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On Solution and Hyeres–Ulam–Rasstas Stability of a Generalized Quadratic Equation

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Abstract: In this paper we study general solutions and Hyers-Ulam-Rassias stability of the following function equation

$$(4-k)f(\sum_{i=1}^{k} x_i) + \sum_{j=1}^{k} f((\sum_{i=1, i\neq j}^{k} x_i) - x_j)) = 4\sum_{i=1}^{k} f(x_i),\ k \geq 3 \tag{1}$$

on Banach spaces. It will be shown that this equation is equivalent to the so-called quadratic functional equation.

Keywords: Hyers–Ulam–Rassias stability; quadratic equation.

1 introduction

A classical question in the theory of functional equations is the following: ”When is it true that a function which approximately satisfies a functional equation ϵ must be close to an exact solution of ϵ?”

If the problem accepts a solution, we say that equation ϵ is stable. The first stability problem concerning group homomorphisms was raised by Ulam [31] in 1940.

We are given a group G and a metric group G' whit metric $d(.,.)$. Given $\epsilon > 0$, dose there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then a homomorphism $h : G \to G'$ exists with $d(f(x), h(x)) < \epsilon$, for all $x \in G$?

Ulam's problem was partially solved by Hyers [11] in 1941. Let E_1 be a normed space, E_2 a Banach space and suppose that the mapping $f : E_1 \to E_2$ satisfies the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon,\ \ x, y \in E_1,$$

where $\epsilon > 0$ is a constant. Then the limit $T(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$ exists for each $x \in E_1$ and T is the unique additive mapping satisfying

$$\|f(x) - T(x)\| \leq \epsilon, \tag{2}$$

for all $x \in E_1$. Also if for each x the function $t \to f(tx)$ from $\mathbb{R}$ to E_2 is continuous on $\mathbb{R}$, then T is linear. If f is continuous at a single point of E_1, then T is continuous everywhere in E_1. Moreover (2) is sharp.

In 1987, Th.M. Rassias [26], formulated and proved the following theorem, which implies Hyers'theorem as a special case. Suppose that E and F are real normed spaces with F a complete normed space, $f : E \to F$ is a mapping such that for each fixed $x \in E$ the mapping $t \to f(tx)$ is continuous on $\mathbb{R}$, and let there exist $\epsilon > 0$ and $p \in [0, 1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p), \tag{3}$$

for all $x, y \in E$. Then there exists a unique linear mapping $T : E \to F$ such that

$$\|f(x) - T(x)\| \leq \frac{\epsilon\|x\|^p}{(1-2^{p-1})},$$

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for all $x \in E$. The case of the existence of a unique additive mapping had been obtained by T. The terminology Hyers-Ulam stability originates from these historical backgrounds. The terminology can also be applied to the case of other functional equations. For more detailed definitions of such terminologies, we can refer to [7], [9], [11] and [16]. In 1994, $P.G\breve{a}vruta$, [8], provided a further generalization of Th. M. Rassias'theorem in which he replaced the bound $\epsilon(\|x\|^p + \|y\|^p)$ in (3) by a general control function $\varphi(x, y)$ for the existence of a unique linear mapping .

The functional equation $f(x+y)+f(x-y) = 2f(x)+2f(y)$ is called the quadratic functional equation. In particular every solution of the quadratic functional equation is said to be a quadratic mapping, see [25, 27]. It is well know that a function f between real vector spaces is quadratic if and only if there exists a unique symmetric bi-additive function B such that $f(x) = B(x, x)$, for all x (see [1, 11, 17]).

A generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [29] for mappings $f : X \to Y$ where X is a normed space and Y is a Banach space. Cholewa [4] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. In [6], Czerwik proved the generalized Hyers-Ulam stability of the quadratic functional equation. Borelli and Forti [3] generalized the stability result as follows (cf.[23, 24]): Let G be an Abelian group, and X a Banach space. Assume that a mapping $f : G \to X$ satisfies the functional inequality

$$\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \leq \varphi(x, y),$$

for all $x, y \in G$, and $\varphi : G \times G \to [0, \infty)$ is a function such that

$$\phi(x, y) := \sum_{i=0}^{\infty} \frac{\varphi(2^i x, 2^i y)}{4^{i+1}} < \infty,$$

for all $x, y \in G$. Then there exists a unique quadratic mapping $Q : G \to X$ with the property $\|f(x) - Q(x)\| \leq \phi(x, x)$, for all $x \in G$.
Stability of the quadratic functional also studied by many other authors in various cases (see for example [5], [13–15], [18–22], [28] and [32]).

Let X and Y be some given vector spaces, and let $f : X \to Y$ be a given function. For any $K \geq 3$, define

$$Df(x_1, ..., x_k) := (4 - k)f(\sum_{i=1}^{k} x_i) + \sum_{j=1}^{k} f((\sum_{i=1, i\neq j}^{k} x_i) - x_j)) - 4\sum_{i=1}^{k} f(x_i), \tag{4}$$

where $x_i \in X, \ \ i = 0, ..., k$. One can see that the quadratic function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ satisfies not only the following functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y) \tag{5}$$

but also

$$Df(x_1, ..., x_k) := 0 \tag{6}$$

for all $x_i \in \mathbb{R}$. So it is natural that these functional equations are called quadratic.

In [2], solutions and Hyers-Ulam-Rassias stability of the functional equation (6) has been studied for $k = 3$.

In Section 2 of this paper, we shall prove that the functional equation (6) is equivalent to the equation (5). The Hyers-Ulam-Rassias stability problem of the functional equation (6) will be also investigated in section 3.

2 Solution of equation (6)

Throughout this section, X and Y will be some vector spaces. The following theorem prove that the functional equation (6) is equivalent to the equation (5). That is every solution of the equation (6) is a quadratic function.

Theorem 1 *Let X and Y be common domain and range of the f's in the equations (5) and (6). Then the equation (6) is equivalent to (5).*

Proof. If we put $x_i = 0$, $i = 1, 2, ..., n$, in the equation (6), we get $f(0) = 0$. By putting $x_1 = x_2 = x$ and $x_i = 0$, $i = 3, 4, ..., n$, in the equation (6), we see that every solution of the equation (6) is even, i.e. $f(x) = f(-x)$. By putting $x_1 = x$, $x_2 = y$ and $x_i = 0$, for $i = 3, ..., n$, and using evenness of f and $f(0) = 0$ we can transform the equation (6) into the equation (5).

Now, suppose a function $f : X \to Y$ satisfies (5) for all $x, y \in X$. Then trivially f is even. Now, using mathematical induction, we are going to show that

$$f(\sum_{i=1}^{k} x_i) + \sum_{j=1}^{k} f((\sum_{i=1, i\neq j}^{k} x_i) - x_j)) = 4\sum_{i=1}^{k} f(x_i) + (k-3)f(\sum_{i=1}^{k} x_i) \tag{7}$$

for any $k \geq 3$. For $k = 3$, see the proof of Theorem 1 [2]. Suppose (7) is holds for $k - 1$, we prove that (7) is valid for k. Let $x_1, x_2, ..., x_n \in X$ be given. By the assumption of induction and the fact that f is even, we have

$$\begin{aligned}
&f(\sum_{i=1}^{k} x_i) + \sum_{j=1}^{k} f((\sum_{i=1,i\neq j}^{k} x_i) - x_j)) = f(\sum_{i=1}^{k} x_i) + \sum_{j=1}^{k-2} f(\sum_{i=1}^{k} x_i - x_j) \\
&+f(x_1 + ... + x_{k-2} - x_{k-1} + x_k) + f(x_1 + ... + x_{k-2} + x_{k-1} - x_k) \\
&= f(x_1 + ... + x_{k-2} + (x_{k-1} + x_k)) + \sum_{j=1}^{k-2} f(\sum_{i=1}^{k} x_i - x_j) \\
&+f(x_1 + ... + x_{k-2} - (x_{k-1} + x_k)) - f(x_1 + ... + x_{k-2} - (x_{k-1} + x_k)) \\
&+f(x_1 + ... + x_{k-2} - x_{k-1} + x_k) + f(x_1 + ... + x_{k-2} + x_{k-1} - x_k) \\
&= 4\sum_{i=1}^{k-2} f(x_i) + 4f(x_{k-1} + x_k) + ((k-1) - 3)f(x_1 + ... + (x_{k-1} + x_k)) \\
&+f(x_1 + ... + x_{k-2} - x_{k-1} + x_k) + f(x_1 + ... + x_{k-2} + x_{k-1} - x_k) \\
&-f(x_1 + ... + x_{k-2} - (x_{k-1} + x_k)) \\
&= 4\sum_{i=1}^{k-2} f(x_i) + 4f(x_{k-1} + x_k) + ((k-1) - 3)f(x_1 + ... + (x_{k-1} + x_k)) \\
&+f((x_{k-1} - x_k) - (x_1 + ... + x_{k-2})) + f((x_{k-1} - x_k) + (x_1 + ... + x_{k-2})) \\
&-f(x_1 + ... + x_{k-2} - (x_{k-1} + x_k)) \\
&= 4\sum_{i=1}^{k-2} f(x_i) + 4f(x_{k-1} + x_k) + ((k-1) - 3)f(x_1 + ... + (x_{k-1} + x_k)) \\
&+2f(x_{k-1} - x_k) + 2f(x_1 + ... + x_{k-2}) - f(x_1 + ... + x_{k-2} - (x_{k-1} + x_k)) \\
&= 4\sum_{i=1}^{k-2} f(x_i) + 2f(x_{k-1} + x_k) + ((k-1) - 3)f(x_1 + ... + (x_{k-1} + x_k)) \\
&+4f(x_{k-1}) + 4f(x_k) + 2f(x_1 + ... + x_{k-2}) - f(x_1 + ... + x_{k-2} - (x_{k-1} + x_k)) \\
&= 4\sum_{i=1}^{k} f(x_i) + ((k-1) - 3)f(x_1 + ... + (x_{k-1} + x_k)) + f(x_1 + ... + (x_{k-1} + x_k)) \\
&= 4\sum_{i=1}^{k} f(x_i) + (k-3)f(x_1 + ... + x_{k-1} + x_k)
\end{aligned}$$

This prove equation (7). Thus (5) and (6) are equivalent. ■

3 Hyers-Ulam-Rassias stability of the equation (6)

In this section, we assume that X and Y are a normed space and a Banach space, respectively.

Theorem 2 *Let $k \in \mathbb{N}$ and $\varphi : \underbrace{X \times ... \times X}_{k-times} :\to [0, \infty)$ be a mapping such that*

$$\tilde{\psi}(x, y) = \sum_{i=0}^{\infty} \frac{\varphi(2^i x, 2^i y, 0, ..., 0)}{4^{i+1}} < \infty, \textit{ for all } x, y \in X. \tag{8}$$

Suppose that the function $f : X \to Y$ satisfy

$$\|Df(x_1, x_2, ..., x_k)\| < \varphi(x_1, ..., x_k) \tag{9}$$

for all $x_i \in X$, $i = 0, ..., k$. Then there exists exactly one quadratic function $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq 2M\tilde{\psi}(0,0) + 2\tilde{\psi}(x,x), \ \ x \in X, \tag{10}$$

where $M = \frac{6k-7}{2(k-1)}$ and the function Q is given by

$$Q(x) = \lim_{n\to\infty} \frac{f(2^n x)}{4^n}, \ \ x \in X.$$

Proof. Suppose that φ satisfies (8). Let x, y be elements of X. From (9) we have

$$\|4(1-k)f(0)\| \qquad < \varphi(0, ..., 0) \tag{11}$$

$$\|f(-x) - f(x) - 4(k-1)f(0)\| \qquad < \varphi(x, 0, ..., 0) \tag{12}$$

These relations imply that

$$\|f(-x) - f(x)\| < \varphi(x, 0, ..., 0) + \varphi(0, ..., 0). \tag{13}$$

Also we get

$$\begin{aligned} &\|2f(x+y) + f(-x+y) + f(x-y) + 4f(x) - 4f(y) - 4(k-2)f(0)\| \\ &< \varphi(x, y, 0, ..., 0) \end{aligned} \tag{14}$$

from (11),(13) and (14), we get

$$\begin{aligned} &\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \\ &\leq \frac{1}{2}\|2f(x+y) - 2f(x-y) - 4f(x) - 4f(y) + f(y-x) - f(x-y) - 4(k-2)f(0)\| \\ &+\frac{1}{2}\|f(y-x) - f(x-y)\| + 2(k-2)\|f(0)\| \\ &\leq \frac{\varphi(x, y, 0, ..., 0)}{2} + \frac{\varphi(x-y, 0, ..., 0)}{2} + \frac{2k-3}{2(k-1)}\varphi(0, ..., 0). \end{aligned} \tag{15}$$

With $\psi(x,y) := \frac{\varphi(x,y,0,...,0)}{2}$, from (15), we have

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| < \frac{2k-3}{(k-1)}\psi(0,0) + \psi(x-y, 0) + \psi(x,y). \tag{16}$$

Thus by (16) we get

$$\|f(2x) + f(0) - 4f(x)\| < \frac{2k-3}{(k-1)}\psi(0,0) + \psi(0,0) + \psi(x,x). \tag{17}$$

Now by (11) and (17)

$$\|f(2x) - 4f(x)\| < \frac{6k-7}{2(k-1)}\psi(0,0) + \psi(x,x), \tag{18}$$

and by definition of M, replacing x by $2^n x$ and dividing by 4^{n+1} we have

$$\|\frac{f(2^{n+1}x)}{4^{n+1}} - \frac{f(2^n x)}{4^n}\| < M\frac{\psi(0,0)}{4^{n+1}} + \frac{\psi(2^n x, 2^n x)}{4^{n+1}}. \tag{19}$$

Hence for any $m, n \in \mathbb{N}$,

$$\|\frac{f(2^m x)}{4^m} - \frac{f(2^n x)}{4^n}\| < M\sum_{i=m}^{n-1}\frac{\psi(0,0)}{4^{i+1}} + \sum_{i=m}^{n-1}\frac{\psi(2^i x, 2^i x)}{4^{i+1}}. \tag{20}$$

We conclude from (20), (8) and definition of ψ that the sequence $\{\frac{f(2^n x)}{4^n}\}$ is a Cauchy sequence in Y, for all $x \in X$. The sequence $\{\frac{f(2^n x)}{4^n}\}$ converges in Y for all $x \in X$, since Y is complete. So we can define the mapping $Q : X \to Y$ by $Q(x) = \lim_{n\to\infty} \frac{f(2^n x)}{4^n}$.

Letting $m = 0$ and passing the limit $n \to \infty$ in (20), we get

$$\|f(x) - Q(x)\| \leq 2M\tilde{\psi}(0,0) + 2\tilde{\psi}(x,x), \ \ x \in X.$$

From the (8) we get

$$\lim_{n\to\infty} \frac{1}{4^n}\psi(2^n x, 2^n y) = \lim_{n\to\infty} \sum_{i=n}^{\infty} \frac{\varphi(2^i x, 2^i y, 0, ..., 0)}{4^{i+1}} = 0, \ \ x, y \in X. \tag{21}$$

Now using definition of Q and relations (15) and (21), one can easily show that

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y), \ \ x, y \in X.$$

On the other hand it follows from Theorem (1) that

$$(4-k)Q(\sum_{i=1}^{k} x_i) + \sum_{j=1}^{k} Q((\sum_{i=1, i\neq j}^{k} x_i) - x_j) = 4\sum_{i=1}^{k} Q(x_i), \text{ for all } x_i \in X, \ \ i = 0, ..., k.$$

To prove the uniqueness of Q, let $T : X \to Y$ be another quadratic mapping satisfying (10). Since Q and T are quadratic mappings, (21) implies that

$$\begin{aligned}\|Q(x) - T(x)\| &= \lim_{n\to\infty} \frac{1}{4^n}\|f(2^n x) - T(2^n x)\| \\ &\leq \lim_{n\to\infty} \frac{2M\tilde{\psi}(0,0)}{4^n} + \lim_{n\to\infty} \frac{2\tilde{\psi}(2^n x, 2^n x)}{4^n} = 0\end{aligned}$$

for all $x \in X$. So $Q = T$.

■

Theorem 3 *Let $k \in \mathbb{N}$ and $\varphi : \underbrace{X \times ... \times X}_{k-times} :\to [0, \infty)$ be a mapping such that*

$$\tilde{\psi}(x,y) = \sum_{i=0}^{\infty} 4^i \varphi(\frac{x}{2^{i+1}}, \frac{y}{2^{i+1}}, 0, ..., 0) < \infty \textit{ for all } x, y \in X. \tag{22}$$

Suppose that the function $f : X \to Y$ satisfy

$$\|(4-k)f(\sum_{i=1}^{k} x_i) + \sum_{j=1}^{k} f((\sum_{i=1, i\neq j}^{k} x_i) - x_j)) - 4\sum_{i=1}^{k} f(x_i)\| < \varphi(x_1, ..., x_k) \tag{23}$$

for all $x_i \in X$, $i = 0, ..., k$. Then there exists exactly one quadratic function $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq 2\tilde{\psi}(x,x), \textit{ for all } x \in X. \tag{24}$$

The function Q is given by

$$Q(x) = \lim_{n\to\infty} 4^n f(\frac{x}{2^n}), \ \ x \in X.$$

Proof. By (22) one can easily see that $\tilde{\psi}(0,0) = 0$. Let $\psi : X \times X \to Y$ be a mapping defined by $\psi(x,y) = \frac{\varphi(x,y,0,...,0)}{2}$, for all $x \in X$. Put $M := \frac{6k-2}{2(k-1)}$. Similar to the proof of Theorem (2) we have

$$\|f(2x) - 4f(x)\| \leq M\psi(0,0) + \psi(x,x),$$

for all $x \in X$. Then we have

$$\|f(x) - 4f(\frac{x}{2})\| \leq M\psi(0,0) + \psi(\frac{x}{2}, \frac{x}{2})$$

for all $x \in X$. Replacing x by $\frac{x}{2^n}$ and multiplying this equation by 4^n, we get

$$\|4^n f(\frac{x}{2^n}) - 4^{n+1} f(\frac{x}{2^{n+1}})\| \le 4^n M\psi(0,0) + 4^n \psi(\frac{x}{2^{n+1}}, \frac{x}{2^{n+1}}), \tag{25}$$

for all $x \in X$. Then from the (25) we get

$$\|4^m f(\frac{x}{2^m}) - 4^n f(\frac{x}{2^n})\| \le \sum_{i=m}^{n-1} 4^i M\psi(0,0) + \sum_{i=m}^{n-1} 4^i \psi(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}), \tag{26}$$

for all $x \in X$. Therefore we conclude from (25), (22) and definition ψ that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence in Y, for all $x \in X$. The sequence $\{4^n f(\frac{x}{2^n})\}$ converges in Y, for all $x \in X$, since Y is complete . So we can define the mapping $Q : X \to Y$ by $Q(x) = \lim_{n\to\infty} 4^n f(\frac{x}{2^n})$, for all $x \in X$.

Letting $m = 0$ and passing the limit $n \to \infty$ in (25) we get

$$\|f(x) - Q(x)\| \le 2\tilde{\psi}(x,x) \text{ for all } x \in X.$$

The rest of the proof is similar to the proof of Theorem (2). ■

The following corollary is a generalization of results of [2] which refine these results.

Corollary 4 *Suppose $k \in \mathbb{N}$, $\epsilon \in \mathbb{R}$ and $f : X \to Y$ satisfies the inequality*

$$\|Df(x_1, ..., x_k)\| \le \epsilon \sum_{i=1}^{k} \|x_i\|^p, \tag{27}$$

for some $2 \ne p \in \mathbb{R}$ and all $x, y, z \in X$. Then there exists a unique quadratic function $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le 4\epsilon \frac{\|x\|^p}{|4 - 2^p|}. \tag{28}$$

Proof. *Define $\varphi : \underbrace{X \times ... \times X}_{k-times} \to [0,\infty)$ by $\varphi(x_1, ..., x_k) = \epsilon \sum_{i=1}^{k} \|x_i\|^p$. If $p < 2$, then with the notations of Theorem 2, we have*

$$\begin{aligned} \widetilde{\psi}(x,y) : &= \sum_{i=0}^{\infty} \frac{\varphi(2^i x, 2^i y, 0, ..., 0)}{4^{i+1}} \\ &= \frac{\epsilon}{4}(\|x\|^p + \|y\|^p) \sum_{i=0}^{\infty} \frac{1}{2^{i(2-p)}} \\ &= \frac{\epsilon}{2^p - 4}(\|x\|^p + \|y\|^p) < \infty. \end{aligned}$$

So by Theorem 2, there exists a unique $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le 2M\widetilde{\psi}(0,0) + 2\widetilde{\psi}(x,x) = \frac{4\epsilon\|x\|^p}{2^p - 4}. \tag{29}$$

Now if $2 < p < \infty$, then with the notations of Theorem 3,

$$\begin{aligned} \tilde{\psi}(x,y) &:= \epsilon \sum_{i=0}^{\infty} 4^i (\|\frac{x}{2^{i+1}}\|^p + \|\frac{y}{2^{i+1}}\|^p) \\ &= \frac{\epsilon}{2^p}(\|x\|^p + \|y\|^p) \sum_{i=0}^{\infty} \frac{4^i}{2^{ip}} \\ &= \epsilon(\|x\|^p + \|y\|^p) \frac{1}{2^p - 4}. \end{aligned}$$

By Theorem 3, there exits a unique $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le 2\widetilde{\psi}(x,x) = \frac{4\epsilon\|x\|^p}{2^p - 4} \tag{30}$$

Now (29) and (30) implies (28), and this completes the proof. ■

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