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NONLINEAR FREE VIBRATION OF SIMPLY SUPPORTED BEAMS CONSIDERING THE EFFECTS OF SHEAR DEFORMATION AND ROTARY INERTIA, A HOMOTOPY PERTURBATION APPROACH

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The objective of this paper is to apply He's homotopy perturbation method (HPM) to analyze nonlinear free vibration of simply supported Timoshenko beams considering the effects of rotary inertia and shear deformation. First, the equation governing the nonlinear free vibration of a Timoshinko beam is nondimensionalized. Galerkin's projection method is utilized to reduce the governing nonlinear partial differential equation to a nonlinear ordinary differential equation. HPM is then used to find analytic expressions for nonlinear natural frequencies of the pre-stretched beam. A parametric study has also been applied in order to investigate the effects of design parameters such as applied axial load and slenderness ratio. Comparison between presented results and numerical results which are in full agreement shows that HPM can significantly improve the accuracy of previously reported results in the literature.

Keywords: Homotopy perturbation method (HPM); modified Lindstedt–Poincaré method; shear deformation; rotary inertia; Timoshenko beam; nonlinear vibration.

1. Introduction

Beams construct the most important building blocks of most engineering structures. For instance, they find application in variety of structures from micro/nano dimensions such as micro/nano resonators to macro dimensions such as airplane wings, flexible satellites, and long span bridges.

In beams systems, nonlinear effects come into play in large amplitude vibrations. The sources of the nonlinearities may be geometric, inertial, or material in nature.[1]

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Pillai and Rao[2] examined the problem of large amplitude free vibrations of simply supported uniform beams and found the frequency response of the system by several methods, the elliptic function method, the harmonic balance method, and the method which one assumes simple harmonic oscillations.

Effects of rotary inertia and shear deformation are not negligible for thick beams or even thin beams that are vibrating at high frequencies like micro/nano scale resonators which are vibrating at extremely high frequencies.[3] Only a few researchers have considered the effect of shear deformation and rotary inertia in vibration of beams. Chen[4] carried out the development of differential quadrature element method to the out of plane vibration analysis of curved nonprismatic beam structures considering the effect of shear deformation. Foda[5] used the method of multiple scales to analyze the nonlinear vibrations of a beam with pinned ends considering the effect of shear deformation and rotary inertia. Ramezani *et al.*[3] used the same method for the same problem with doubly clamped boundary conditions.

The current paper makes use of the HPM to analyze nonlinear free vibration of Timoshenko beams. Recently, several attempts have been made to develop the new techniques for obtaining analytical solutions for nonlinear problems which reasonably approximate the exact solutions. For instance, Adomian decomposition method (ADM)[6,7] the variational iteration method (VIM)[8] and the homotopy perturbation method (HPM)[9] have drawn great attention of scientists and engineers. HPM is one of the most effective techniques among these methods, which has been employed to solve a large variety of linear and nonlinear problems.[10]

He[11] presented a new perturbation technique which does not depend upon the assumption of small parameters. His new method takes full advantages of the traditional perturbation methods and homotopy techniques. Blendez *et al.*[12] solved the nonlinear differential equations which govern the nonlinear oscillation of a simple pendulum and showed that even only one iteration leads to the relative error of less than 2% for the approximated period even for amplitudes as high as 130°. Blendez *et al.*[13] found improved approximate solutions to conservative truly nonlinear oscillators using He's HPM. They found that for the second-order approximation the relative error in the analytical approximate frequency is approximately 0.03% for any parameter values involved. Ganji *et al.*[14] applied the HPM to solve the second kind of nonlinear integral equations. Their results revealed that the HPM is very effective and simple. Sadighi and Ganji[15] introduced analytical methods, called HPM, ADM, and VIM to obtain the exact solutions of Laplace equation with Dirichlet and Neumann boundary conditions. The comparison among these methods showed that HPM is much easier, more convenient, and efficient than ADM and VIM. Momani and Odibat[16] presented an efficient and reliable treatment of the HPM for nonlinear partial differential equations with fractional time derivative and showed that HPM is robust, efficient, and easy to implement.

Ganji *et al.*[17–19] applied homotopy perturbation for analyzing heat transfer problems. Their results reveal that HPM is very effective, convenient, and quite accurate.

As it is observed in the literature of the HPM, this method overcomes the limitations of classical perturbation methods and at the same time provides an accurate prediction of the behavior of the nonlinear systems. So here, this method has been used in conjunction with the modified Lindstedt–Poincaré method to solve the problem of nonlinear free vibration of Timoshenko beams considering the effects of shear deformation and rotary inertia.

2. Problem Formulation

Utilizing the Hamilton's principal and considering the effects of rotary inertia and shear deformation, Foda[5] found the nonlinear partial differential equation governing the beams behavior as follows:

$$EI\frac{\partial^4\hat{w}}{\partial\hat{x}^4}+m\frac{\partial^2\hat{w}}{\partial\hat{t}^2}-\left(mr^2+\frac{mEI}{kAG}\right)\frac{\partial^4\hat{w}}{\partial\hat{x}^2\partial\hat{t}^2}+\frac{m^2r^2}{kAG}\frac{\partial^4\hat{w}}{\partial\hat{t}^4}$$
$$-N\left[\frac{\partial^2\hat{w}}{\partial\hat{x}^2}-\frac{EI}{kAG}\frac{\partial^4\hat{w}}{\partial\hat{x}^4}+\frac{mr^2}{KAG}\frac{\partial^4\hat{w}}{\partial\hat{x}^2\partial\hat{t}^2}\right]=0\,. \tag{1}$$

In this equation, E is the Young's modulus of elasticity of the beam material, I is the second moment of area of the cross section with respect to the bending axis, $\hat{w}$ is the beam deflection, m is the longitudinal density, $\hat{t}$ is the time, A is the cross sectional area of the beam, G is the shear modulus, and k is the shear correction factor that depends only on the geometric properties of the cross section of the beam. The parameters N and r are defined as follows:

$$N=N_0+\frac{EA}{2L}\int_0^L\left(\frac{\partial\hat{w}}{\partial\hat{x}}\right)^2d\hat{x}\,, \tag{2}$$

$$r^2=\frac{I}{A}\,, \tag{3}$$

where N_0 is the pretension of the beam and L is the length of the beam. For simply supported beams the following boundary conditions must be met.

$$\hat{w}(0,\hat{t})=\hat{w}(L,\hat{t})=\frac{\partial^2\hat{w}(0,\hat{t})}{\partial\hat{t}^2}=\frac{\partial^2\hat{w}(L,\hat{t})}{\partial\hat{t}^2}=0\,. \tag{4}$$

The initial conditions are chose as Eqs. (5) and (6).

$$\hat{w}(\hat{x},0)=W_{\max}\sin\left(\frac{n\pi\hat{x}}{L}\right), \tag{5}$$

$$\frac{\partial\hat{w}(\hat{x},0)}{\partial\hat{t}}=\frac{\partial^2\hat{w}(\hat{x},0)}{\partial\hat{t}^2}=\frac{\partial^3\hat{w}(\hat{x},0)}{\partial\hat{t}^3}=0\,. \tag{6}$$

The initial conditions indicated in Eq. (5) imply that the beam is initially deflected in its nth vibrational mode. Introducing the nondimensionalized variables t, x, w,

and T which are defined in Eqs. (7)–(9), Eq. (1) is nondimensionalized in the form of Eq. (11).

$$t = \frac{\hat{t}}{T}\,, \tag{7}$$

$$x = \frac{\hat{x}}{L}\,, \tag{8}$$

$$w = \frac{\hat{w}}{L}\,, \tag{9}$$

$$T = \frac{1}{\beta^2}\sqrt{\frac{mL^4}{EI}}\,, \tag{10}$$

$$\frac{\partial^4 w}{\partial x^4} + \beta^4 \frac{\partial^2 w}{\partial t^2} - \left(1 + \frac{E}{kG}\right)\left(\frac{r}{L}\right)^2 \beta^4 \frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{E\beta^8}{kG}\left(\frac{r}{L}\right)^4 \frac{\partial^4 w}{\partial t^4}$$
$$- \frac{N}{EA}\left(\frac{L}{r}\right)^2 \left[\frac{\partial^2 w}{\partial x^2} - \frac{E}{kG}\left(\frac{r}{L}\right)^2 \frac{\partial^4 w}{\partial x^4} + \frac{E\beta^4}{KG}\left(\frac{r}{L}\right)^4 \frac{\partial^4 w}{\partial x^2 \partial t^2}\right] = 0\,. \tag{11}$$

Note that for simply supported boundary conditions

$$\beta_n = n\pi\,. \tag{12}$$

The nondimensionalized form of boundary conditions is Eq. (13) and the nondimensionalized form of initial conditions are Eqs. (14) and (15).

$$w(0,t) = w(1,t) = \frac{\partial^2 w(0,t)}{\partial t^2} = \frac{\partial^2 w(1,t)}{\partial t^2} = 0\,, \tag{13}$$

$$w(x,0) = \frac{W_{\max}}{L}\sin(n\pi x)\,, \tag{14}$$

$$\frac{\partial w(x,0)}{\partial t} = \frac{\partial^2 w(x,0)}{\partial t^2} = \frac{\partial^3 w(x,0)}{\partial t^3} = 0\,. \tag{15}$$

Now the Galerkin projection method is used to convert the nonlinear partial differential Eq. (11) to a nonlinear ordinary differential equation. So the solution Eq. (11) is assumed as

$$w(x,t) = \phi_1(x) q(t)\,, \tag{16}$$

where $\phi_n(x)$ is the nth linear undamped vibrational mode of the beam which satisfy boundary conditions (13). For simply support boundary conditions $\phi_n(x)$ can be stated as Eq. (17).

$$\phi_n(x) = \sin(n\pi x)\,. \tag{17}$$

By substituting Eq. (16) into Eq. (11) and integrating the residual by weight $\phi_n(x)$ over the problem domain, one may arrive to the following nonlinear ODE.

$$\ddddot{q} + (\alpha_1 + \alpha_2 q^2)\ddot{q} + \alpha_3 q + \alpha_4 q^3 = 0\,, \tag{18}$$

where

$$\alpha_1 = \frac{1}{\beta^4 F_1}\left(\frac{L}{r}\right)^4 \left[\frac{kG}{E}F_1 + \left(1 + \frac{kG}{E}\right)\left(\frac{r}{L}\right)^2 F_2 + \frac{N_0}{EA}\left(\frac{r}{L}\right)^2 F_2\right], \tag{19}$$

$$\alpha_2 = \frac{1}{2\beta^4 F_1}\left(\frac{L}{r}\right)^2 F_2^2 \tag{20}$$

$$\alpha_3 = \frac{1}{\beta^8 F_1}\left(\frac{L}{r}\right)^4 \left[\frac{kG}{E}F_3 + \frac{kG}{E}\frac{N_0}{EA}\left(\frac{L}{r}\right)^2 F_2 + \frac{N_0}{EA}F_3\right], \tag{21}$$

$$\alpha_4 = \frac{1}{2\beta^8 F_1}\left(\frac{L}{r}\right)^4 \left[\frac{kG}{E}\left(\frac{L}{r}\right)^2 F_2^2 + F_2 F_3\right]. \tag{22}$$

In Eqs. (19)–(22), F_i's, $1 \leqslant i \leqslant 3$ are defined as follows.

$$F_1 = \int_0^1 \phi_n^2 dx\,, \tag{23}$$

$$F_2 = \int_0^1 \phi_n'^2 dx\,, \tag{24}$$

$$F_3 = \int_0^1 \phi_n''^2 dx\,. \tag{25}$$

For simply supported boundary conditions, the parameters α_i's $(1 \leq i \leq 4)$ are simplified as

$$\alpha_1 = \left[\frac{kG}{E}\left(\frac{L}{n\pi r}\right)^4 + \left(1 + \frac{kG}{E}\right)\left(\frac{L}{n\pi r}\right)^2 + \frac{N_0}{EA}\left(\frac{L}{n\pi r}\right)^2\right], \tag{26}$$

$$\alpha_2 = \frac{n^2\pi^2}{4}\left(\frac{L}{n\pi r}\right)^2, \tag{27}$$

$$\alpha_3 = \left[\frac{kG}{E}\left(\frac{L}{n\pi r}\right)^4 + \frac{kG}{E}\frac{N_0}{EA}\frac{n^2\pi^2}{2}\left(\frac{L}{n\pi r}\right)^6 + \frac{N_0}{EA}\left(\frac{L}{n\pi r}\right)^4\right], \tag{28}$$

$$\alpha_4 = \frac{n^2\pi^2}{4}\left[\frac{kG}{E}\left(\frac{L}{n\pi r}\right)^6 + \left(\frac{L}{n\pi r}\right)^4\right]. \tag{29}$$

The initial conditions (14) and (15) are transformed to the following initial conditions for $q(t)$.

$$q(0) = \frac{W_{\max}}{L}\,. \tag{30}$$

$$\dot{q}(0) = 0\,, \tag{31}$$

$$\ddot{q}(0) = 0\,, \tag{32}$$

$$\ddddot{q}(0) = 0\,. \tag{33}$$

Now the HPM is used to solve nonlinear ODE (18). For this purpose at the first, homotopy form is constructed as Eq. (34).

$$\ddddot{q} + \alpha_1\ddot{q} + \alpha_3 q + P[\alpha_2 q^2\ddot{q} + \alpha_4 q^3] = 0\,. \tag{34}$$

From Eq. (34), linear frequencies of the Timoshenko beams are calculated as Eqs. (35) and (36).

$$\omega_{10}^2 = \frac{\alpha_1}{2} - \sqrt{\frac{\alpha_1^2}{4} - \alpha_3}\,, \tag{35}$$

$$\omega_{20}^2 = \frac{\alpha_1}{2} + \sqrt{\frac{\alpha_1^2}{4} - \alpha_3}\,, \tag{36}$$

where ω_{10} is the bending natural frequency and ω_{20} is the rotary natural frequency. Equations (35) and (36) can be solved for α_1 and α_3. The results are as follows:

$$\alpha_1 = \omega_{10}^2 + \omega_{20}^2\,, \tag{37}$$

$$\alpha_3 = \omega_{10}^2\omega_{20}^2\,. \tag{38}$$

Using the modified Lindstedt–Poincaré method, $q(t)$, ω_{10}^2, and ω_{20}^2 are perturbed using homotopy parameter P.

$$q(t) = q_0(t) + Pq_1(t) + O(P^2)\,, \tag{39}$$

$$\omega_{10}^2 = \omega_1^2 + P\omega_{11} + O(P^2)\,, \tag{40}$$

$$\omega_{20}^2 = \omega_2^2 + P\omega_{21} + O(P^2)\,. \tag{41}$$

Equations (40) and (41) are then substituted into Eqs. (37) and (38) to find first-order perturbation expansion of coefficients α_1 and α_3.

$$\alpha_1 = C_1 + C_2P + O(P^2)\,, \tag{42}$$

$$\alpha_3 = C_3 + C_4P + O(P^2)\,, \tag{43}$$

where

$$C_1 = \omega_1^2 + \omega_2^2 \,, \tag{44}$$

$$C_2 = \omega_{11} + \omega_{21} \,, \tag{45}$$

$$C_3 = \omega_1^2 \omega_2^2 \,, \tag{46}$$

$$C_4 = \omega_1^2 \omega_{21} + \omega_2^2 \omega_{11} \,. \tag{47}$$

Substituting Eqs. (39), (42), and (43) into Eq. (34) and setting the coefficient of like powers of P equal to zero leads to the following sets of differential equations.

$$\ddddot{q}_0 + C_1 \ddot{q}_0 + C_3 q_0 = 0 \,, \tag{48}$$

$$\ddddot{q}_1 + C_1 \ddot{q}_1 + C_3 q_1 + C_2 \ddot{q}_0 + C_4 q_0 + \alpha_2 q_0^2 \ddot{q}_0 + \alpha_4 q_0^3 = 0 \,. \tag{49}$$

The initial conditions (30)–(33) are translated to initial conditions (50)–(53) for Eq. (48) and initial conditions (54)–(57) for Eq. (49).

$$q_0(0) = \frac{W_{\max}}{L} \,, \tag{50}$$

$$\dot{q}_0(0) = 0 \,, \tag{51}$$

$$\ddot{q}_0(0) = 0 \,, \tag{52}$$

$$\ddddot{q}_0(0) = 0 \,, \tag{53}$$

$$q_1(0) = 0 \,, \tag{54}$$

$$\dot{q}_1(0) = 0 \,, \tag{55}$$

$$\ddot{q}_1(0) = 0 \,, \tag{56}$$

$$\ddddot{q}_1(0) = 0 \,, \tag{57}$$

Solution of Eq. (48) can be expressed as Eq. (58) for $q_0(t)$.

$$q_0(t) = A \cos \omega_1 t + B \cos \omega_2 t \,, \tag{58}$$

where

$$A = \frac{W_{\max}}{L} \frac{\omega_2^2}{\omega_2^2 - \omega_1^2} \,, \tag{59}$$

$$B = \frac{W_{\max}}{L} \frac{\omega_1^2}{\omega_1^2 - \omega_2^2} \,. \tag{60}$$

Substituting $q_0(t)$ from Eq. (58) to Eq. (49), leads to the following differential equation for $q_1(t)$.

$$\begin{aligned}\ddddot{q}_1 &+ C_1\ddot{q}_1 + C_3 q_1 - C_2(A\omega_1^2\cos\omega_1 t + B\omega_2^2\cos\omega_2 t) + C_4(A\cos\omega_1 t + B\cos\omega_2 t)\\ &- \alpha_2(A\cos\omega_1 t + B\cos\omega_2 t)^2(A\omega_1^2\cos\omega_1 t + B\omega_2^2\cos\omega_2 t)\\ &+ \alpha_4(A\cos\omega_1 t + B\cos\omega_2 t)^3\,. \end{aligned} \tag{61}$$

Elimination of secular terms in Eq. (61) yields to Eqs. (62) and (63).

$$\begin{aligned} C_4 A - C_2 A\omega_1^2 - \frac{3}{4}\alpha_2 A^3\omega_1^2 - \alpha_2 AB^2\omega_2^2 - \frac{1}{2}\alpha_2 AB^2\omega_1^2\\ + \frac{3}{2}\alpha_4 AB^2 + \frac{3}{4}\alpha_4 A^3 = 0\,, \end{aligned} \tag{62}$$

$$\begin{aligned} C_4 B - C_2 B\omega_2^2 - \frac{3}{4}\alpha_2 B^3\omega_2^2 - \alpha_2 A^2 B\omega_1^2 - \frac{1}{2}\alpha_2 A^2 B\omega_2^2\\ + \frac{3}{2}\alpha_4 A^2 B + \frac{3}{4}\alpha_4 B^3 = 0\,. \end{aligned} \tag{63}$$

Letting $P = 1$ in Eqs. (42) and (43) and noting that $C_1 = \omega_1^2 + \omega_2^2$ and $C_3 = \omega_1^2\omega_2^2$, C_2 and C_4 are determined as follows:

$$C_2 = \alpha_1 - (\omega_1^2 + \omega_2^2)\,, \tag{64}$$

$$C_4 = \alpha_3 - \omega_1^2\omega_2^2\,. \tag{65}$$

By substituting Eqs. (64) and (65) into Eqs. (62) and (63), two coupled fourth-order polynomial algebraic equations in terms of ω_1 and ω_2 are obtained.

$$\begin{aligned} A\omega_1^4 - \left(\alpha_1 A + \frac{3}{4}\alpha_2 A^3 + \frac{1}{2}\alpha_2 AB^2\right)\omega_1^2 - (\alpha_2 AB^2)\omega_2^2\\ + \alpha_3 A + \frac{3}{2}\alpha_4 AB^2 + \frac{3}{4}\alpha_4 A^3 = 0\,, \end{aligned} \tag{66}$$

$$\begin{aligned} B\omega_2^4 - \left(\alpha_1 B + \frac{3}{4}\alpha_2 B^3 + \frac{1}{2}\alpha_2 A^2 B\right)\omega_2^2 - (\alpha_2 A^2 B)\omega_1^2\\ + \alpha_3 B + \frac{3}{2}\alpha_4 A^2 B + \frac{3}{4}\alpha_4 B^3 = 0\,. \end{aligned} \tag{67}$$

Equations (66) and (67) can be stated in a more simplified manner as

$$\begin{aligned} \omega_1^8 - \left(2\omega_2^2 + \alpha_1 + \frac{1}{2}\alpha_2 a^2\right)\omega_1^6 + \left(\omega_2^4 + (2\alpha_1 - \alpha_2 a^2)\omega_2^2 + \alpha_3 + \frac{3}{2}\alpha_4 a^2\right)\omega_1^4\\ - \left(\left(\alpha_1 + \frac{3}{4}\alpha_2 a^2\right)\omega_2^4 + 2\alpha_3\omega_2^2\right)\omega_1^2 + \frac{3}{4}\alpha_4 a^2\omega_2^4 + \alpha_3\omega_2^4 = 0\,, \end{aligned} \tag{68}$$

$$\omega_2^8 - \left(2\omega_1^2 + \alpha_1 + \frac{1}{2}\alpha_2 a^2\right)\omega_2^6 + \left(\omega_1^4 + (2\alpha_1 - \alpha_2 a^2)\omega_1^2 + \alpha_3 + \frac{3}{2}\alpha_4 a^2\right)\omega_2^4$$
$$- \left(\left(\alpha_1 + \frac{3}{4}\alpha_2 a^2\right)\omega_1^4 + 2\alpha_3\omega_1^2\right)\omega_2^2 + \frac{3}{4}\alpha_4 a^2\omega_1^4 + \alpha_3\omega_1^4 = 0\,, \tag{69}$$

where

$$a = \frac{W_{\max}}{L}\,. \tag{70}$$

Equations (68) and (69) are then solved for finding the natural frequencies ω_1 and ω_2.

3. Results and Discussion

In order to demonstrate the accuracy and effectiveness of the procedure explained in previous section, a numerical example is solved for the case of simply supported beams. Table 1 shows the characteristics of the beam under investigation.

Considering the first spatial vibrational mode of the beam, the ratio of the nonlinear period of vibration, T, to the linear period, T_0, of classical beam theory has been computed for different values of $W_{\max}/h$ at different slenderness ratios.

Table 1. Characteristic properties of the material used.[5]

Material properties	E	G	ρ	k
Property value	70 Gpa	27 Gpa	2700 kg/m^3	0.85

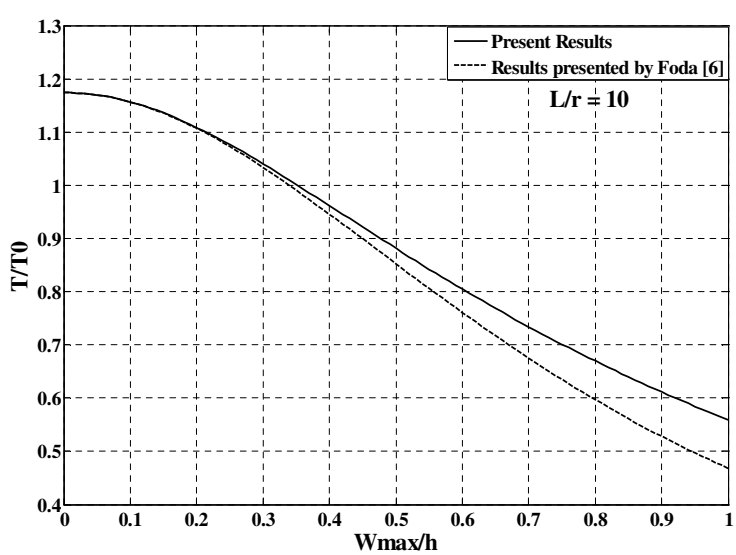

Fig. 1. Variation of the nonlinear nondimensionalized period of vibration with $W_{\max}/h$ for $L/r = 10$.

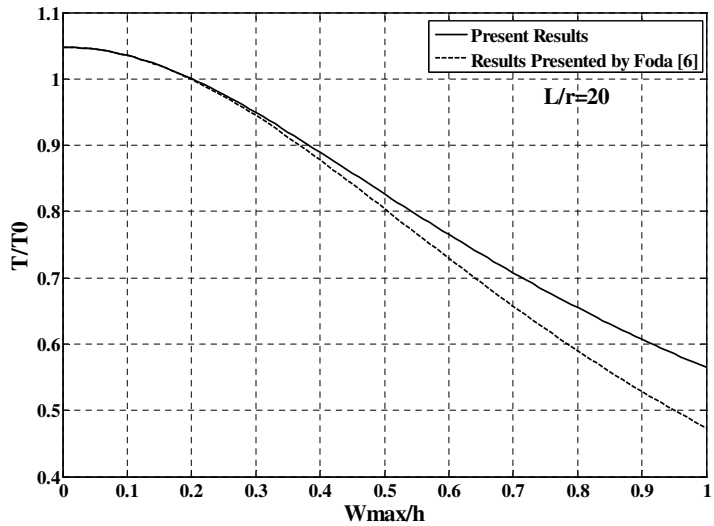

Fig. 2. Variation of the nonlinear nondimensionalized period of vibration with $W_{\max}/h$ for $L/r = 20$.

Figures 1 and 2 show a comparison between results of HPM and results presented by Foda.[6]

For verification purpose, a numerical approach has also been implemented. In the numerical approach, Eq. (13) is solved for each $W_{\max}/h$ numerically and the results in time domain are searched for nonlinear period. In Fig. 3, the present

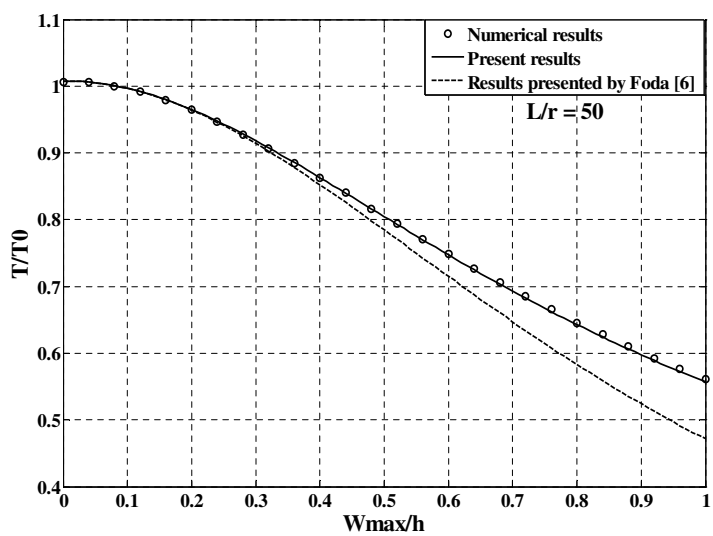

Fig. 3. Variation of the nonlinear nondimensionalized period of vibration with $W_{\max}/h$ for $L/r = 50$.

results have also been compared with numerical results. As it is seen HPM gives better prediction of nonlinear period than the method used by Foda.[6] This figure shows that when the slenderness ratio is relatively large or in other words the problem is strongly nonlinear, HPM can improve the accuracy of the solution up to nearly 15%.

Figures 4–6 show the effect of the parameter $N_0/(E.A)$ on the nonlinear period of vibration. It is seen that applying pre-tensile loads will reduce the nonlinear

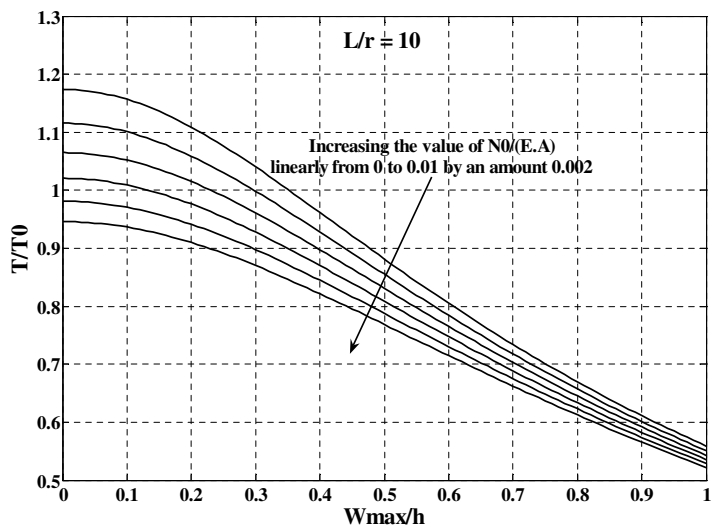

Fig. 4. Effect of the parameter $N_0/(E.A)$ on the nonlinear period of vibration for $L/r = 10$.

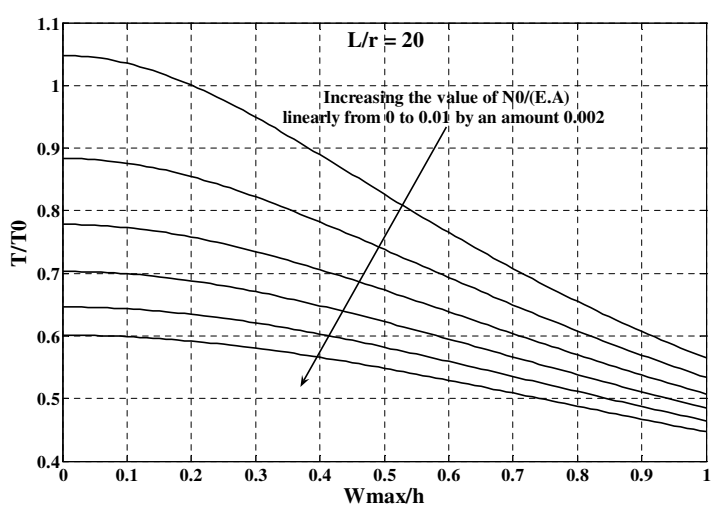

Fig. 5. Effect of the parameter $N_0/(E.A)$ on the nonlinear period of vibration for $L/r = 20$.

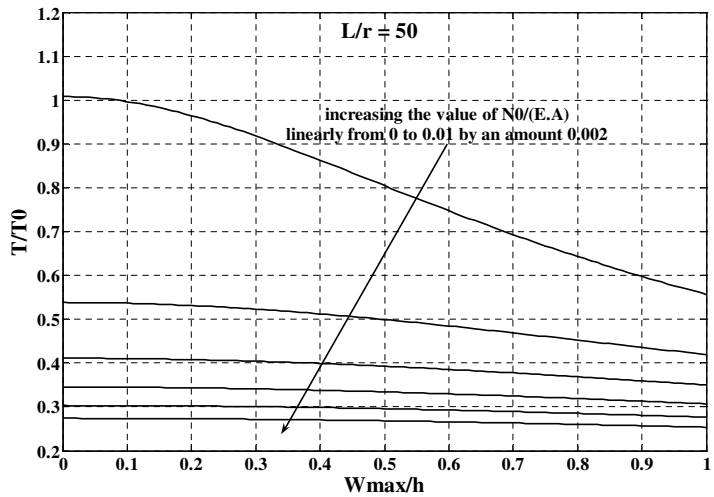

Fig. 6. Effect of the parameter $N_0/(E.A)$ on the nonlinear period of vibration for $L/r = 50$.

period or increase the nonlinear frequency of the system. It is also evident that the effect of the pre-tensile load would decrease as the initial deflection of the beam increased.

Figures 7–9 have been predicted to investigate the effect of the slenderness ratio to the nonlinear period of the system. It can be seen that when $W_{\max}/h$ is less than about 0.95, increasing the slenderness ratio would decrease the nonlinear period of vibration regardless of the value of the pre-tensile axial load applied on the beam.

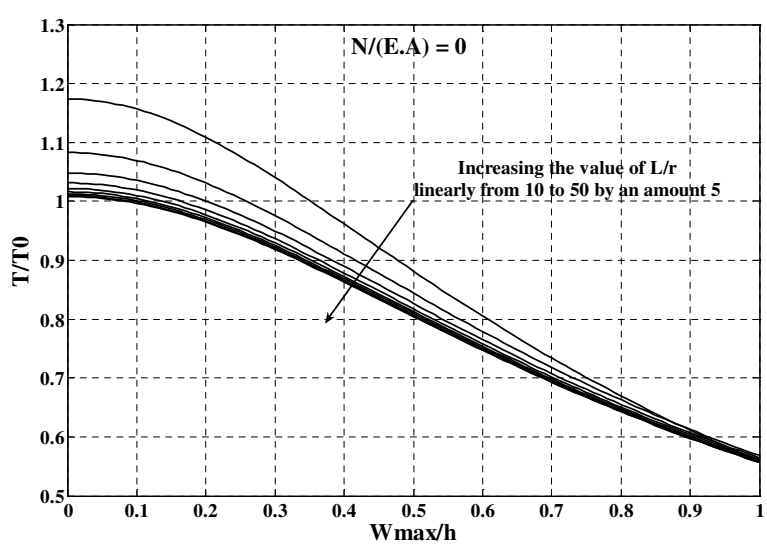

Fig. 7. Effect of the slenderness ratio to the nonlinear period of the system when $N_0/(E.A) = 0$.

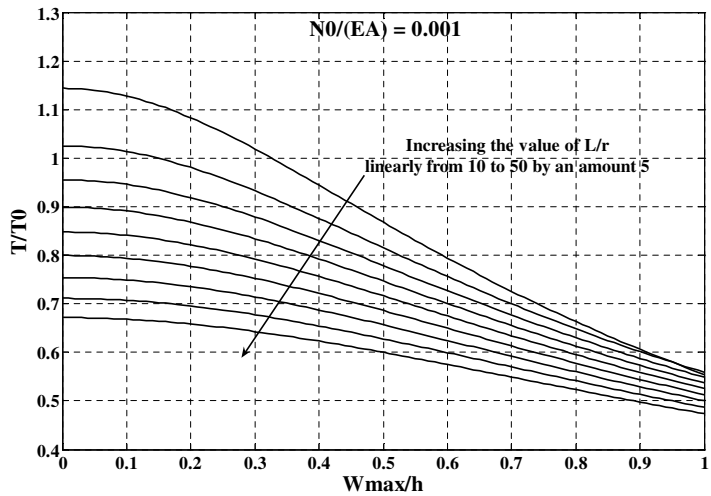

Fig. 8. Effect of the slenderness ratio to the nonlinear period of the system when $N_0/(E.A) = 0.001$.

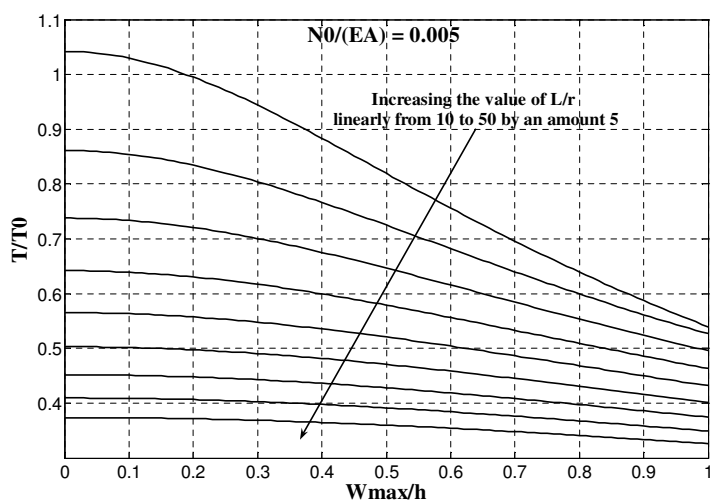

Fig. 9. Effect of the slenderness ratio to the nonlinear period of the system when $N_0/(E.A) = 0.005$.

4. Conclusion

In this study, the method of homotopy perturbation has been applied in conjunction with modified Lindstedt–Poincaré method in order to investigate the nonlinear free vibrational behavior of beams, considering the effects of shear deformation and rotary inertia. It has been showed that the results of the HPM is significantly more accurate than the previously reported analytical results in the literature. A parametric study has also been applied in order to characterize the behavior of the

beam due to changes in applied pre-tensile loads and changes in slenderness ratio. It has been observed that increasing the applied pre-tensile loads and slenderness ratio would increase the nonlinear natural frequency of the beam.

Appendix A

Nomenclature

List of English symbols

Symbol	Description
A	Cross sectional area of the beam.
E	Young's modulus of elasticity of the beam material.
G	Shear modulus of elasticity.
I	Second moment of area of the cross section with respect to the bending axis.
k	Shear correction factor.
L	Length of the beam.
m	Longitudinal density.
N_0	Pretension of the beam.
P	Perturbation parameter.
t	Nondimensionalized time.
$\hat{t}$	Time.
T	Nonlinear period of vibration.
T_0	Linear period on vibration of classical beam theory.
w	Nondimensionalized beam deflection.
$\hat{w}$	Beam deflection.
$W_{\max}$	Maximum initial displacement on midpoint of the beam.
x	Nondimensionalized coordinate along beam axis.
$\hat{x}$	Coordinate along beam direction.

List of Greek symbols

Symbol	Description
$\phi_n(x)$	nth linear undamped vibrational mode of the beam.
ω_{10}	Bending natural frequency.
ω_{20}	Rotary natural frequency.

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