ON VARIETAL CAPABILITY OF DIRECT PRODUCTS OF GROUPS AND PAIRS OF GROUPS

HANIEH MIREBRAHIMI¹ AND BEHROOZ MASHAYEKHY²

Abstract. In this paper we give some conditions in which a direct product of groups is $\mathcal{V}$-capable if and only if each of its factors is $\mathcal{V}$-capable for some varieties $\mathcal{V}$.

1. INTRODUCTION AND PRELIMINARIES

R. Baer [1] initiated an investigation of the question which conditions a group $G$ must fulfill in order to be the group of inner automorphisms of a group $E$, that is $(G \cong E/Z(E))$. Following M. Hall and J. K. Senior [5], such a group $G$ is called capable. Baer [1] determined all capable groups which are direct sums of cyclic groups. As P. Hall [4] mentioned, characterizations of capable groups are important in classifying groups of prime-power order.

F. R. Beyl, U. Felgner and P. Schmid [2] proved that every group $G$ possesses a uniquely determined central subgroup $Z^*(G)$ which is minimal subject to being the image in $G$ of the center of some central extension of $G$. This $Z^*(G)$ is the smallest central subgroup of $G$ whose factor group is capable [2, Corollary 2.2]. Hence $G$ is capable if and only if $Z^*(G) = 1$ [2, Corollary 2.3]. They showed that the class of all capable groups is closed under the direct products [2, Proposition 6.1]. Also, they presented a condition in which the capability of a direct product of finitely many of groups implies the capability of each of the factors [2, Proposition 6.2]. Moreover, they proved that if

¹ 2010 Mathematics Subject Classification. Primary 20E10, 20K25; Secondary 20E34, 20D15, 20F18.

² Key words and phrases. Capable group; Direct product; Variety of groups; $\mathcal{V}$-capable group; Pair of groups; Capable pair of groups.

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N is a central subgroup of G, then \( N \subseteq Z^*(G) \) if and only if the mapping \( M(G) \to M(G/N) \) is monomorphic [2, Theorem 4.2].

Then M. R. R. Moghadam and S. Kayvanfar [8] generalized the concept of capability to \( V \)–capability for a group \( G \). They introduced the subgroup \( (V^*)^*(G) \) which is associated with the variety \( V \) defined by the set of laws \( V \) and a group \( G \) in order to establish a necessary and sufficient condition under which \( G \) can be \( V \)–capable [8, Corollary 2.4]. They also showed that the class of all \( V \)–capable groups is closed under the direct products [8, Theorem 2.6]. Moreover, they exhibited a close relationship between the groups \( VM(G) \) and \( VM(G/N) \), where \( N \) is a normal subgroup contained in the marginal subgroup of \( G \) with respect to the variety \( V \). Using this relationship, they gave a necessary and sufficient condition for a group \( G \) to be \( V \)–capable [8, Theorem 4.4].

In this note, we present some conditions in which the \( V \)–capability of a direct product of finitely many groups implies the \( V \)–capability of each of its factors.

2. Main results

Suppose that \( V \) is a variety of groups defined by the set of laws \( V \). A group \( G \) is said to be \( V \)–capable if there exists a group \( E \) such that \( G \cong E/V^*(E) \). If \( \psi : E \to G \) is a surjective homomorphism with \( ker\psi \subseteq V^*(E) \), then the intersection of all subgroups of the form \( \psi(V^*(E)) \) is denoted by \( (V^*)^*(G) \). It is obvious that \( (V^*)^*(G) \) is a characteristic subgroup of \( G \) contained in \( V^*(G) \).

If \( V \) is the variety of abelian groups, then the subgroup \( (V^*)^*(G) \) is the same as \( Z^*(G) \) and in this case \( V \)–capability is equal to capability [8].

**Theorem 2.1.** [8] (i) A group \( G \) is \( V \)–capable if and only if \( (V^*)^*(G) = 1 \).
(ii) \( (V^*)^*(\prod_{i \in I} G_i) \leq \prod_{i \in I}(V^*)^*(G_i) \).

As a consequence, if \( G_i \)'s are \( V \)–capable groups, then \( G = \prod_{i \in I} G_i \) is also \( V \)–capable. In the above theorem, the equality does not hold in general (Example 2.3).

**Theorem 2.2.** [8] Let \( N \) be a normal subgroup contained in the marginal subgroup of \( G, V^*(G) \). Then \( N \subseteq (V^*)^*(G) \) if and only if the homomorphism induced by the natural map \( VM(G) \to VM(G/N) \) is a monomorphism.

In this section we verify the equation \( (V^*)^*(A \times B) = (V^*)^*(A) \times (V^*)^*(B) \) for some famous varieties.
In general, for an arbitrary variety of groups \( V \), and groups \( A \) and \( B \), 
\( \mathcal{V}M(A \times B) \cong \mathcal{V}M(A) \times \mathcal{V}M(B) \times T \), where \( T \) is an abelian group [7]. For some particular varieties, the group \( T \) is trivial with some conditions. For instance, some famous varieties as variety of abelian groups [7], variety of nilpotent groups [3], and some varieties of polynilpotent groups [6] have the property that: for any two groups \( A \) and \( B \) with \((|A^{ab}|, |B^{ab}|) = 1\) the isomorphism 
\( \mathcal{V}M(A \times B) \cong \mathcal{V}M(A) \times \mathcal{V}M(B) \) \((*)\) holds.

Now, suppose that \( V \) is a variety, \( A \) and \( B \) are two groups with the property 
\( \mathcal{V}M(A \times B) \cong \mathcal{V}M(A) \times \mathcal{V}M(B) \).

By Theorem 2.2, we have the following monomorphism
\[
\mathcal{V}M(A) \times \mathcal{V}M(B) \hookrightarrow \mathcal{V}M\left(\frac{A}{(V*)^*(A)}\right) \times \mathcal{V}M\left(\frac{B}{(V*)^*(B)}\right).
\]

Moreover, we have the following inclusion
\[
\mathcal{V}M\left(\frac{A}{(V*)^*(A)}\right) \times \mathcal{V}M\left(\frac{B}{(V*)^*(B)}\right) \hookrightarrow \mathcal{V}M\left(\frac{A}{(V*)^*(A)} \times \frac{B}{(V*)^*(B)}\right).
\]

Finally, we get the monomorphism
\[
\mathcal{V}M(A \times B) \hookrightarrow \mathcal{V}M\left(\frac{A \times B}{(V*)^*(A) \times (V*)^*(B)}\right).
\]

Thus, by Theorem 2.2, we conclude that
\[
(V*)^*(A) \times (V*)^*(B) \leq (V*)^*(A \times B).
\]

This note leads us to our main result.

**Theorem 2.3.** Let \( V \) be a variety, \( A \) and \( B \) be two groups with \( \mathcal{V}M(A \times B) \cong \mathcal{V}M(A) \times \mathcal{V}M(B) \), then \((V*)^*(A \times B) = (V*)^*(A) \times (V*)^*(B)\). Consequently \( A \times B \) is \( V \)-capable if and only if \( A \) and \( B \) are both \( V \)-capable.

**Remark 2.4.** In some famous varieties as the variety of abelian groups and the variety of nilpotent groups, the isomorphism \( \mathcal{V}M(A \times B) \cong \mathcal{V}M(A) \times \mathcal{V}M(B) \) holds, where \((|A^{ab}|, |B^{ab}|) = 1\) ([3, 9]). Thus, using Theorem 2.3, for a family of groups \( \{A_i \mid 1 \leq i \leq n\} \) whose abelianizations have mutually coprime orders, \( \prod_{i=1}^{n} A_i \) is capable (\( \mathcal{N}_c \)-capable) if and only if every \( A_i \) is capable (\( \mathcal{N}_c \)-capable). Note that in these varieties, for finitely generated groups \( A \) and \( B \), 
\( \mathcal{V}M(A \times B) \cong \mathcal{V}M(A) \times \mathcal{V}M(B) \) if and only if \(|A^{ab}|\) and \(|B^{ab}|\) are finite with \((|A^{ab}|, |B^{ab}|) = 1\) ([3, 9]).
Corollary 2.5. Let \( \{A_i \mid 1 \leq i \leq n\} \) be a family of groups whose abelianizations have mutually coprime orders. If \( \prod_{i=1}^{n} A_i \) is nilpotent of class at most \( c_1 \), then it is \( N_{c_1, \cdots, c_s} \)-capable if and only if every \( A_i \) is \( N_{c_1, \cdots, c_s} \)-capable.

References


\(^{1,2}\) Department of Pure Mathematics, Center of Excellence in Analysis on Algebraic Structures, Ferdowsi University of Mashhad, P. O. Box 1159-91775, Mashhad, Iran.

E-mail address: \texttt{h.mirebrahimi@um.ac.ir}
E-mail address: \texttt{bmashf@um.ac.ir}