A criterion for $c$-capability of pairs of groups

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Abstract

The notion of capability for pairs of groups was defined by Ellis in 1996. In this paper, we extend the theory of $c$-capability for pairs of groups and introduce a criterion, denoted by $Z^c_G(N)$, for $c$-capability of a pair $(G,N)$ of groups. We also study the behavior of $Z^c_G(N)$ with respect to direct products of groups.

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1 Introduction and Motivation

In 1940, P. Hall [6] remarked that characterization of groups which are the central quotient groups of other groups, is important in classifying groups of prime-power order. This kind of groups was named capable by Hall and Senior [5]. So a group $G$ is called capable if there exists a group $E$ such that $G \cong E/Z(E)$. Capability of groups was first studied by R. Baer [1] who determined all capable groups which are direct sums of cyclic groups. In 1996, Ellis [4] extended the theory of capability in an interesting way to a theory for pairs of groups. By a pair of groups we mean a group $G$ and a normal subgroup $N$ and this is denoted by $(G,N)$. He also introduced the exterior $G$-center subgroup of $N$, $Z^G_\wedge(N)$, for any pair $(G,N)$ and proved that the pair $(G,N)$ is capable if and only if $Z^G_\wedge(N) = 1$. The capability of pairs of groups has been also studied more by the authors in [8].

On the other hand, in 1997 Burns and Ellis [3] introduced the notion of $c$-capability of groups. A group $G$ is said to be $c$-capable if there exists a group $E$ such that $G \cong E/Z_c(E)$. They also introduced the subgroup $Z^c_\wedge(G)$ with the property that $G$ is $c$-capable if and only if $Z^c_\wedge(G) = 1$. In this paper following Burns and Ellis [3] and Ellis [4], we extend the theory of $c$-capability for pairs of groups. We also introduce a subgroup of $N$, shown by $Z^c_\wedge(G,N)$, that can be used as a criterion for $c$-capability of a pair $(G,N)$ of groups. The properties of $Z^c_\wedge(G,N)$ and its behavior with respect to the products of groups will also be studied. Finally, a set of examples of $c$-capable pairs shall be given. In other words, the paper actually generalizes the works [3, 4, 8] somehow.
2 Main Results

Let $M$ and $G$ be two arbitrary groups and $\alpha_1 : G \rightarrow \text{Aut}(M)$ be a group homomorphism whose image contains $\text{Inn}(M)$. Then $G$ acts on $M$ by $m^g = \alpha_1(g)(m)$, for all $g \in G, m \in M$. The $G$-commutator subgroup of $M$ is defined the subgroup $[M, G]$ generated by all the $G$-commutators $[m, g] = m^{-1}m^g$, where $m^g$ is the action of $g$ on $m$, for all $g \in G, m \in M$ and the $G$-center of $M$ is defined to be the subgroup

$$Z(M, G) = \{m \in M | m^g = m, \forall g \in G\}.$$ 

Existence of the homomorphism $\alpha_1$ implies that $Z(M, G) \subseteq Z(M)$. Also it is easy to see that there is a group homomorphism $\alpha_2 : G \rightarrow \text{Aut}(M/Z(M, G))$ whose image contains $\text{Inn}(M/Z(M, G))$ and hence $G$ acts on $M/Z(M, G)$. Then we can define the normal subgroup $Z_2(M, G)$ of $M$ as follows:

$$\frac{Z_2(M, G)}{Z(M, G)} = Z\left(\frac{M}{Z(M, G)}, G\right).$$

Now by continuing this process, we shall get to the following definition.

Definition 2.1. For $c \geq 1$, we define the $c$th $G$-center subgroup of $M$ as follows:

$$Z_1(M, G) = Z(M, G), \quad \frac{Z_c(M, G)}{Z_{c-1}(M, G)} = Z\left(\frac{M}{Z_{c-1}(M, G)}, G\right) \quad (c \geq 2).$$

So we have the upper $G$-central series of $M$,

$$1 = Z_0(M, G) \leq Z_1(M, G) \leq Z_2(M, G) \leq \ldots \leq Z_c(M, G) \leq \ldots .$$

It is easy to see that for all $c \geq 1$,

$$Z_c(M, G) = \{m \in M | [\ldots [[m, g_1], g_2], \ldots, g_c] = 1, \forall g_1, g_2, \ldots, g_c \in G\}.$$ 

Now using the above definition we define a relative $c$-central extension of a pair $(G, N)$ of groups.

Definition 2.2. Let $(G, N)$ be a pair of groups. A relative $c$-central extension of the pair $(G, N)$ is a group homomorphism $\varphi : E \rightarrow G$, together with an action of $G$ on $E$ such that

(i) $\varphi(E) = N$,
(ii) $\varphi(e^g) = g^{-1}\varphi(e)g$, for all $g \in G, e \in E$,
(iii) $e^g e'^e = e^{-1}e'e$, for all $e, e' \in E$,
(iv) $\ker \varphi \subseteq Z_c(E, G)$.

Note that conditions (ii) and (iii) in Definition 2.2 assert that $\varphi$ is a crossed module. A pair $(G, N)$ is said to be $c$-capable, if there exists a relative $c$-central extension
\( \varphi : E \to G \) with \( \ker \varphi = Z_c(E,G) \).

Let \((G,N)\) be a \(c\)-capable pair of groups. So there exists a relative \(c\)-central extension \( \varphi : M \to G \) with \( \ker \varphi = Z_c(M,G) \). Then it is straightforward to see that \( \varphi : M/Z(M,G) \to G \), defined by \( \varphi(mZ(M,G)) = \varphi(m) \), is a relative \((c-1)\)-central extension of \((G,N)\) such that \( \ker \varphi = Z_{c-1}(M,G) \). Hence the pair \((G,N)\) is \((c-1)\)-capable. This implies that every \(c\)-capable pair is a capable pair. But the converse is not true generally. For instance, let \( G = \langle x,y,z \mid x = yx^{-1}y^3, y = zy^{-1}z^3, z = xz^{-1}x^3, x^{16} = 1 \rangle \) and put \( Q = G/Z^*(G,G) \). Then Theorem 1.4 in [3] shows that the pair \((Q,Q)\) is capable but it is not \(2\)-capable.

It is interesting to find a useful way for determining all \(c\)-capable pairs of groups. The following definition provides a criterion for characterizing the \(c\)-capability of pairs of groups.

**Definition 2.3.** Let \((G,N)\) be a pair of groups. Then we define the \(c\)th precise center of the pair \((G,N)\) to be

\[
Z_c^*(G,N) = \bigcap \{ \varphi(Z_c(E,G)) \mid \varphi : E \to G \text{ is a relative } c-\text{central extension of } (G,N) \}.
\]

In particular \(Z_c^*(G,G)\) coincides with the subgroup \(Z_c^*(G)\) defined in [3].

The above definition helps us to state a necessary and sufficient condition for the \(c\)-capability of a pair of groups. For doing this, we need the following theorem.

**Theorem 2.4.** For any pair \((G,N)\) of groups, there exists a relative \(c\)-central extension \( \varphi : E \to G \) such that \( \varphi(Z_c(E,G)) = Z_c^*(G,N) \).

**Proof.** Let \( \{ \varphi_i : E_i \to G \mid i \in I \} \) be the set of all relative \(c\)-central extensions of a pair \((G,N)\). Put

\[
E = \{ (e_i)_{i \in I} \in \prod_{i \in I} E_i \mid \exists n \in N \ \forall i \in I; \ \varphi_i(e_i) = n \}.
\]

Define \( \varphi : E \to G \) by \( \varphi((e_i)_{i \in I}) = n \) such that \( \varphi_i(e_i) = n \), for all \( i \in I \). It is easy to check that \( \varphi \) is a relative \(c\)-central extension of the pair \((G,N)\). So \(Z_c^*(G,N) \subseteq \varphi(Z_c(E,G))\).

On the other hand, if \((e_i)_{i \in I} \in Z_c(E,G) = \prod_{i \in I} Z_c(E_i,G)\), then \( e_j \in Z_c(E_j,G) \), for all \( j \in I \). This implies that \( \varphi((e_i)_{i \in I}) = \varphi_j(e_j) \in \varphi_j(Z_c(E_j,G)) \), for all \( j \in I \) and so \( \varphi((e_i)_{i \in I}) \in \bigcap_{i \in I} \varphi_i(Z_c(E_i,G)) = Z_c^*(G,N) \). Therefore \( \varphi(Z_c(E,G)) \subseteq Z_c^*(G,N) \) and this completes the proof.

The following important corollary is an immediate consequence of Theorem 2.4.
Corollary 2.5. Let \((G, N)\) be a pair of groups. Then the pair \((G, N)\) is \(c\)-capable if and only if \(Z^*_c(G, N) = 1\).

The next theorem states another property of the \(c\)th precise center subgroup \(Z^*_c(G, N)\).

Theorem 2.6. Let \((G, N)\) be a pair of groups and \(K\) be a normal subgroup of \(G\) contained in \(N\). Then
\[
Z^*_c(G, N)K K \subseteq Z^*_c(G K, N K).
\]

Proof. By Theorem 2.4, there exists a relative \(c\)-central extension \(\varphi : M \to G/K\) of \((G/K, N/K)\) such that \(\varphi(Z_c(M, G/K)) = Z^*_c(G/K, N/K)\). Put \(H = \{(m, n) \in M \times N | \varphi(m) = nK\}\) with an action of \(G\) on \(H\) defined by \((m, n)^g = (m^{gK}, n^g)\), for all \(g \in G\), \(n \in N\) and \(m \in M\). Then the group homomorphism \(\psi : H \to G\) defined by \(\psi(m, n) = n\), is a relative \(c\)-central extension of \((G, N)\). Also \((m, n) \in Z_c(H, G)\) implies that \(m \in Z_c(M, G/K)\). So \(\psi(Z_c(H, G))K/K \subseteq \varphi(Z_c(M, G/K))\). Hence the result follows.

The following theorem shows that the class of all \(c\)-capable pairs is closed under direct products.

Theorem 2.7. Let \(\{(G_i, N_i)\}_{i \in I}\) be a family of pairs of groups. Then
\[
Z^*_c(\prod_{i \in I} G_i, \prod_{i \in I} N_i) \subseteq \prod_{i \in I} Z^*_c(G_i, N_i).
\]

Proof. Let \(\varphi_i : M_i \to G_i\) be a relative \(c\)-central extension of \((G_i, N_i)\) with \(\varphi(Z_c(M_i, G_i)) = Z^*_c(G_i, N_i)\), for all \(i \in I\). Define
\[
\psi : \prod_{i \in I} M_i \to \prod_{i \in I} G_i,
\]
\[
\{m_i\}_{i \in I} \mapsto \{\varphi_i(m_i)\}_{i \in I}
\]

It is easy to check that \(\psi\) is a relative \(c\)-central extension of \((\prod_{i \in I} G_i, \prod_{i \in I} N_i)\) and
\[
\psi(Z_c(\prod_{i \in I} M_i, \prod_{i \in I} G_i)) = \prod_{i \in I} \varphi_i(Z_c(M_i, G_i)) = \prod_{i \in I} Z^*_c(G_i, N_i).
\]

In the above theorem, equality does not hold in general. A counterexample is given by \(I = \{1, 2\}\), \(G_1 = G_2 = \mathbb{Z}_4\) and \(N_1 = N_2 = \mathbb{Z}_2\). The pair \((G_1 \times G_2, N_1 \times N_2)\) is 1-capable whereas \((G_1, N_1)\) and \((G_2, N_2)\) are not capable (See Theorem 5.4 in [8]). Also we are going to give a condition under which the equality holds. But first we need to state the following lemma which has a straightforward proof.

Lemma 2.8. Let \(M\) and \(G\) be groups with an action of \(G\) on \(M\). Then for all \(m, n \in M\) and \(g, h \in G\), we have
Corollary 2.10. Let \( \{(G_i, N_i)\}_{i \in I} \) be a family of pairs of groups such that \(|G_i|, |G_j| = 1\), for all \( i, j \in I \) with \( i \neq j \). Then

\[
Z_c^*(\prod_{i \in I} G_i, \prod_{i \in I} N_i) = \prod_{i \in I} Z_c^*(G_i, N_i).
\]

Proof. Put \( M_i = Z_c^*(G_i, N_i) \), for all \( i \in I \). Let \( \varphi : E \to G \) be a relative \( c \)-central extension of \((G, N)\). It is enough to show that for all \( i \in I \), \( \varphi^{-1}(M_i) \subseteq Z_c(E, G) \). Suppose \( i \in I \) and put \( E_i = \varphi^{-1}(N_i) \). The homomorphism \( \varphi \) induces a relative \( c \)-central extension \( \varphi_i : E_i \to G_i \) of the pair \((G_i, N_i)\). It follows that \( M_i \subseteq \varphi(Z_c(E_i, G_i)) \) and hence

\[
[\varphi^{-1}(M_i), _cG_i] = 1,
\]

in which \( [\varphi^{-1}(M_i), _cG_i] \) is \( \{\cdots[[\varphi^{-1}(M_i), G_i, G_i], \ldots, G_i\} \). On the other hand, for all \( j \in I \), with \( j \neq i \), \( [G_i, G_j] = 1 \) and so \( [E_i, G_j] \subseteq \ker \varphi \subseteq Z_c(E, G) \). Thus by Lemma 2.8, for any nonnegative integer \( k \),

\[
[[E_i, _kG_i], G_j] \subseteq [[E_i, (k-1)G_i, G_j], G_i] \subseteq \cdots \subseteq [E_i, G_j, _kG_i].
\]

Let \( m^* \in \varphi^{-1}(M_i) \) and \( h_1^*, \ldots, h_c^* \) be elements of \( G_i \)'s \((t \in I)\), where there exists an integer \( k \), \( 1 \leq k \leq c \), such that \( h_1^*, \ldots, h_{k-1}^* \in G_i \) and \( h_k^* \in G_j \), with \( j \neq i \). Then Lemma 2.8 and inequality (2) imply that \( \theta : \varphi^{-1}(M_i) \to [\varphi^{-1}(M_i), _cG] \) defined by \( \theta(m) = [m, h_1^*, \ldots, h_c^*] \), for all \( m \in \varphi^{-1}(M_i) \), and also \( \gamma : G_j \to [\varphi^{-1}(M_i), _cG] \) defined by \( \gamma(g) = [m^*, h_1^*, \ldots, h_{k-1}^*, g, h_{k+1}^*, \ldots, h_c^*] \), for all \( g \in G_j \), are homomorphisms with \( \ker \varphi \subseteq \ker \theta \). It follows that the order of \( [m^*, h_1^*, \ldots, h_c^*] \) divides \( |\varphi^{-1}(M_i)|/\ker \varphi = |M_i| \) and \( |G_j| \). Since \(|M_i|, |G_j| = 1\), then we have \( [m^*, h_1^*, \ldots, h_c^*] = 1 \). Using this fact and (1), we have \( [\varphi^{-1}(M_i), _cG] = 1 \). This completes the proof.

Corollary 2.10. Let \( \{(G_i, N_i)\}_{i \in I} \) be a family of pairs of groups.

(i) If for all \( i \in I \), \((G_i, N_i)\) is a \( c \)-capable pair, then the pair \((\prod_{i \in I} G_i, \prod_{i \in I} N_i)\) is \( c \)-capable.

(ii) If for all \( i, j \in I \) with \( i \neq j \), we have \(|G_i|, |G_j| = 1\), then all the pairs \((G_i, N_i)\) are \( c \)-capable if and only if the pair \((\prod_{i \in I} G_i, \prod_{i \in I} N_i)\) is \( c \)-capable.
The authors [8] gave a description of \( Z^*_c(G, N) \) in terms of a free presentation of \( G \) and applied it to obtain a number of interesting results. So it might be useful to find a relationship between \( Z^*_c(G, N) \) and a free presentation of \( G \). Let \((G, N)\) be a pair of groups. Suppose that \( G \cong F/R \) is a free presentation of \( G \) and \( S \) is the preimage of \( N \) in \( F \). First, let us define

\[
\gamma_{c+1}^*(G, N) = \left[ S, \underbrace{cF, cF, \ldots, cF}_c \right],
\]

where \([S, \underbrace{cF, cF, \ldots, cF}_c]\) denotes a left normed commutator (\( c \geq 1 \)). It is easy to see that this definition is independent of the free presentation for \( G \). Also we need to recall that the \( c \)-nilpotent multiplier of \( G \) is defined to be

\[
M^{(c)}(G) = \frac{R \cap \gamma_{c+1}^*(F)}{[R, cF]}.
\]

This multiplier is also an abelian group and independent of the chosen free presentation. In order to make a relation between the subgroup \( Z^*_c(G, N) \) and a free presentation of \( G \), a straightforward way is to show that the natural homomorphism \( \sigma : S/[R, cF] \to G \) is a relative \( c \)-central extension. But the problem which arises here is that the natural action on \( S/[R, cF] \) is not well defined generally. Hence we are forced to add an extra condition. Therefore, we suppose that \( G \) is a group with a free presentation

\[
1 \to R \to F \xrightarrow{\pi} G \to 1
\]

and a normal subgroup \( N \cong S/R \) such that \([R, S] \subseteq [R, cF]\) (Corollary 2.13 gives an example of a pair \((G, N)\) which satisfies in this condition). Then the action of \( G \) on \( S/[R, cF] \), defined by \((s[R, cF])^g = s \pi(f) \) with \( \pi(f) = g \), is well defined. So the group homomorphism

\[
\sigma : \frac{S}{[R, cF]} \to G,
\]

\[
s[R, cF] \mapsto \pi(s)
\]

is a relative \( c \)-central extension of the pair \((G, N)\). Therefore

\[
Z^*_c(G, N) \subseteq \sigma(Z_c(S/[R, cF], G)).
\]

This inequality yields the following interesting results.

**Theorem 2.11.** With the above assumption, if \( K \subseteq Z^*_c(G, N) \) then
(i) the natural homomorphism \( M^{(c)}(G) \to M^{(c)}(G/K) \) is injective,
(ii) \( K \subseteq Z^*_c(G) \cap N \),
(iii) \( \gamma_{c+1}^*(G, N) \cong \gamma_{c+1}^*(G/K, N/K) \).
Proof. Let $T$ be the preimage of $K$ in $F$. Then $K \subseteq Z_c^*(G, N)$ implies that $\sigma(T/[R, cF]) \subseteq \sigma(Z_c(S/[R, cF], G))$. It follows that $[T, cF]/[R, cF] = 1$. On the other hand $[T, cF]/[R, cF]$ is the kernel of the natural homomorphism $M^{(c)}(G) \to M^{(c)}(G/K)$ and also the natural homomorphism $[S, cF]/[R, cF] \to [S, cF]/[T, cF]$. So (i) and (iii) hold. By [3, Lemma 2.1] $K \subseteq Z_c^*(G)$ if and only if the natural homomorphism $M^{(c)}(G) \to M^{(c)}(G/K)$ is injective. Hence (ii) follows by (i).

The following corollary is an immediate consequence of Theorem 2.11.

Corollary 2.12. With the previous assumption, if $Z_c^*(G, N) = N$, then $\gamma_{c+1}(G, N) = 1$.

Finally, Theorem 2.11 helps us to provide a set of examples of $c$-capable groups. But for this, we need to recall the definition of $n$th nilpotent product for cyclic groups. Thus, let $\{G_i\}_{i \in I}$ be a family of cyclic groups. Then the $n$th nilpotent product of the family $\{G_i\}_{i \in I}$ is defined to be the group $\prod_{i \in I} G_i = \prod_{i \in I} G_i/\langle \prod_{i \in I} G_i \rangle$, where $\prod_{i \in I} G_i$ is the free product of the family $\{G_i\}_{i \in I}$.

Corollary 2.13. Let $\{F_i\}_{i \in I}$ be a family of infinite cyclic groups. Put $G = \prod_{i \in I} c^{+n} F_i$ and $N = \gamma_{c+k}(G)$, for $0 < k \leq n$. Then the pair $(G, N)$ is $c$-capable.

Proof. The result easily follows for $i = 1$. Assume that $i \geq 2$. The groups $G$ and $N$ have free presentations $G \cong F/R$ and $N \cong S/R$, where $F = \prod_{i \in I} F_i$, $R = \gamma_{c+n+1}(F)$ and $S = \gamma_{c+k}(F)$. So the condition $[R, S] \subseteq [R, cF]$ holds for the pair $(G, N)$ and $Z_c^*(G, N) \subseteq Z_c^*(G \cap N)$, by Theorem 2.11. On the other hand, using [7, Theorem 3.8] we have $Z_c^*(G) = 1$, for $i \geq 2$. Hence the result follows by Corollary 2.5.

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References


