ON SOLUTIONS OF A GENERALIZED QUADRATIC FUNCTIONAL EQUATION OF PEXIDER TYPE

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Abstract
In this paper we study general solutions of the following Pexider functional equation

\[(4 - k)f_1\left(\sum_{i=1}^{k} x_i\right) + \sum_{j=2}^{k} f_j\left(\sum_{i=1, i \neq j}^{k} x_i\right) - x_j\right)\]

\[+ f_{k+1}\left(-x_1 + \sum_{i=2}^{k} x_i\right) = 4\sum_{j=1}^{k} f_{k+j+1}(x_j),\]

on a vector space over a field of characteristic different from 2, for \(k \geq 3\).

Keywords: Quadratic functional equation, Pexiderized quadratic functional equation.


1. Introduction
It is easy to see that the quadratic function \(f(x) = x^2\) is a solution for each of the following functional equations

(1.1) \(f(x + y) + f(x - y) = 2f(x) + 2f(y)\)

(1.2) \(f(x + y + z) + f(x - y + z) + f(x + y - z) + f(-x + y + z) = 4f(x) + 4f(y) + 4f(z).\)

So, it is natural that these equations are called quadratic functional equations. In particular, every solution of the original quadratic functional equation (1.1) is said to be a quadratic function.

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It is well known that a function \( f \) between real vector spaces is quadratic if and only if there exists a unique symmetric bi-additive function \( B \) such that \( f(x) = B(x, x) \), for all \( x \) (see [1], [2], [7], [14], [15]).

It was proved in [12] that the functional equations (1.1) and (1.2) are equivalent. Solutions and the Hyers–Ulam–Rassias stability of the following generalization of (1.2),

\[
Df(x_1, \ldots, x_k) := (4 - k)f(\sum_{i=1}^{k} x_i) + \sum_{j=1}^{k} f(\sum_{i=1, i \neq j}^{k} x_i) - 4 \sum_{i=1}^{k} f(x_i),
\]

also were investigated in [8].

Solutions and the stability of various functional equations in several variables are also studied by many authors (see for example [3]–[6], [9]–[11], [13], [16], [17]–[21]).

**1.1. Theorem.** Assume that \( X \) and \( Y \) are vector spaces over fields of characteristic different 2, respectively. The functions \( f_i : X \to Y, i = 1, \ldots, 7 \), satisfy the functional equation (1.4) if and only if there exist a quadratic function \( Q : X \to Y \), constants \( c_i \in Y, i = 1, \ldots, 7 \), and additive functions \( a_i : X \to Y, i = 1, \ldots, 4 \), such that

\[
\begin{align*}
&f_1(x) = Q(x) + 2a_1(x) + a_2(x) + a_3(x) - a_4(x) + c_1, \\
f_2(x) = Q(x) - a_2(x) + a_3(x) + a_4(x) + c_2, \\
f_3(x) = Q(x) + a_2(x) - a_3(x) + a_4(x) + c_3, \\
f_4(x) = Q(x) - 2a_1(x) + a_2(x) + a_3(x) + a_4(x) + c_4, \\
f_5(x) = Q(x) + a_1(x) + c_5, \\
f_6(x) = Q(x) + a_2(x) + c_6, \\
f_7(x) = Q(x) + a_3(x) + c_7,
\end{align*}
\]

with

\[
c_1 + c_2 + c_4 + c_4 = 4c_5 + 4c_6 + 4c_7.
\]

To prove our main result, we need the proof of this theorem. In this proof with \( F_i(x) = f_i(x) - f_i(0) \), if \( F^e_i \) and \( F^o_i \) denote the even part and odd part of \( F_i \), respectively, then it is proved that there exist additive functions \( a_i : X \to Y, i = 1, \ldots, 4 \), such that

\[
\begin{align*}
&F^e_2 = 2a_1 + a_2 + a_3 - a_4, \\
&F^o_2 = -a_2 + a_3 + a_4, \\
&F^e_3 = a_2 - a_3 + a_4, \\
&F^o_3 = -a_2 + a_3 + a_4, \\
&F^e_4 = -2a_1 + a_2 + a_3 + a_4, \\
&F^o_4 = a_1, \\
&F^e_5 = a_2, \\
&F^o_5 = a_3
\end{align*}
\]

and there exists a quadratic function \( Q : X \to Y \) such that

\[
F^e_1 = F^o_2 = F^o_3 = F^o_4 = F^e_5 = F^o_6 = F^o_7 = Q.
\]

In this paper we consider the following Pexiderized form of (1.3)

\[
(4 - k)f(\sum_{i=1}^{k} x_i) + \sum_{j=2}^{k} f_j(\sum_{i=1, i \neq j}^{k} x_i) - 4 \sum_{i=1}^{k} f_i(-x_i + \sum_{i=1}^{k} x_i) = 4 \sum_{j=1}^{k} f_{k+j+1}(x_j),
\]

(1.8)
for $k \geq 3$, and general solutions of this functional equation will be investigated in Section 2. Note that $f_1$ does not exist when $k = 4$.

2. Solution of equation (1.8)

2.1. Theorem. Assume that $X$ and $Y$ are vector spaces over fields of characteristic different from 2, respectively and $k$ is a natural number with $k \geq 3$. Then functions $f_i : X \to Y$, $i = 1, \ldots, 2k+1$, satisfy the functional equation (1.8) if and only if there exist a quadratic function $Q : X \to Y$, constants $c_i \in Y$, $i = 1, \ldots, 2k+1$, and additive functions $a_i : X \to Y$, $i = 1, \ldots, k$, and $a_i' : X \to Y$, $i = 3, \ldots, k$ such that

$$f_1(x) = Q(x) + \frac{1}{4-k}(2a_1 + (5-k)a_2 + \sum_{i=3}^{k} a_i - \sum_{i=3}^{k} a'_i(x)) + c_1, \quad k \neq 4,$$

$$f_2(x) = Q(x) - a_2(x) + a_3(x) + c_2,$n

$$f_i(x) = Q(x) + a_2(x) - a_j(x) + a'_j(x) + c_j, \quad j = 3, 4, \ldots, k,$n

$$f_{k+j+1}(x) = Q(x) + a_j(x) + c_{k+j+1}, \quad j = 0, 1, 2, \ldots, k$$

with

$$\sum_{i=1}^{k+1} c_i = 4 \sum_{i=k+2}^{2k+1} c_i.$$

Proof. Define $c_i = f_i(0)$, $i = 1, \ldots, 2k+1$. By letting $x_i = 0$, $i \geq 1$, in (1.8) it is clear that the $c_i$'s satisfy the relation (2.2). For $i = 1, \ldots, 2k+1$, define $F_i(x) = f_i(x) - c_i$. It then follows from (1.8) and (2.2) that the $F_i$'s satisfy the functional equation (1.8) with $F_1(0) = 0$.

Denote by $F_i^n(x)$ and $F_i^o(x)$ the even part and the odd part of $F_i(x)$, respectively. If we replace $x_i$ in (1.8) by $-x_i$, for any $i = 1, \ldots, k$, and if we add (subtract) the resulting equation to (from) (1.8), we can see that the $F_i^n$'s as well as the $F_i^o$'s also satisfy (1.8).

Let us consider (1.8) for the $F_i^n$'s

$$(4-k)F_i^n(\sum_{i=1}^{k} x_i) + \sum_{j=2}^{k} F_j^n((- \sum_{i=1, i \neq j}^{k} x_i) - x_j)) + F_{k+1}^n(-x_1 + \sum_{i=2}^{k} x_i)$$

$$= 4 \sum_{j=1}^{k} F_{k+j+1}^o(x_j).$$

Step I By letting $x_i = 0$, $i \geq 4$, in (2.3) we get

$$((4-k)F_i^n + F_j^n + \cdots + F_k^n)(x_1 + x_2 + x_3) + F_2^n(x_1 - x_2 + x_3)$$

$$+ F_3^n(x_1 + x_2 - x_3) + F_{k+1}^o(-x_1 + x_2 + x_3)$$

$$= 4F_{k+2}^n(x_1) + 4F_{k+3}^o(x_2) + 4F_{k+4}^o(x_3).$$

So, by relations (1.5) and (1.6), there exist additive functions $a_i : X \to Y$, $i = 1, 2, 3$, and an additive function $a'_3 : X \to Y$ such that

$$F_{k+2}^n = a_1,$n

$$F_{k+3}^o = a_2,$n

$$F_{k+4}^o = a_3,$n

$$F_3^o = a_2 - a_3 + a'_3,$n

$$F_2^o = -a_2 + a_3 + a'_3.$$
(2.4) \((4-k)F_1^o + F_3^o + F_5^o + \cdots + F_k^o = 2a_1 + a_2 + a_3 - a'_3\)

and also
\[ F_{k+1}^o = -2a_1 + a_2 + a_3 + a'_3. \]

**Step II** By putting \(x_i = 0\), for \(i \geq 3, i \neq 4\), in (2.3) we have
\[
((4-k)F_1^o + F_3^o + F_5^o + \cdots + F_{k-1}^o)(x_1 + x_2 + x_3) + F_2^o(x_1 - x_2 + x_4) \\
+ F_4^o(x_1 + x_2 - x_4) + F_{k+1}^o(-x_1 + x_2 + x_4) \\
= 4F_{k+2}^o(x_1) + 4F_{k+3}^o(x_2) + 4F_{k+5}^o(x_4).
\]

Then by relations (1.5) and (1.6), there exist additive functions \(a_4 : X \to Y\) and \(a'_4 : X \to Y\), \(i = 3, 4\), such that
\[
F_{k+2}^o = a_1 \\
F_{k+3}^o = a_2 \\
F_{k+5}^o = a_4 \\
F_4^o = a_2 - a_4 + a'_4 \\
F_2^o = -a_2 + a_3 + a'_3 \\
(4-k)F_1^o + F_3^o + F_5^o + \cdots + F_k^o = 2a_1 + a_2 + a_4 - a'_4.
\]

**Step III** If we put \(x_i = 0\) for \(i \geq 3, i \neq j\), in (2.3) then we get
\[
((4-k)F_1^o + F_3^o + \cdots + F_{j-1}^o + F_{j+1}^o + F_{j+2}^o + \cdots + F_k^o)(x_1 + x_2 + x_j) \\
+ F_2^o(x_1 - x_2 + x_j) + F_4^o(x_1 + x_2 - x_j) + F_{k+1}^o(-x_1 + x_2 + x_j) \\
= 4F_{k+2}^o(x_1) + 4F_{k+3}^o(x_2) + 4F_{k+j+1}^o(x_j).
\]

So, by relations (1.5) and (1.6), there exist additive functions \(a_j : X \to Y\), and \(a'_j : X \to Y\) such that
\[
F_{k+2}^o = a_1 \\
F_{k+3}^o = a_2 \\
F_{k+j+1}^o = a_j \\
F_j^o = a_2 - a_j + a'_j \\
F_2^o = -a_2 + a_3 + a'_3 \\
(4-k)F_1^o + F_3^o + \cdots + F_{j-1}^o + F_{j+1}^o + F_{j+2}^o + \cdots + F_k^o = 2a_1 + a_2 + a_j - a'_j.
\]

**Step IV** If we put \(x_i = 0\) for \(i \geq 3, i \neq k\), in (2.3) then we have
\[
((4-k)F_1^o + F_3^o + F_5^o + \cdots + F_{k-1}^o)(x_1 + x_2 + x_k) + F_2^o(x_1 - x_2 + x_k) \\
+ F_4^o(x_1 + x_2 - x_k) + F_{k+1}^o(-x_1 + x_2 + x_k) \\
= 4F_{k+2}^o(x_1) + 4F_{k+3}^o(x_2) + 4F_{2k+1}^o(x_k).\]
Using relations (1.5) and (1.6) we may find additive functions $a_k : X \to Y$, and $a'_k : X \to Y$ such that 
\[
\begin{align*}
F'_{k+2} &= a_1 \\
F'_{k+3} &= a_2 \\
F'_{2k+1} &= a_k \\
F'_k &= a_2 - a_k + a'_k \\
F'_2 &= -a_2 + a_3 + a'_3 \\
(4\cdot k - 1)F'_{1} + F'_{3} + F'_{4} + \cdots + F'_{k-1} &= 2a_1 + a_2 + a_k - a'_k.
\end{align*}
\]
By these steps we get all $F'_i$, $i = 2, \ldots, 2k + 1$. If $k = 4$, then there is no need to consider $F'_4$. For $k \neq 4$, using (2.4) we may find $F'_i$ as follows 
\[
F'_i(x) = \frac{1}{4 - k}(2a_1 + (5 - k)a_2 + (a_3 + a_4 + a_5 + \cdots + a_k) - (a'_3 + a'_4 + a'_5 + \cdots + a'_k)).
\]
We will now deal with (1.8) associated with the $F'_i$'s; 
\[
(4\cdot k - 1)F'_1 + F'_3 + F'_4 + \cdots + F'_k = F'_k = F'_{k+1} = F'_{k+2} = F'_{k+3} = F'_{k+4} = Q.
\]
Also for any $i = 4, 5, \ldots, k$, by letting $x_i = 0$, for $3 \leq i \leq k$, $i \neq j$, in (2.5) we get 
\[
((4\cdot k - 1)F'_1 + F'_3 + F'_4 + \cdots + F'_k)(x_1 + x_2 + x_3) + F'_2(x_1 - x_2 + x_3) + F'_3(x_1 + x_2 - x_3) + F'_4(-x_1 + x_2 + x_3)
\]
\[
= 4F'_{k+2}(x_1) + 4F'_{k+3}(x_2) + 4F'_{k+4}(x_3).
\]
Hence, by (1.7), there exists a quadratic function $Q : X \to Y$ such that 
\[
(4\cdot k - 1)F'_1 + F'_3 + F'_4 + \cdots + F'_k = F'_k = F'_{k+1} = F'_{k+2} = F'_{k+3} = F'_{k+4} = Q.
\]
Thus using (1.7), we get 
\[
(4\cdot k - 1)F'_1 + F'_3 + F'_4 + \cdots + F'_{k-1} + F'_j = F'_k = F'_{k+1} = F'_{k+2} = F'_{k+3} = F'_{k+j+1} = Q.
\]
Therefore, for $i \geq 2$, $F'_i$ is concluded by (2.6) and (2.7). For getting $F'_1$, when $k \neq 4$, we note that 
\[
(4\cdot k - 1)F'_1 + F'_3 + F'_4 + \cdots + F'_{k-1} = Q.
\]
Also from relations (2.6) and (2.7), we have 
\[
F'_k = \cdots = F'_{k-1} = Q,
\]
and so we get 
\[
F'_1 = Q.
\]
Conversely, if there exist a quadratic function $Q : X \to Y$, constants $c_i \in Y$, $i = 1, \ldots, 2k + 1$, satisfying (2.2), and if there exist additive functions $a_i : X \to Y$, $i = 1, \ldots, k$, and $a'_i : X \to Y$, $i = 3, \ldots, k$, such that each of the equations in (2.1) holds true, then it is obvious that the $f_i$'s satisfy the functional equation (1.8). \[\square\]
References


