cases} 
(1, 2, 3, 4, 5, 6) & n = 7, \\
(1, 2, 3) & n = 5, \\
(1, 2, 3) & n = 6, \\
(1, 2, 3) & n = 8, \\
(1, 2) & n = 9, \\
(1, 2) & n = 10, \\
(1, 2) & n = 11, \\
(1, 2) & n = 12, \\
(1, 2) & n = 13, \\
(1, 2) & n = 14, \\
(1, 2) & n = 15, \\
(1, 2) & n = 16, \\
(1, 2) & n = 17, \\
(1, 2) & n = 18, \\
(1, 2) & n = 19, \\
(1, 2) & n = 20, \\
(1, 2) & n = 21. 
\end{cases}
\]
and for \( n = 4, 5 \)

\[
\delta(\Gamma(A_n)) = d((1 2 3)) = 1.
\]

We will prove that:

(a) If \( n \geq 9 \), then \( \Delta(\Gamma(A_n)) = d((1 2 3, 3 4 n - 1)) = \frac{3}{2}(n - 3)! - 2 \).

(b) If \( n \geq 6 \) is even, then \( \delta(\Gamma(A_n)) = d((1 2)(3 4 \cdots n)) = n - 4 \) and if \( n \) is odd, then \( \delta(\Gamma(A_n)) = d((1 2)(3 4 \cdots n - 1)) = n - 5 \).

First we need the following Lemma.

**Lemma 3.14.** Let \( x \in A_n \) be a non-identity element with the cyclic structure \( k_1 \cdot k_2 \cdots k_n \). Then \( |C_{A_n}(x)| = \frac{1}{2}|C_{S_n}(x)| \) if and only if \( x \) commutes with an odd permutation in \( S_n \) if and only if there exists an even natural number \( j \), \( 1 \leq j \leq n \), such that \( k_j \geq 2 \).

**Proof.** It is easy to prove that \( |C_{A_n}(x)| = \frac{1}{2}|C_{S_n}(x)| \) if and only if \( x \) commutes with an odd permutation in \( S_n \) (see [10]). Suppose that there exists an even natural number \( j \) such that \( k_j \geq 2 \) and that \( \sigma \) is one of the cycles of length \( j \) that appears in the cycle decomposition of \( x \). Then \( \sigma x \sigma = x \) and \( \sigma \) is an odd permutation in \( S_n \). Now suppose that there exists an odd natural number \( j \) such that \( k_j \geq 2 \) and that \( \lambda \) and \( \mu \) are two of the cycles of length \( j \) that appear in the cycle decomposition of \( x \). Suppose \( \lambda = (a_1 a_2 \cdots a_j) \) and \( \mu = (b_1 b_2 \cdots b_j) \) and set \( \sigma = (a_1 b_1)(a_2 b_2) \cdots (a_j b_j) \).

Then \( \sigma \) is an odd permutation in \( S_n \) and \( (\lambda \mu) \sigma = (\lambda \mu \sigma) \) and therefore \( \sigma x \sigma = x \). For the converse, suppose \( |C_{A_n}(x)| = \frac{1}{2}|C_{S_n}(x)| \) and by contraposition, suppose that there exists no even natural number \( 1 \leq j \leq n \) such that \( k_j \geq 2 \) and no odd natural number \( 1 \leq j \leq n \) such that \( k_j \geq 2 \). Therefore \( k_j = 0 \) if \( i \) is even and \( k_j = 0 \) or \( 1 \) if \( j \) is odd. Suppose \( j_1, j_2, \ldots, j_r \) are natural numbers with \( 1 \leq j_1 < j_2 < \cdots < j_r \leq n \) such that \( k_{j_1} = 1 \) (\( 1 \leq j \leq r \)) and \( k_t = 0 \) for \( t \notin \{j_1, j_2, \ldots, j_r\} \). Therefore \( x = \sigma_j x \sigma_j \cdots \sigma_{j_r} \), such that \( \sigma_j, \sigma_{j_2}, \ldots, \sigma_{j_r} \) are disjoint cycles with length \( j_1, j_2, \ldots, j_r \), respectively. If \( C = \langle \sigma_j, \sigma_{j_2}, \ldots, \sigma_{j_r} \rangle \) is the internal direct product of subgroups of \( S_n \), then obviously we have \( C \subseteq C_{A_n}(x) \subseteq C_{S_n}(x) \). But \( |C_{S_n}(x)| = k_{j_1} \cdot k_{j_2} \cdots k_{j_r} = j_1 j_2 \cdots j_r = |C| \). Thus \( C_{A_n}(x) = C_{S_n}(x) \) and this is a contradiction and the proof is complete. \( \Box \)

**Corollary 3.15.** (a) If \( n \geq 9 \), then \( \Delta(\Gamma(A_n)) = d((1 2 3)) = \frac{3}{2}(n - 3)! - 2 \).

(b) If \( n \geq 6 \) is even, then \( \delta(\Gamma(A_n)) = d((1 2)(3 4 \cdots n)) = n - 4 \) and if \( n \) is odd, then \( \delta(\Gamma(A_n)) = d((1 2)(3 4 \cdots n - 1)) = n - 5 \).

**Proof.** Let

\[
X_n = \left\{ (k_1, k_2, \ldots, k_n) \in (M_n) \mid k_1 < n, \sum_{i=1}^{n} i k_i = n, \sum_{i=1}^{n} k_{2i} \text{ is even} \right\}.
\]

It is obvious that if \( x \) is an element in \( S_n \) with cyclic structure \( 1^{k_1} 2^{k_2} \cdots n^{k_n} \), then \( x \in A_n \) if and only if \( (k_1, k_2, \ldots, k_n) \in X_n \). Also if \( (k_1, k_2, \ldots, k_n) \in X_n \), then we must have \( k_1 \leq n - 3 \). Define sets \( P_{n,1} \), \( P_{n,2} \) and \( P_{n,3} \) as follows:

\[
P_{n,1} = \{(k_1, k_2, \ldots, k_n) \in X_n \mid \text{there exists an even number } i \text{ such that } k_i \geq 1 \};
P_{n,2} = \{(k_1, k_2, \ldots, k_n) \in X_n \mid \text{there exists an odd number } j \text{ such that } k_j \geq 2 \};
P_{n,3} = X_n \setminus (P_{n,1} \cup P_{n,2}).
\]

Define the function \( f_n \) as Lemma 3.5. We have

\[
\Delta(\Gamma(A_n)) = \max \left\{ \frac{1}{2} f_n(K) \mid K \in P_{n,1} \cup P_{n,2} \right\} \cup \{f_n(K) \mid K \in P_{n,3}\} - 2 \]

and

\[
\delta(\Gamma(A_n)) = \min \left\{ \frac{1}{2} f_n(K) \mid K \in P_{n,1} \cup P_{n,2} \right\} \cup \{f_n(K) \mid K \in P_{n,3}\} - 2.
\]

To prove \( a \), we use induction on \( n \geq 9 \). We can simply use GAP, to check that \( a \) is true for \( n = 9, 10 \). Suppose that \( n \geq 10 \) and \( a \) is true for every \( 5 \leq m \leq n \). We have

\[
P_{n+1,2} \subseteq C_{X_n} \subseteq \{k_1, k_2, \ldots, k_n \} \in X_n + 1 \text{ is an arbitrary element. If } \]

\[
k_{n+1} = 1, \text{ then } f_n(A_{n+1}) = n + 1 \leq \frac{3}{2}(n - 2). \]

Also if \( k_n = 1 \), then \( n \) must be an odd number and we must have \( k_1 = 1 \) and \( k_i = 0 \) for \( 1 \leq i \leq n \). Therefore \( K \in P_{n+1,2} \) and we have \( f_n(A_{n+1}) = n \leq \frac{3}{2}(n - 2) \). Hence we can assume that \( k_{n+1} = k_n = 0 \). First let \( K \in P_{n+1,2} \setminus P_{n+2,2} \). There exists an even number \( 1 \leq i \leq n \), such that \( k_i \geq 1 \). If \( k_2 \geq 2 \), then we have \( k_1 + 2 + 2(k_2 - 2) + 3k_3 + 3 \cdots + 3(n - 1)k_{n-1} = n - 1 \) and thus \( k_1 + 2 + 2 - 1, \ldots, k_{n-1}, k_n \in P_{n+1,2} \). Therefore \( \frac{1}{2} f_n(A_{n+1}) = n - 4 \leq \frac{3}{2}(n - 4) \) and so \( \frac{1}{2} f_n(A_{n+1}) = \frac{3}{2}(n - 4) \). We can simply check that for \( n \geq 10 \) we have \( k_2(k_2 - 1) \leq n - 2 \) and therefore \( \frac{1}{2} f_n(A_{n+1}) \leq \frac{3}{2}(n - 4) \). If \( k_2 \leq 1 \), then we can assume that \( i \leq 4 \). Therefore \( (k_1 + i - 3, k_2 + k_3, k_4, \ldots, k_n) \in X_n \) and hence \( (k_1 + i - 3, k_2 + k_3, k_4, \ldots, k_n) \in A_n \). Therefore \( \frac{1}{2} f_n(A_{n+1}) = \frac{3}{2}(n - 4) \). We can simply check that for \( n \geq 10 \) we have \( k_2(k_2 - 1) \leq n - 2 \) and therefore \( \frac{1}{2} f_n(A_{n+1}) \leq \frac{3}{2}(n - 4) \).
Lemma 3.16. For \( n \geq 4 \), we have \( \vartheta(\Gamma(A_n)) = 3 \), \( \omega(\Gamma(A_n)) = 4 \) and
\[
\begin{align*}
3^{n-1} - 1, & \quad n = 3k; \\
4.3^{3k-1} - 1, & \quad n = 3k + 1; \\
16.3^{2k-2} - 1, & \quad n = 3k + 2 \text{ and } n \neq 5,
\end{align*}
\]
for \( n \neq 5 \).

Proof. Obviously \( \{(1,2),(3,4),(1,2),(4,3)\} \) is a cycle in \( \Gamma(A_n) \) and so \( \text{girth}(\Gamma(A_n)) = 3 \). Now, \( \omega(\Gamma(A_n)) \) is the size of an abelian subgroup of \( A_n \) of maximal order minus one. In Theorem 1.1 in [21], it is proved that an abelian subgroup of \( A_n \) of maximal order is conjugate to one of the following groups:

1. \( \{(1,2,3),(4,5,6),\ldots,(3k - 2,3k - 1,3k)\} \) if \( n = 3k; \\
2. \( \{(1,2,3),(4,5),(1,2,4),(3,6,7),\ldots,(3k - 1,3k,3k + 1)\} \) if \( n = 3k + 1; \\
3. \( \{(1,2,3),(4,5),(2,3,4),(5,6,7),(3,6,8),(2,3,5),(4,7,6,10,11),\ldots,(3k - 2,3k - 1,3k)\}\) if \( n = 3k + 2 \)

and the second statement is proved.

Also, we can use [15], to prove the following Corollary:

Corollary 3.17. For \( n \geq 4 \), we have
\[
\alpha(\Gamma(A_n)) \geq \frac{\ln(n^2)}{\ln(c)}.
\]

In [3], the authors discussed the commuting graph of subsets of rings. Here we investigate the commuting graph of two interesting subsets of the symmetric group, \( I_n \) and \( T_n \), and find out some information about their commuting graph:

Theorem 3.18. For \( n \geq 5 \), \( \Gamma(T_n) \) is a connected graph and \( \text{diam}(\Gamma(T_n)) = 2 \). In addition, \( \Gamma(T_n) \) is a regular graph of degree \( (n-2) \) and so \( \delta(\Gamma(T_n)) = \Delta(\Gamma(T_n)) = (n-2) \). Also \( \omega(\Gamma(T_n)) = n-1 \) and \( \omega(\Gamma(T_n)) = \frac{\ln(n)}{\ln(c)} \) and if \( n \geq 6 \), then \( \text{girth}(\Gamma(T_n)) = 3 \), but \( \text{girth}(\Gamma(T_5)) = 5 \).

Proof. Suppose \((a,b),(c,d) \in T_n\) are two distinct transpositions. If they are disjoint, then \((a,b) = (c,d) \) is a path in \( \Gamma(T_n) \), otherwise there are distinct letters \( x, y \) so that \((a,b) = (x,y) \) is a path in \( \Gamma(T_n) \). Therefore \( \Gamma(T_n) \) is connected and we have \( \text{diam}(\Gamma(T_n)) = 2 \). But \( d(1,2), (2,3) \) is a 2 and therefore \( \text{diam}(\Gamma(T_n)) = 2 \). If \( n \geq 5 \), then \((1,2),(3,4)\) and \((5,6)\) form a cycle and then \( \text{girth}(\Gamma(T_5)) = 3 \). In the case \( n = 5 \), \((1,2),(3,4),(1,5),(2,4)\) and \((3,5)\) form a cycle and we can simply check that no cycle of length 3 or 4 exists, so \( \text{girth}(\Gamma(T_5)) = 5 \).

Theorem 3.19. For \( n \geq 4 \), \( \Gamma(I_n) \) is a connected graph with \( \text{girth}(\Gamma(I_n)) = 3 \) and \( \text{diam}(\Gamma(I_n)) = 3 \).

Proof. The remaining cases \((1,2),(3,4)\) and \((1,2)(3,4)\) form a cycle in \( \Gamma(I_n) \) and so \( \text{girth}(\Gamma(I_n)) = 3 \). For \( n = 4 \), it is obvious that \( \text{diam}(\Gamma(I_n)) = 3 \). For \( n = 5 \), suppose \( \sigma, \mu \in I_5 \), then \( \sigma \) and \( \mu \) belong to \( T_5 \) in this case we know by Theorem 3.18 that \( d(\sigma, \mu) \leq 2 \).

(a) \( \sigma, \mu \in I_5 \) but \( \sigma \notin T_5 \). In this case there exists \( \mu_1 \in T_5 \) such that \( \mu \mu_1 = \mu_1 \mu \) and \( d(\mu_1, \sigma) \leq 2 \), therefore \( d(\sigma, \mu) \leq 3 \).

(3) \( \sigma, \mu \notin T_5 \). In this case suppose \( \sigma = (a)(b)(c) \) and \( \sigma \notin \text{supp}(\sigma) \). Then we have the following subcases for \( \mu = \sigma(\mu, \mu) \):

\[
(a)(b)(c), (a)(b)(c) \in \text{I}(a)(b)(c).
\]

For each of these subcases we have the following paths respectively:

\[
\sigma = (a)(b)(c) \in \text{I}(a)(b)(c), \quad \sigma = (a)(b)(c) \in \text{I}(a)(b)(c).
\]

Thus \( d(\sigma, \mu) \leq 3 \).
Therefore \( \Gamma(I_3) \) is connected and \( \text{diam}(\Gamma(I_3)) \leq 3 \).

Let \( n \geq 6 \). For each \( \sigma \in I_n \), there are at least three disjoint transpositions \( t_1, t_2 \) and \( t_3 \) such that \( t_1, t_2, t_3 \in C_{n}(\sigma) \). Now for each \( \mu \in I_n \), choose a transposition \( t \) such that \( \mu t = \mu \). There exists \( 1 \leq i \leq 3 \) such that \( \mu t_i = \mu t_t \), and therefore \( \sigma t_i - t_t = \mu \) is a path in \( \Gamma(I_n) \). Thus \( \Gamma(I_n) \) is connected and \( \text{diam}(\Gamma(I_n)) \leq 3 \). Now we will prove that \( \text{diam}(\Gamma(I_n)) = 3 \) for every \( n \geq 5 \). We will find distinct elements \( \mu \) and \( \sigma \) in \( I_n \) such that \( \text{d}(\mu, \sigma) = 3 \). First let \( 5 \leq n = 2k + 1 \) be an odd number and suppose \( \mu = (1 \ 2 \ (2k - 1 \ \cdots \ 2k) \ \sigma = (1 \ 2 \ (2k - 2) \ \cdots \ 2k - 1) \) are two elements in \( I_n \). By contrary, suppose that \( \text{d}(\mu, \sigma) < 3 \). Therefore we must have \( \text{d}(\mu, \sigma) = 3 \) and there should be an element \( \alpha \in I_n \) such that \( \alpha \sigma = C_{n}(\mu) \cap C_{n}(\sigma) \). If \( \sigma = 2k + 1 \), then \( \sigma = 2k + 1 \alpha = (2k + 1) \sigma = (2k + 1) \alpha = (2k + 1) \alpha = (m) \mu = (m) \mu = \alpha \sigma \). Hence \( \sigma = 2k + 1 \). Similarly let \( \sigma = 2k \), then \( \sigma = 2k \alpha = (2k) \alpha = (2k) \alpha = (m) \sigma = (m) \sigma = \alpha \sigma \). Hence \( \sigma = 2k \). Now suppose that \( 1 \leq i \leq 2k - 1 \) and \( \text{d}(\mu, \sigma) = \alpha \sigma = \alpha \sigma = \alpha \sigma \). Therefore \( \alpha \sigma = \alpha \sigma \) and this contradicts \( \sigma \). Therefore \( \text{d}(\mu, \sigma) = 3 \). If \( 5 \leq n = 2k + 2 \) is even, then with the same \( \mu \) and \( \sigma \) as above, we have \( C_{n}(\mu) \cap C_{n}(\sigma) = C_{n}(\mu) \cap C_{n}(\sigma) \). Therefore \( \text{d}(\mu, \sigma) = \alpha \sigma = \alpha \sigma \). Therefore \( \alpha \sigma = \alpha \sigma \) and this contradicts \( \sigma \). Therefore \( \text{d}(\mu, \sigma) = 3 \.

4. The Relation Between the Commuting Graph and the Prime Graph of Groups

Definition. Let \( V \) be a finite set. An abstract simplicial complex \( K \) is a family of nonempty subsets of \( V \), called simplexes, such that

1. if \( v \in V \), then \{v\} \in K;
2. if \( s \in K \) and \( s' \subseteq s \), then \( s' \in K \).

One calls \( V \) the vertex set of \( K \); a simplex \( s \in K \) having \( q + 1 \) distinct vertices is called a \( q \)-simplex. (See \[16\].)

To every finite partially ordered set \( X \) it is associated an abstract simplicial complex, the order complex of \( X \), that we still call \( X \), by taking the elements of \( X \) as vertices (0-simplexes) and as \( n \)-simplexes the chains of \( n \) elements of \( X \), for \( n \geq 0 \). Every abstract simplicial complex \( X \), has topology as well as graph theoretical aspects. To \( X \) we can assign a topological space which is called geometric realization of \( X \) and \( X \) is called connected if its geometric realization is topologically connected. Also we can assign a graph to \( X \) whose vertices are exactly vertices of \( X \) and \( x, y \) are joined if and only if \( \{x, y\} \) is a 2-simplex in \( X \).

Let \( G \) be a finite group. We define the following partially ordered sets (poets):

\[ N(G) = \{1 \neq N \leq G : N \text{ is a nilpotent subgroup of } G\} \text{ ordered by inclusion,} \]
\[ Ab(G) = \{1 \neq H \leq G : H \text{ is an abelian subgroup of } G\} \text{ ordered by inclusion.} \]

We consider \( N(G) \) and \( Ab(G) \) as both simplicial complexes and graphs.

In [13], the author proved that \( N(G) \) is connected if and only if the prime graph of \( G \), \( \Pi(G) \), is connected. Also the author proved that the complexes associated to \( Ab(G) \) and \( N(G) \) are \( G \)-homotopy equivalent. This means that \( Ab(G) \) is connected if and only if \( N(G) \) is connected if and only if \( \Pi(G) \) is connected. We prove the following Lemma:

**Lemma 4.1.** Let \( G \) be a group with \( Z(G) = \{1\} \). \( \Gamma(G) \) is connected if and only if \( \Pi(G) \) is connected.

**Proof.** We prove that \( \Gamma(G) \) is connected if and only if \( Ab(G) \) is connected. Observe that if \( A \) is a nontrivial abelian subgroup of \( G \), then \( A \setminus \{1\} \) lies completely in one connected component of \( \Gamma(G) \), because \( Z(G) = \{1\} \). First suppose that \( Ab(G) \) is connected, but \( \Gamma(G) \) is not. Let \( C_1, C_2 \) be distinct connected components of \( \Gamma(G) \). If \( x \in C_1 \) and \( y \in C_2 \), then \( \{x\} \setminus \{1\} \subseteq C_1 \) and \( \{y\} \setminus \{1\} \subseteq C_2 \). According to Zorn’s Lemma, there are nontrivial abelian subgroups \( A, B \) of \( G \) such that \( A \setminus \{1\} \subseteq C_1 \) and \( B \setminus \{1\} \subseteq C_2 \) and \( d_{\text{Ab}(G)}(A, B) = n \) is minimum. Therefore there exists a path \( A_0 = A - A_1 - \cdots - A_n = B \) of length \( n \) in \( Ab(G) \). By the definition of \( Ab(G) \), we have

\[ A_0 = A \subseteq A_1 \subseteq \cdots \subseteq A_n = B, \]

where \( X \subseteq Y \) means \( X \subseteq Y \) or \( Y \subseteq X \). Since \( Z(G) = \{1\} \), \( A_0 \setminus \{1\} \subseteq C_1 \) and \( A_0 \subseteq A_1 \) imply \( A_1 \setminus \{1\} \subseteq C_1 \). We have \( d_{\text{Ab}(G)}(A_1, B) = n - 1 \) and this contradicts to the choice of \( n \). Hence \( \Gamma(G) \) must be connected.

For the converse, suppose \( \Gamma(G) \) is connected, but \( Ab(G) \) is not, and let \( C_1, C_2 \) be two distinct connected components of \( Ab(G) \). Assume that \( A, B \) are two proper abelian subgroups of \( G \) such that \( A \setminus \{1\} \subseteq C_1 \) and \( B \setminus \{1\} \subseteq C_2 \), then \( A \cap B = \{1\} \). Let \( a \in A \setminus \{1\} \) and \( b \in B \setminus \{1\} \). Since \( \Gamma(G) \) is connected, there is a path \( a = a_0 - a_1 - \cdots - a_n = b \) of length \( n \) in \( \Gamma(G) \). Put \( A_1 = \langle a_0, a_1 \rangle \), then \( a_0 \) is a subgroup of abelian subgroups \( A_1 \) and \( A_2 \) and so \( A_1 \setminus \{1\} \subseteq C_1 \). Now let \( A_2 = \langle a_1, a_2 \rangle \), then \( a_1 \) is a subgroup of abelian subgroups \( A_1 \) and \( A_2 \) and so \( A_2 \setminus \{1\} \subseteq C_1 \). Suppose that if \( A_1 = \langle a_4, a_5 \rangle \), then \( a_4 \) is a subgroup of abelian subgroups \( A_1 \) and \( A_5 \) and so \( A_5 \setminus \{1\} \subseteq C_1 \). By induction on \( n \), we obtain \( A_n \setminus \{1\} = \langle a_n, \cdots, a_1 \rangle \setminus \{1\} \subseteq C_1 \), but \( \emptyset \) is a subgroup of abelian subgroups \( A_n \) and \( B \) so \( B \setminus \{1\} \subseteq C_1 \) and this is a contradiction. Hence \( Ab(G) \) must be connected.

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References