Approximately Orthogonal Additive Set-valued Mappings

Alireza Kamel Mirmostafaee* and Mostafa Mahdavi
Center of Excellence in Analysis on Algebraic Structures, Department of pure Mathematics, School of Mathematical Sciences, Ferdowsi University of Mashhad 91775, Iran
e-mail: mirmostafaei@um.ac.ir and m_mahdavi1387@yahoo.com

Abstract. We investigate the stability of orthogonally additive set-valued functional equation

\[ F(x + y) = F(x) + F(y) \quad (x \perp y) \]

in Hausdorff topology on closed convex subsets of a Banach space.

1. Introduction

A functional equation \( \mathcal{E} \) is called stable if for any function \( f \) satisfying approximately to the equation \( \mathcal{E} \), there is a true solution of \( \mathcal{E} \) near to \( f \). In 1940, S. M. Ulam [24] proposed the first stability problem for group homomorphisms. Hyers [9] gave the first significant partial solution to his problem for linear functions. Th. M. Rassias [20] improved Hyers’ theorem by weakening the condition for the Cauchy difference controlled by \( \|x\|^p + \|y\|^p \), \( p \in [0,1) \). For some recent developments in this area, we refer the reader to the articles [5, 6, 11, 12, 15, 19] and the references therein.

In 1985, Rätz [21] gave a generalization of Birkhoff-James orthogonality [1, 10] in vector spaces. He also investigated some properties of orthogonally additive functional equation. This definition motivated some Mathematicians to discuss about the orthogonal stability of functional equations (see e. g. [8, 13, 16, 22]). On the other hand, set-valued mappings and their stability have been investigated by some authors from different point of view [2, 7, 14, 17, 23].

In the next section, we prove the stability of set-valued orthogonal additive functional equation

\[ F(x + y) = F(x) + F(y) \quad (x \perp y). \]

* Corresponding Author.
Received June 11, 2012; accepted August 30, 2012.
2010 Mathematics Subject Classification: 39B22, 39B55, 39B62, 39B82.
Key words and phrases: Set-valued mappings, orthogonal space, Hausdorff metric, Hyers-Ulam stability.
This research was supported by a grant from Ferdowsi University of Mashhad No. MP91281 MIM.
In fact, we will show if \((X, \bot)\) is an orthogonal space, \(Y\) is a Banach space and \(F : X \to \text{CC}(Y)\) is an even function such that

\[
\mathcal{H}\left(F(x + y), F(x) + F(y)\right) \leq \varepsilon \quad (x, y \in X, x \bot y),
\]

for some \(\varepsilon > 0\). Then there exists a unique quadratic function \(Q : X \to \text{CC}(Y)\) such that

\[
\mathcal{H}\left(F(x), Q(x)\right) \leq \frac{7\varepsilon}{4} \quad (x \in X).
\]

In this case, we will show that there is a quadratic function \(q : X \to Y\) such that

\[
q(x) \in F(x) + \frac{7\varepsilon}{3} \overline{B(0, 1)} \quad (x \in X).
\]

2. Main Results

Throughout the paper, unless otherwise stated, we will assume that \(X\) and \(Y\) are topological vector spaces over \(\mathbb{R}\). If \(A, B \subseteq Y\) and \(\lambda \in \mathbb{R}\), we use the following notions

\[
A + B = \{a + b : a \in A, \ b \in B\}, \quad \lambda A = \{\lambda a : a \in A\}.
\]

The following properties will often be used in the sequel:

For each \(A, B \subseteq Y\) and \(\lambda, \mu \geq 0\), we have

\[
\lambda(A + B) = \lambda A + \lambda B, \quad (\lambda + \mu)A \subseteq \lambda A + \mu A.
\]

Moreover, if \(A\) is convex, \((\lambda + \mu)A = \lambda A + \mu A\).

**Definition 2.1.** Let \(Y\) be a normed space and \(A_1, A_2 \subseteq Y\) be non-empty closed bounded sets. Then the Hausdorff distance between \(A_1\) and \(A_2\) is defined by

\[
\mathcal{H}(A_1, A_2) := \inf \{s > 0 : A_1 \subseteq A_2 + sB(0, 1) \text{ and } A_2 \subseteq A_1 + sB(0, 1)\}.
\]

It is known that \(\mathcal{H}\) defines a metric on closed convex subsets of \(Y\), which is called Hausdorff metric topology\([3, 4]\). Moreover, if \(Y\) is a Banach space, \((\text{CC}(Y), \mathcal{H})\), the space of all non-empty compact convex subsets of \(Y\) with the Hausdorff metric topology is a complete metric space \([3]\).

In 1985, Rätz \([21]\) introduced the following notion:

**Definition 2.2.** Let \(X\) be a real topological vector space of dimension \(\geq 2\). A binary relation \(\bot \subseteq X \times X\) is called an **orthogonal relation** if the following properties hold.

1. \(x \bot 0, \ 0 \bot x\) for every \(x \in X\),
(2) if \( x, y \in X \setminus \{0\} \), \( x \perp y \), then \( x \) and \( y \) are linearly independent;

(3) if \( x, y \in X \), \( x \perp y \), \( \alpha x \perp \beta y \) for all \( \alpha, \beta \in \mathbb{R} \),

(4) if \( P \) is a two dimensional subspace of \( X \), \( x \in P \), \( \lambda \in \mathbb{R}^+ \), then there exists some \( y \in P \) such that \( x \perp y \) and \( x + y \perp \lambda x - y \).

The space \( X \) with an orthogonal relation \( \perp \) is called an orthogonally space and is denoted by \((X, \perp)\).

**Definition 2.3.** Let \( X \) and \( Z \) be two sets. A function \( Q : X \to Z \) is called **quadratic** if \( Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y) \) for all \( x, y \in X \).

We need to the following result due to Rådström [18].

**Lemma 2.4.** Let \( A, B \) and \( C \) be nonempty subsets of a topological vector space \( Y \). Suppose that \( B \) is closed and convex and \( C \) is bounded. If \( A + C \subseteq B + C \), then \( A \subseteq B \). If moreover, \( A \) is closed and convex and \( A + C = B + C \), then \( A = B \).

Now, we are ready to state the main result of this paper.

**Theorem 2.5.** Let \( X \) be a topological vector space over \( \mathbb{R} \) which is also an orthogonal space and let \( Y \) be a Banach space. Let \( F : X \to CC(Y) \) be an even function and for some \( \varepsilon > 0 \),

\[
\mathcal{H}\left( F(x + y), F(x) + F(y) \right) \leq \varepsilon \quad (x, y \in X, x \perp y).
\]

Then there exists a unique quadratic and orthogonal additive function \( Q : X \to CC(Y) \) such that

\[
\mathcal{H}(F(x), Q(x)) \leq \frac{7\varepsilon}{3} \quad (x \in X).
\]

**Proof.** We divide the proof into several steps.

**Step 1.** For each \( x \in X \),

\[
\mathcal{H}\left( F(2x), 4F(x) \right) \leq 7\varepsilon.
\]

**Proof of step 1.** By Definition 2.2, for each \( x \in X \), there is some \( y \in X \) such that \( x \perp y \) and \( x + y \perp x - y \). Take some \( y \in X \) with this property. Then
\[
F(x) = F\left(\frac{x+y}{2} \pm \frac{x-y}{2}\right)
\]
\[
\subseteq F\left(\frac{x+y}{2}\right) + F\left(\frac{x-y}{2}\right) + \varepsilon B(0,1)
\]
\[
= F\left(\frac{x+y}{2}\right) + F\left(\frac{y-x}{2}\right) + \varepsilon B(0,1) \quad (\therefore F \text{ is even})
\]
\[
\subseteq F\left(\frac{x+y}{2} \pm \frac{y-x}{2}\right) + 2\varepsilon B(0,1)
\]
\[
= F(y) + 2\varepsilon B(0,1).
\]

Since \(x + y \perp y - x\), by interchanging the role of \(x\) and \(y\), we see that
\[
F(y) \subseteq F(x) + 2\varepsilon B(0,1).
\]

On the other hand,
\[
F(2x) = F(x + y + x - y) \subseteq F(x + y) + F(x - y) + \varepsilon B(0,1)
\]
\[
\subseteq 2F(x) + 2F(y) + 3\varepsilon B(0,1)
\]
\[
\subseteq 4F(x) + 7\varepsilon B(0,1)
\]
and
\[
4F(x) = 2F(x) + 2F(x) \subseteq 2F(x) + 2F(y) + 4\varepsilon B(0,1)
\]
\[
\subseteq F(x) + F(y) + F(x) + F(-y) + 4\varepsilon B(0,1) \quad (\text{since } x \perp y)
\]
\[
\subseteq F(x + y) + F(x - y) + 6\varepsilon B(0,1) \quad (\text{since } x + y \perp x - y)
\]
\[
\subseteq F(2x) + 7\varepsilon B(0,1).
\]

Therefore (2.2) holds.

**Step 2.** There is a unique orthogonal additive function \(Q : X \rightarrow CC(Y)\) such that

\[
Q(2x) = 4Q(x) \quad \text{and}
\]

(2.3) \[
\mathcal{H}(F(x), Q(x)) \leq \frac{7\varepsilon}{3}
\]
for each \(x \in X\).

**Proof of step 2.** Replace \(x\) by \(2^n x\) in (2.2) and multiply both sides of the obtained inequality by \(4^{-(n+1)}\) to obtain the following inequality

\[
\mathcal{H}\left(4^{-(n+1)} F(2^{n+1} x), 4^{-n} F(2^n x)\right) \leq \frac{7\varepsilon}{4^{n+1}} \quad (n \geq 0, \ x \in X).
\]
It follows that for each \( n > m \geq 0 \), we have

\[
\mathcal{H}\left(4^{-n}F(2^n x), 4^{-m}F(2^m x)\right) \leq \sum_{k=m}^{n-1} 2^k 2^{k+1} + \sum_{k=m}^{n-1} \frac{\varepsilon}{2^{k+1}} (x \in X).
\]

(2.4)

Since the right hand side of the above inequality tends to zero as \( n \to \infty \), \( \{4^{-n}F(2^n x)\} \) is a Cauchy sequence in \((CC(Y), \mathcal{H})\). Completeness of \( CC(Y) \) with respect to the Hausdorff metric topology insures that

\[
Q(x) = \lim_{n \to \infty} 4^{-n}F(2^n x) (x \in X)
\]
defines a function from \( X \) to \( CC(Y) \). Put \( m = 0 \) in (2.4) to obtain

\[
\mathcal{H}\left(Q(x), F(x)\right) = \lim_{n \to \infty} \mathcal{H}\left(4^{-n}F(2^n x), F(x)\right)
\]

(2.5)

\[
\leq \sum_{k=0}^{\infty} \frac{\varepsilon}{2^{k+1}} = \frac{\varepsilon}{3} (x \in X).
\]

Moreover, for every \( x \in X \), we have

\[
Q(2x) = \lim_{n \to \infty} 4^{-n}F(2^{n+1} x)
\]

(2.6)

\[
= 4 \lim_{n \to \infty} 4^{-(n+1)}F(2^{n+1} x) = 4Q(x).
\]

If \( x \perp y \), we have

\[
\mathcal{H}\left(Q(x) + Q(y), Q(x + y)\right) = \lim_{n \to \infty} \mathcal{H}\left(4^{-n}F(2^n x) + 4^{-n}F(2^n y), 4^{-n}F(2^n(x + y))\right) \leq \lim_{n \to \infty} 4^{-n}\varepsilon = 0.
\]

Hence \( Q \) is orthogonal additive. Suppose that \( Q' : X \to CC(Y) \) satisfies the following properties:

(i) \( \mathcal{H}\left(Q'(x), F(x)\right) \leq \frac{\varepsilon}{3} \) and

(ii) \( Q'(2x) = 4Q'(x) \) for each \( x \in X \).

Then for each \( x \in X \), we have

\[
\mathcal{H}\left(Q'(x), Q(x)\right) = \lim_{n \to \infty} \mathcal{H}\left(4^{-n}Q'(2^n x), 4^{-n}F(2^n x)\right)
\]

\[
= \lim_{n \to \infty} 4^{-n}\mathcal{H}\left(Q'(2^n x), F(2^n x)\right) \leq \lim_{n \to \infty} 4^{-n}\frac{\varepsilon}{3} = 0.
\]

Thus the uniqueness assertion of step 2 follows.

**Step 3.** The function \( Q : X \to CC(Y) \) is quadratic.
Proof of step 3. Let $x, y \in X$. Then the following cases may happen.

(i) $y = \alpha x$, where $\alpha \geq 0$. In this case, by property (4) of Definition 2.2, for each $x \in X$, there is some $z \in X$ such that $x \perp z$ and $x + z \perp \alpha x - z$. Therefore

$$Q(x + y) + Q(x - y) = Q(x + \alpha x) + Q(x - \alpha x) = Q(x + z + \alpha x - z) + Q(\alpha x - x).$$

It follows that

$$Q(x + \alpha x) + Q(x - \alpha x) + Q(2z) = Q(x + z) + Q(\alpha x - z) + Q(\alpha x - x + 2z) = Q(x) + 2Q(z) + Q(\alpha x) + Q(x + z + z - \alpha x) = Q(x) + 2Q(z) + Q(\alpha x) + Q(x + z) + Q(z - \alpha x) = 2Q(x) + 2Q(\alpha x) + 4Q(z) = 2Q(x) + 2Q(\alpha x) + Q(2z).$$

Thanks to Lemma 2.4, the result follows in this case.

(ii) $y = \alpha x$, where $\alpha < 0$. Let $\beta = -\alpha$. Then $\beta > 0$. Hence,

$$Q(x + \alpha x) + Q(x - \alpha x) = Q(x - \beta x) + Q(x + \beta x) = 2Q(x) + 2Q(\beta x) = 2Q(x) + 2Q(\alpha x)$$

since $Q$ is even.

(iii) $x$ and $y$ are linearly independent.

By Definition 2.2, there is some $z$ in linear span of $\{x, y\}$ such that $x \perp z$. Let $y = \alpha x + \beta z$. Then

$$Q(x + y) + Q(x - y) = Q[(x + \alpha x) + \beta z] + Q[x - (\alpha x + \beta z)] = Q(x + \alpha x) + Q(\beta z) + Q(x - \alpha x) + Q(-\beta z) = 2Q(x) + 2Q(\alpha x) + 2Q(\beta z) = 2Q(x) + 2Q(\alpha x + \beta z) = 2Q(x) + 2Q(y).$$

This completes the proof of the theorem.

Example 2.6. Let $X$ be an inner product space and $\varepsilon > 0$. Define $F : X \to CC(\mathbb{R})$ by $F(x) = [0, \|x\|^2 + \varepsilon]$. It is easy to see that $F$ is $[0, \varepsilon]$-orthogonal additive even function. According to Theorem 2.5, there is a quadratic function $Q : X \to CC(\mathbb{R})$ such that

$$\mathcal{H}(F(x), Q(x)) \leq \frac{7\varepsilon}{3} (x \in X).$$

Definition 2.7. Let $X$ and $Y$ be two sets. By a selection of a set-valued function $F : X \to 2^Y$, we mean a single-valued mapping $f : X \to Y$ such that $f(x) \in F(x)$ for each $x \in X$.

Corollary 2.8. Under conditions of Theorem 2.5, there is a quadratic function $q : X \to Y$ such that

$$q(x) \in F(x) + \frac{7\varepsilon}{3} B(0, 1) (x \in X).$$
Proof. It is known that if $X$ is an abelian group with division by two and $Y$ is a topological vector space, then every subquadratic set-valued function $Q : X \to CC(Y)$ admits a quadratic selection $q : X \to Y$ [4, Theorem 35.2]. So the result follows from Theorem 2.5.

Acknowledgements. The authors would like to thank the two anonymous reviewers for their helpful comments.

References


