Sliding mode controllers for second order and extended Heisenberg systems

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Abstract: This paper presents a method to design an SMC for extended Heisenberg systems which is a class of non-linear systems. To stabilise the system with the proposed SMC, some sufficient conditions are introduced. In fact, the system stability is achieved using an appropriate Lyapunov equation. To illustrate the theoretical results, three examples are presented.

Keywords: sliding mode control; SMC; extended Heisenberg system; sliding surface; stabilisation, Lyapunov function.

Reference to this paper should be made as follows: Zarrabi, M.R., Farahi, M.H., Effati, S. and Koshkouei, A.J. (xxxx) 'Sliding mode controllers for second order and extended Heisenberg systems', *Int. J. Modelling, Identification and Control*, Vol. X, No. Y, pp.000–000.

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1 Introduction

Sliding mode control (SMC) is an effective method for controlling linear and non-linear dynamical systems operating under uncertainly conditions by enforcing the state trajectories onto a surface on which the system dimension is normally reduced. A traditional SMC is usually designed in two phases. First, a sliding surface is designed or specified such that the state trajectories tend to an equilibrium point along this surface. Then, a control is designed so that the state trajectories reach the sliding surface in a finite time. During the sliding mode, the effective controller is called the equivalent control. In this paper, a SMC control is designed which guarantees that the trajectories hits the sliding surface in a finite time and remain within an ε-neighbourhood of the sliding surface and eventually tend to an equilibrium point, say the origin. Note that, the system stability and behaviour depend on the selection of sliding surface (Utkin, 1977; Zinober, 1994). Several methods have been proposed to design a stable sliding surface, such neural network by Chatchanayuenyong and Parnichkun (2006), and Mihoub et al. (2009), fuzzy SMC by Le and Kuo, (1998), discrete-time SMC by Bandyopadhyay and Janardhanan (2005), and Qian et al. (2011) minimising integral absolute error by Thoma et al. (2009), passivity-based SMC by Koshkouei (2007, 2008) and flatness, backstepping with SMC by Koshkouei et al. (2008). Also, there are many systems affected by uncertainties which do not satisfy the matching condition (i.e., uncertainties are not in the same channel as the input maps are) to solve these problems, various methods for with mismatched uncertainties have been proposed by Zinober and Liu (1996), Swaroop et al. (2000), Jang and Kim (2005), and Drakunov et al. (2005). Zinober and Liu (1996) and Swaroop et al. (2000) proposed SMC based on the backstepping design which was used to relax the matching conditions. However, since the integrator backstepping technique suffers from the problem of 'explosion of terms', an additional procedure is needed to solve this problem. Jang and Kim (2005) also proposed a SMC to transmit mismatched part to the sliding surface, but their method needs to have some initial conditions that many dynamical systems fail to fulfil these requirements. Drakunov et al. (2005) considered an extended Heisenberg dynamical system with a drift and proposed a sliding plane by considering complex conditions in order to force its trajectories such that the system state is stabilised on an ε-neighbourhood of the origin

In this paper, a SMC control design method for a dynamical system in an extended Heisenberg forms with uncertainly conditions are presented. A sliding surface is designed so that using a suitable Lyapunov function the system stability is achieved without imposing any complexity.

The paper is organised as follows: Section 2, a sliding surface design method for a class of non-linear dynamical system, is considered. Section 3, addresses an applicable method to design a SMC for an extended Heisenberg system. In Section 4, several examples are given to illustrate the design procedures as well as their effectiveness. Finally, conclusions are presented in Section 5.

2 Design a sliding surface for regular form of a dynamical system

Definition 1: A dynamical system in regular form is presented as follow:

$$\begin{cases} \dot{x}_1 = x^T Q^1 x + a_{11}x_1 + \cdots + a_{1n}x_n + \mu_1 \\ \dot{x}_2 = x^T Q^2 x + a_{21}x_1 + \cdots + a_{2n}x_n + \mu_2 \\ \vdots \\ \dot{x}_{n-m} = x^T Q^{n-m} x + a_{n-m1}x_1 + \cdots + a_{n-m}x_n + \mu_{n-m} \\ \dot{x}_{n-m+1} = x^T Q^{n-m+1} x + a_{n-m+11}x_1 + \cdots + a_{n-m+1n} \\ \qquad x_n + b_{11}u_1 + \cdots + b_{1m}u_m + \mu_{n-m+1} \\ \vdots \\ \dot{x}_n = x^T Q^n x + a_{n1}x_1 + \cdots + a_{nn}x_n + b_{m1}u_1 + \cdots + \\ \qquad b_{mm}u_m + \mu_{n'} (m < n), \end{cases} \tag{2.1}$$

where the trajectory function $x(\cdot)$ satisfies $x(t) \in \mathcal{A}$ (a known compact subset of $\mathbb{R}^n$) and is absolutely continuous, and the control function $u(\cdot)$satisfies $u(t) \in U$ (a bounded, closed subset of $\mathbb{R}^m$), and Q^i's for $i = 1, 2, \ldots, n$, are square $n \times n$ matrices with continuous time varying entries. μ_i, $i = 1, 2, \ldots, n$, are function of time. Also the constants a_{ij}, $i, j = 1, 2, \ldots, n$ and b_{kl}, $k, l = 1, 2, \ldots, m$. are the state and input coefficients, respectively. The dynamical system (2.1) can be written in the following matrix form

$$\dot{x} = \begin{bmatrix} x^T & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & x^T \end{bmatrix} \begin{bmatrix} Q^1 x \\ \vdots \\ Q^n x \end{bmatrix} + Ax + Bu + \rho, \tag{2.2}$$

where

$$\begin{bmatrix} x^T & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & x^T \end{bmatrix} \in \mathbb{R}^{n\times(n*n)}, \ A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \in \mathbb{R}^{n\times n},$$

$$B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \in \mathbb{R}^{n\times m}, \ B_2 = \begin{bmatrix} b_{11} \\ \vdots \\ b_{m1} \end{bmatrix} \text{ and } \rho \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix} \in \mathbb{R}^n.$$

The system (2.2) still can be written in compact form as:

$$\dot{y}_1 = Y^{n-m} Q^{n-m}(y) + A_{11}(t)y_1(t) + A_{12}(t)y_2(t) + \rho_1(t) \tag{2.3}$$

$$\dot{y}_2 = Y^m Q^m(y) + A_{21}(t)y_1(t) + A_{22}(t)y_2(t) + B_2(t)u(t) + \rho_2(t) \tag{2.4}$$

where

$$y=\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\in\mathbb{R}^n,\ y_1=\begin{bmatrix} x_1 \\ \vdots \\ x_{n-m} \end{bmatrix},\ y_2=\begin{bmatrix} x_{n-m+1} \\ \vdots \\ x_n \end{bmatrix},$$

$$Q^{n-m}(y)=\begin{bmatrix} Q^1 x \\ \vdots \\ Q^{n-m}x \end{bmatrix}\in\mathbb{R}^{(n*(n-m))\times 1},$$

$$Q^m(y)=\begin{bmatrix} Q^{n-m+1}x \\ \vdots \\ Q^n x \end{bmatrix}\in\mathbb{R}^{(n+m)\times 1},$$

and Q^j is a partitioned matrix and defined as

$$Q^j=\begin{bmatrix} Q_{11}^j & Q_{12}^j \\ Q_{21}^j & Q_{22}^j \end{bmatrix}$$

for $j = 1, 2, \ldots, n$,

$$Q_{11}^j\in\mathbb{R}^{(n-m)\times(n-m)},\ Q_{12}^j\in\mathbb{R}^{(n-m)\times m},$$
$$Q_{21}^j\in\mathbb{R}^{m\times(n-m)},\ Q_{22}^j\in\mathbb{R}^{m\times m}$$

and

$$(Q_{12}^j)^T=Q_{21}^j,\ A=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\in\mathbb{R}^{n\times n},$$
$$A_{11}\in\mathbb{R}^{(n-m)\times(n-m)},\ A_{12}\in\mathbb{R}^{(n-m)\times m},\ A_{21}\in\mathbb{R}^{m\times(n-m)},$$

and $A_{22}\in\mathbb{R}^{m\times m}$,

$$Y^{n-m}=\begin{bmatrix} x^T & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & x^T \end{bmatrix}\in R^{(n-m)\times(n*(n-m))}$$

and

$$Y^m=\begin{bmatrix} x^T & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & x^T \end{bmatrix}\in\mathbb{R}^{m\times(n*m)},\ \rho=\begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix}\in\mathbb{R}^n,$$

$$\rho_1=\begin{bmatrix} \mu_1 \\ \vdots \\ \mu_{n-m} \end{bmatrix}\in\mathbb{R}^{n-m},\ \text{and}\ \rho_2=\begin{bmatrix} \mu_{n-m+1} \\ \vdots \\ \mu_n \end{bmatrix}\in\mathbb{R}^m.$$

Now for the dynamical system (2.3) to (2.4), one can define a sliding surface as follow:

$$S(y)=y_2+K\left(2\sum_{j=1}^{n-m}Q_{12}^{j^T}y_1^j+A_{12}^T\right)y_1=0, \tag{2.5}$$

where K is an appropriate positive number (will be defined after), and y_1^j is j^{th} component of y_1.

In order to design the reaching subcontroller, u_r, consider the derivative of (2.5) as follow:

$$\begin{aligned}\dot S(y)=\dot y_2+K\Bigg(&\left(2\sum_{j=1}^{n-m}\left(\dot Q_{12}^{j^T}y_1^j+Q_{12}^{j^T}\dot y_1^j\right)+A_{12}^T\right)y_1\\ &+\left(2\sum_{j=1}^{n-m}Q_{12}^{j^T}y_1^j+Q_{12}^T\right)\dot y_1\Bigg).\end{aligned} \tag{2.6}$$

From (2.3) to (2.6), it is easy to show that

$$\dot S=\dot S(y)=B_2u_r+M,$$

where M is a bounded $m\times 1$ vector as

$$\begin{aligned}M=&\,Y^mQ^m(y)+A_{21}y_1+A_{22}y_2+\rho_2\\ &+\Bigg(\left(2\sum_{j=1}^{n-m}\left(\dot Q_{12}^{j^T}y_1^j+Q_{12}^j\dot y_1^j\right)+\dot A_{12}^T\right)y_1\\ &+\left(2\sum_{j=1}^{n-m}Q_{12}^{j^T}y_1^j+A_{12}^j\right)\dot y_1\Bigg).\end{aligned} \tag{2.7}$$

One may select u_r such that the condition $\dot S=-\alpha sign(S)$, ($\alpha > 0$) is fulfilled. In this case, the reaching subcontroller can be defined by $u_r = (-B_2)^{-1}(\alpha sign(S) + M)$ which satisfies the existence sliding mode condition $S^T\dot S<0$.

By considering $\dot S(y)=0$, another subcontroller, so-called the equivalent control, $u_{eq'}$ can be designed, which guarantees that trajectories remain on the sliding surface after a finite time t_s, where t_s is the reaching time to the *SS*, i.e., $S(t_s) = 0$. Thus,

$$\begin{aligned}u_{eq}=&\,(-B_2)^{-1}\Big(Y^mQ^m(y)+A_{21}y_1+A_{22}y_2+\rho_2\\ &+K\Bigg(\left(2\sum_{j=1}^{n-m}\left(\dot Q_{12}^{j^T}y_1^j+Q_{12}^j\dot y_1^j\right)+\dot A_{12}^j\right)y_1\\ &+\left(2\sum_{j=1}^{n-m}Q_{12}^{j^T}y_1^j+A_{12}^T\right)\Big(Y^{n-m}Q^{n-m}(y)\\ &+A_{11}y_1+A_{12}y_2+\rho_1\Big)\Bigg)\Big).\end{aligned} \tag{2.8}$$

So the SMC can be defined as

$$u=\begin{cases} u_r & \text{if} \quad 0\le t<t_s \\ u_{eq} & \text{if} \quad t\ge t_s. \end{cases}$$

Now, the system stability is proved.

Theorem 1: Consider the sliding surface (2.5), and select the gain $K\in\mathbb{R}^+$ to be satisfied the following condition

$$\begin{aligned}&-\frac{1}{K}y_2^2+\sum_{j=1}^{n-m}\left(y_1^TQ_{11}^{j^T}y_1^jy_1+y_2^TQ_{22}^{j^T}y_1^jy_2\right)\\ &+y_1^TA_{11}y_1+\rho_1^Ty_1\le 0.\end{aligned}$$

Then the dynamical system (2.3) to (2.4) is asymptotically stable.

Proof: Define the Lyapunov function $V(x)$ from R^n to R as

$$V(x)=\frac{1}{2}y_1^Ty_1.$$

Now, by considering (2.3),

$$\begin{aligned}\dot{V}(x) &= y_1^T \dot{y}_1 = y_1^T \left(Y^{n-m} Q^{n-m} + A_{11} y_1 + A_{12} y_2 + \rho_1 \right) \\ &= \sum_{j=1}^{n-m} \left(y_1^T Q_{11}^{j^T} y_1^j y_1 + y_1^T Q_{12}^{j^T} y_1^j y_2 + y_2^T Q_{21}^{j^T} y_1^j y_1 + y_2^T Q_{22}^{j^T} y_1^j y_2 \right) \\ &+ y_1^T A_{11} y_{12} + y_1^T A_{12} y_2 + \rho_1^T y_1 = \left(\sum_{j=1}^{n-m} 2 y_1^T Q_{12}^{j^T} y_1^j + y_1^T A_{12} \right) y_2 \\ &+ y_1^T A_{11} y_1 + \sum_{j=1}^{n-m} \left(y_1^T Q_{11}^{j^T} y_1^j y_1 + y_2^T Q_{22}^{j^T} y_1^j y_2 \right) + \rho_1^T y_1,\end{aligned}$$

where y_i^j is j^{th} component of yi (i = 1, 2). From (2.5),

$$y_2 = -K \left(2 \sum_{j=1}^{n-m} Q_{12}^j y_1^j + A_{12}^T \right) y_1,$$

so

$$\begin{aligned}\dot{V}(x) &= -K \left(\left(2 \sum_{j=1}^{n-m} Q_{12}^{j^T} y_1^j + A_{12}^T \right) y_1 \right)^T \\ &\left(\left(2 \sum_{j=1}^{n-m} Q_{12}^{j^T} y_1^j + A_{12}^T \right) y_1 \right) + \sum_{j=1}^{n-m} \left(y_1^T Q_{11}^{j^T} y_1^j y_1 + y_2^T Q_{22}^{j^T} y_1^j y_2 \right) \\ &+ y_1^T A_{11} y_1 + \rho_1^T y_1.\end{aligned}$$

Now one can choose the positive number K, such that, $\dot{V}(x) < 0$. In this case, $V(x)\dot{V}(x) < 0$ which guarantee the asymptotic stability of the system. □

3 Design a sliding surface for an extended Heisenberg system

Consider an extended Heisenberg system as follow (Drakunov et al., 2005):

$$\dot{z} = Y^T J_1(t, z, X) X + \rho_1(t) \tag{3.1}$$

$$\dot{X} = Y + J_2(t, z, X) X + \rho_2(t) \tag{3.2}$$

$$\dot{Y} = u + \rho_3(t), \tag{3.3}$$

where the state vector is

$$\left(z, X^T, Y^T \right)^T \in \mathbb{R}^{2n+1}, z \in \mathbb{R}^1, X, Y \in \mathbb{R}^n.$$

$\rho i(t)$, i = 1, 2, 3, are continuous functions. Also, $J_1(t, z, X)$ and $J_2(t, z, X)$ are $n \times n$ skew symmetric matrices, which can be state and time dependent.

Now, define the sliding surface as

$$S(z, X, Y) = Y + K \left(X + J_1 X z \right) = 0, \tag{3.4}$$

where K is an appropriate positive number. In order to design the reaching subcontroller u_r, consider

$$\begin{aligned}\dot{S}(t) &= \dot{Y} + K \left(\dot{X} + \dot{J}_1 X z + J_1 \dot{X} z + J_1 X \dot{z} \right) \\ &= u + \rho_3 + K \left(Y + J_2 X + \rho_2 + \dot{J}_1 X z + J_1 Y z \right. \\ &\left. + J_1 J_2 X z + J_1 \rho_2 z + J_1 X Y^T J_1 X + J_1 X \rho_1 \right).\end{aligned}$$

Now, the reaching subcontroller $u_r(t)$ is selected such that the condition $\dot{S} = -\alpha sign(S)$ with $\alpha > 0$ is satisfied. In this case, $S^T \dot{S} < 0$, and the trajectories tend to the sliding surface.

Using $\dot{S}(s) = 0$, gives

$$\begin{aligned}u + \rho_3 &= -K \left(Y + J_2 X + \dot{J}_1 X z + J_1 \left(Y + J_2 X + \rho_2 \right) z \right. \\ &\left. + J_1 X \left(Y^T J_1 X + \rho_1 \right) + \rho_2 \right).\end{aligned}$$

Hence,

$$\begin{aligned}u_{eq} &= -K \left(Y + J_2 X + \dot{J}_1 X z + J_1 \left(Y + J_2 X + \rho_2 \right) z \right. \\ &\left. + J_1 X \left(Y^T J_1 X + \rho_1 \right) + \rho_2 \right) - \rho_3.\end{aligned} \tag{3.5}$$

Thus, in this case, the traditional SMC is defined as:

$$u = \begin{cases} u_r & \text{if} \quad 0 \le t < t_s \\ u_{eq} & \text{if} \quad t \ge t_s, \end{cases} \tag{3.6}$$

where t_s is the reaching time to the SS, i.e., $S(t_s) = 0$.

Theorem 2: Consider the dynamical system (3.1) to (3.3). The control (3.6) with the sliding surface (3.4) asymptotically stabilises the system.

Proof: To show the stability of the dynamical system (3.1) to (3.3) via the control (3.6) with the sliding surface (3.4), the Lyapunov function $V(z, X)$ from R^{n+1} to R as

$$V(z, X) = \frac{1}{2} \left(z^2 + X^T X \right),$$

is considered. Now,

$$\begin{aligned}\dot{V}(z, X) &= \dot{z} z + \dot{X}^T X = \left(Y^T J_1 X + \rho_1 \right) z \\ &+ \left(Y + J_2 X + \rho_2 \right)^T X \\ &= Y^T \left(X + J_1 X z \right) + \rho_1 z + \rho_2^T X.\end{aligned}$$

Note that since J_i is a skew matrix, hence $X^T J_i X = 0$. Thus, if one choose $Y = -K(X + J_1 X z)$, since z and X are bounded functions, the positive number K can be selected such that

$$-K \left(X + J_1 X z \right)^T \left(X + J_1 X z \right) + \rho_1^T x_1 + \rho_2^T x_2 < 0.$$

Therefore $\dot{V}(x) < 0$. In this case $V(x)\dot{V}(x) < 0$, which ensures the asymptotically stability of the Heisenberg system (3.1) to (3.3). □

Remark: If J_i be a negative definite matrix, then the dynamical system (3.1) to (3.3) is stable within an ε-neighbourhood of the origin when the sliding surface (3.4) is selected, since $X^T J_i X$ is less than or equal to zero.

4 Numerical examples

In this section, three numerical examples are presented to demonstrate the applicability of our method.

Example 1: Consider the following second order form dynamical system:

$$\dot{x}_1 = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} -2 & 1 & -1 \\ 1 & 1.5 & 0.5 \\ -1 & 0.5 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + x_1 - x_3 + \rho_1$$

$$\dot{x}_2 = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} -2 & -1 & -2 \\ -1 & 2 & 1 \\ -2 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + 0.5x_2 + x_3 + \rho_2$$

$$\dot{x}_3 = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 0.5 & 0 & 1 \\ 0 & 1.5 & 2 \\ 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + x_1 + 2x_2$$
$$+1.5x_3 + 5u + \rho_3,$$

where $\rho_1 = \rho_2 = \rho_3 = 5\sin t$. In this example,

$$Q^1 = \begin{pmatrix} -2 & 1 & -1 \\ 1 & 1.5 & 0.5 \\ -1 & 0.5 & 2 \end{pmatrix}, Q_{11}^1 = \begin{pmatrix} -2 & 1 \\ 1 & 1.5 \end{pmatrix}, Q_{12}^1 = \begin{pmatrix} -1 \\ 0.5 \end{pmatrix},$$

$$Q_{21}^1 = \begin{pmatrix} -1 & 0.5 \end{pmatrix}, Q_{22}^1 = (2), Q^2 = \begin{pmatrix} -2 & -1 & -2 \\ -1 & 2 & 1 \\ -2 & 1 & -1 \end{pmatrix}$$

$$Q_{11}^2 = \begin{pmatrix} -2 & -1 \\ -1 & 2 \end{pmatrix}, Q_{12}^2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, Q_{21}^2 = \begin{pmatrix} -2 & 1 \end{pmatrix}, Q_{22}^2 = (-1)$$

$$Q^3 = \begin{pmatrix} 0.5 & 0 & 1 \\ 0 & 1.5 & 2 \\ 1 & 2 & -1 \end{pmatrix}, Q_{11}^3 = \begin{pmatrix} 0.5 & 0 \\ 0 & 1.5 \end{pmatrix}, Q_{12}^3 = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

$$Q_{21}^3 = \begin{pmatrix} 1 & 2 \end{pmatrix}, Q_{22}^3 = (-1), A_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix}, A_{12} = \begin{pmatrix} -1 \\ 1 \end{pmatrix},$$

$$A_{21} = \begin{pmatrix} 1 & 2 \end{pmatrix}, A_{22} = (1.5).$$

Define the sliding surface as (2.5) which is

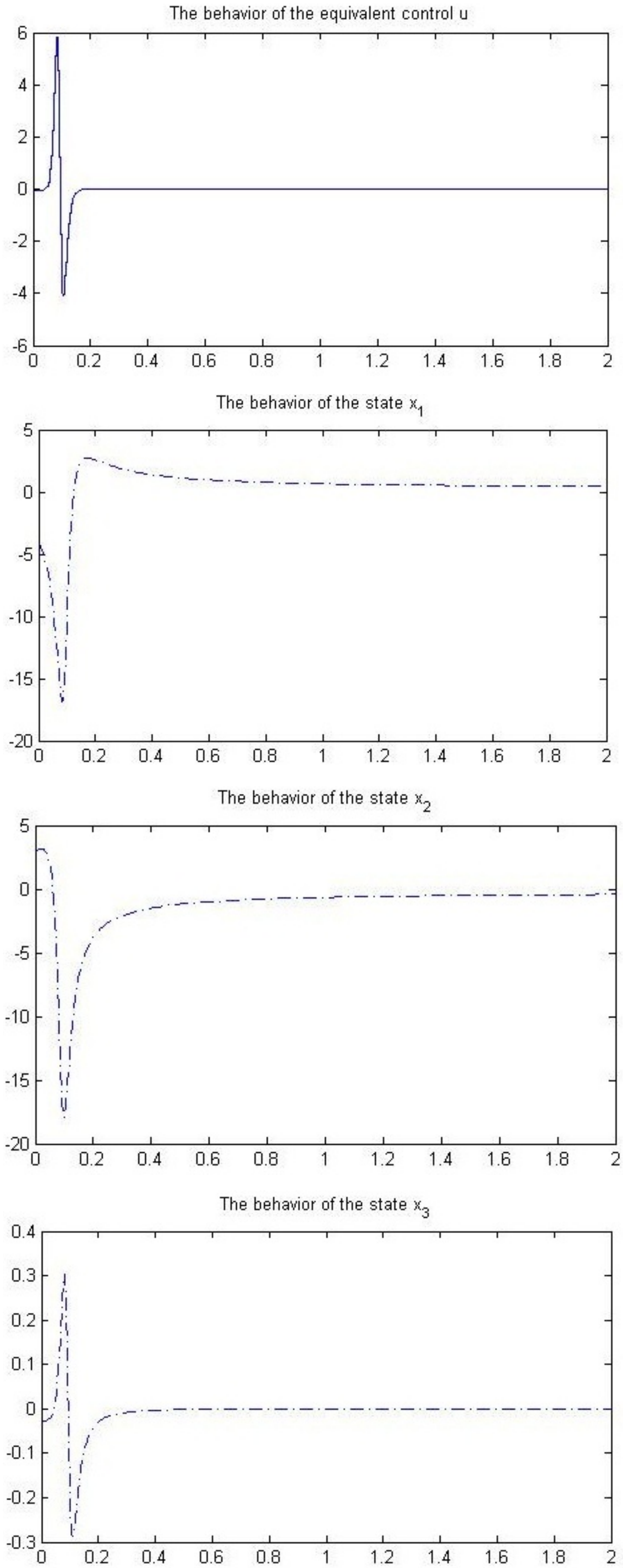
$$x_3 = -K\left(-2x_1^2 + x_2^2 - x_1x_2 - x_1 + x_2\right),$$

and the equivalent control from (2.8) is now as

$$u_{eq} = \frac{-1}{5}\left(4x_1 + 3x_2 - x_3 + 1.5 + 5\sin t\right)x_3.$$

Select $K = 0.0005$ which guarantees the stability condition. Consider the reaching point (–4, 3, –0.0262) on the sliding surface. Figure 1 shows the action of the equivalent control, and the behaviour of the states using the proposed SMC.

Figure 1 Trajectories and control of Example 1 (see online version for colours)



Example 2: Consider the following dynamical system:

$$\dot{x}_1 = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} \sin t & 2 & 1 \\ 2 & 1 & \cos t \\ 1 & \cos t & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$
$$+2\sin \pi t \; x_1 - x_2 + 0.2\sin 2\pi t$$

$$\dot{x}_2 = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} -1 & 0.2\sin \pi t & \cos t \\ 0.2\sin \pi t & 0 & 2 \\ \cos t & 2 & 0.5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$
$$-\cos \pi t \; x_2 - x_3 + 0.2\sin 2\pi t$$

$$\dot{x}_3 = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 1 & 0.5 & \sin t \\ 0.5 & -1 & 0.5\cos t \\ \sin t & 0.5\cos t & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$
$$+x_1 - \sin 2\pi t \; x_2 + \cos t \; x_3 + 10u + 0.2\sin 2\pi t.$$

In this example,

$$Q^1 = \begin{pmatrix} \sin t & 2 & 1 \\ 2 & 1 & \cos t \\ 1 & \cos t & -3 \end{pmatrix}, Q_{11}^1 = \begin{pmatrix} \sin t & 2 \\ 2 & 1 \end{pmatrix}, Q_{12}^1 = \begin{pmatrix} 1 \\ \cos t \end{pmatrix},$$

$$Q_{21}^1 = \begin{pmatrix} 1 & \cos t \end{pmatrix}, Q_{22}^1 = (-3), Q^2 = \begin{pmatrix} -1 & 0.2\sin \pi t & \cos t \\ 0.2\sin \pi t & 0 & 2 \\ \cos t & 2 & 0.5 \end{pmatrix},$$

$$Q_{11}^2 = \begin{pmatrix} -1 & 0.2\sin \pi t \\ 0.2\sin \pi t & 0 \end{pmatrix}, Q_{12}^2 = \begin{pmatrix} \cos t \\ 2 \end{pmatrix}, Q_{21}^2 = \begin{pmatrix} \cos t & 2 \end{pmatrix},$$

$$Q_{22}^2 = (0.5), Q^3 = \begin{pmatrix} 1 & 0.5 & \sin t \\ 0.5 & -1 & 0.5\cos t \\ \sin t & 0.5\cos t & -1 \end{pmatrix}, Q_{11}^3 = \begin{pmatrix} 1 & 0.5 \\ 0.5 & -1 \end{pmatrix},$$

$$Q_{12}^3 = \begin{pmatrix} \sin t \\ 0.5\cos t \end{pmatrix}, Q_{21}^3 = \begin{pmatrix} \sin t & 0.5\cos t \end{pmatrix}, Q_{22}^3 = (-1),$$

$$A_{11} = \begin{pmatrix} 2\sin \pi t & -1 \\ 0 & -\cos \pi t \end{pmatrix}, A_{12} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, A_{21} = \begin{pmatrix} 1 & -\sin 2\pi t \end{pmatrix},$$
$$A_{22} = (\cos t).$$

Define the sliding surface as (2.5),

$$x_3 = -K\left(3x_1^2 + 5x_2^2 + (4\cos t + 1)x_1 x_2 + x_1 - \left(1 + \sin(2\pi t)\right) x_2\right),$$

and from (2.8), one can find u_{eq} as well. For this example, we select $K = 0.1$, and the reaching point as (10, –6, –19.6) on the sliding surface. Figure 2 shows the control u_{eq}, and the behaviour of the states using the SMC.

Example 3: Consider the following extended Heisenberg system (Drakunov et al., 2005):

$$\dot{z} = x_1 y_2 - x_2 y_1 + 0.1\sin(2\pi t)$$
$$\dot{x}_1 = y_1 - x_2 + 0.1\sin(2\pi t)$$
$$\dot{x}_2 = y_2 + x_1 + 0.1\sin(2\pi t)$$
$$\dot{y}_1 = u_1 + 0.1\sin(2\pi t)$$
$$\dot{y}_2 = u_2 + 0.1\sin(2\pi t).$$

For this case, define the sliding surface as (3.4),

$$y_1 = -K_1\left(x_1 - x_2 z\right),$$
$$y_2 = -K_2\left(x_2 - x_1 z\right),$$

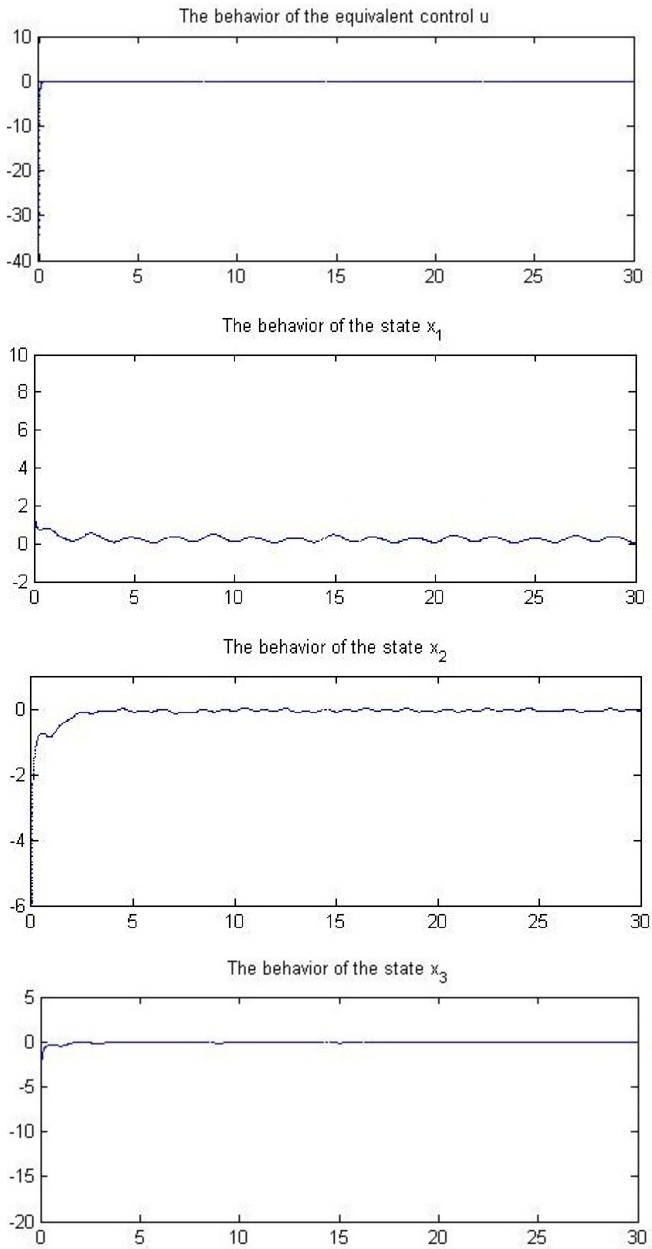
and, $u_{eq} = (u_{1eq}, u_{2eq})$ can be defined by (3.5) as

$$u_{1eq} = -\Big(y_1 - x_2 - \left(y_2 + x_1 + 0.1\sin(2\pi t)\right) z + x_2^2 y_1$$
$$-x_1 x_2 y_2 - 0.1\sin(2\pi t) x_2 + 0.1\sin(2\pi t)\Big) - 0.1\sin(2\pi t),$$

and

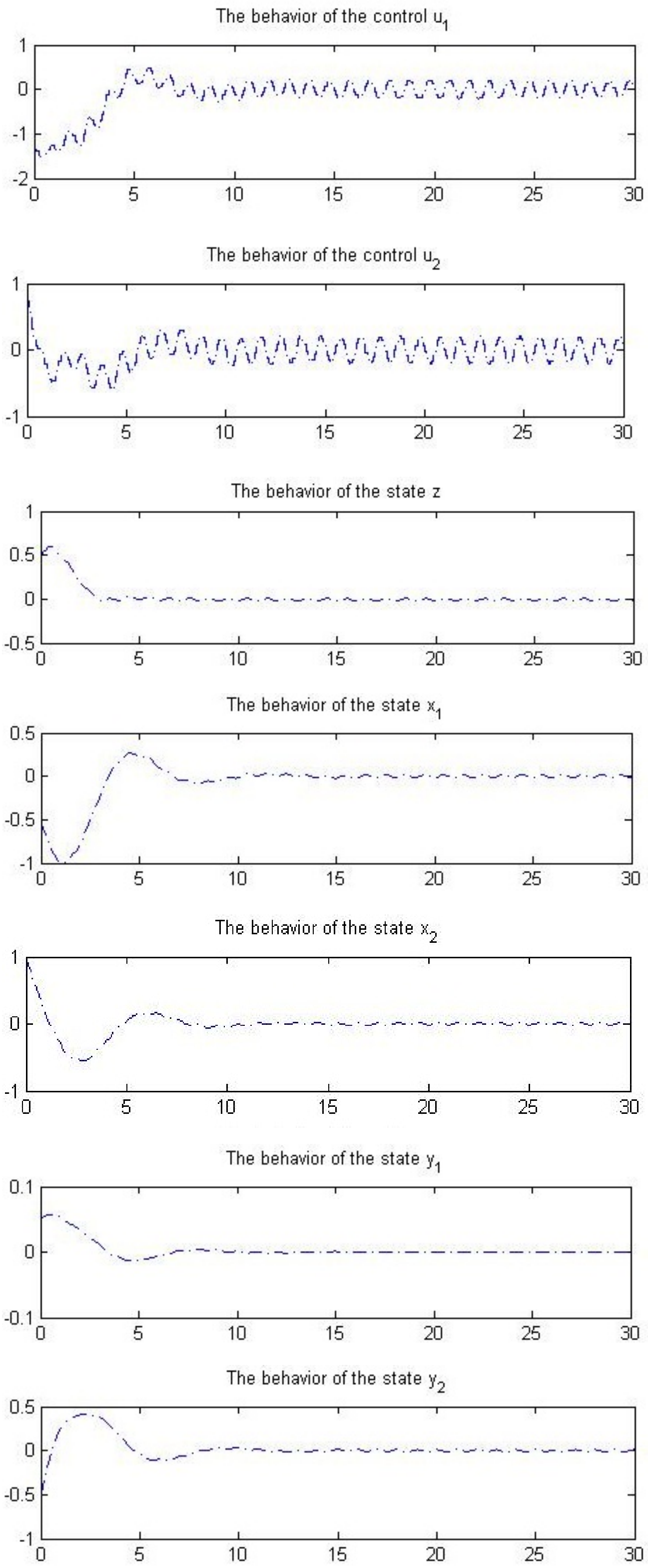
$$u_{2eq} = -\Big(y_2 - x_1 + \left(y_1 - x_2 + 0.1\sin(2\pi t)\right) z + x_1^2 y_2$$
$$-x_1 x_2 y_1 - 0.1\sin(2\pi t) x_2 + 0.1\sin(2\pi t)\Big) - 0.1\sin(2\pi t).$$

Figure 2 Trajectories and control of Example 2 (see online version for colours)



Select $K_1 = 0.05$, $K_2 = 0.7$. The reaching point on the sliding surface is selected as $(z, x_1, x_2, y_1, y_2) = (0.5, 1, -0.5, 0.05, -0.0375)$. Figure 3 shows the system responses and the SMC action.

Figure 3 Trajectories and controls of Example 3 (see online version for colours)



5 Conclusions

A SMC has been designed for stabilising an extended Heisenberg system using an appropriate Lyapunov function. In this method, an appropriate sliding surface is defined and two subcontrollers are designed to achieve the system stability. In fact, one subcontroller guarantees the state trajectories reach the sliding surface in a finite time while another subcontroller ensures that the trajectories stay on the sliding surface and tends to an equilibrium point. A Lyapunov function is employed to achieve the systems stability under given condition.

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