Optimization of a variant design of dynamic vibration absorber for suppressing random vibration in beams

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Abstract
An effective way to suppressing vibration in mechanical and civil structures is using a passively dynamic vibration absorber (DVA). Arrangement of mass, spring and damper besides frequency and damping ratios of DVA play an important role in the performance of the absorbers, i.e., use of damping element between absorber mass and primary mass suppresses vibration of single-degree-of-freedom (sdof) systems better than traditional DVA which reported in literature but according to author’s knowledge it has not been analyzed for continues structures yet.

In this article, the derivation of solution to $H_2$ and $H_{-\infty}$ optimization problems of this non-traditional DVA applied to suppress random vibration in beam structures will be reported analytically, as well as, a set of optimum frequency and damping ratios of absorbers for both optimization procedures. Effectiveness of using the non-traditional DVA in improving vibration suppression performance in comparison to traditional one will be shown by use of numerical results. Furthermore, comparing $H_{-\infty}$ and $H_2$ optimization procedures shows that for a beam which is excited by random force, use of $H_2$ optimization resulting in better vibration suppression than the other optimization.

Keywords: Dynamic Vibration Absorber, Random Vibration, Passive Control, Beam Structures.

Introduction
Dynamic Vibration Absorbers are a well-established passive vibration control device which, when tuned correctly and attached properly could suppress structures vibration in a good manner. This cheap and easy-to-maintain solution for suppressing vibration besides its simple design and high reliability is found to be very useful in the fields of civil and mechanical engineering.

The first invented DVA had no damping element and it was useful only in a narrow range of frequencies close to the natural frequency of the DVA [1]. Ormondroyd and Den Hartog pointed out that energy dissipation mechanism could widen the frequency band of the DVA’s efficient operation [2], they used damping element parallel to the spring, and this configuration is known as traditional DVA now, “Figure 1b”.

As frequency and damping ratio of absorbers affect their performance, several optimization criteria are proposed to find optimum parameters of the DVAs. In following briefly review $H_2$ and $H_{\infty}$ optimizations.

$H_2$ optimization:
Ormondroyd and Den Hartog [2] pointed out that the damping of the DVA had an optimum value for the minimization of the amplitude response of system, in 1926; such optimization criterion is known as $H_2$ optimization and is based on fixed point theory. The objective is to minimize the maximum amplitude magnification factor of the primary system.

$H_{\infty}$ optimization:
This optimization criterion was proposed by Crandall and Mark [3] in 1963. The objective is to reduce the total vibration energy of the system over all frequency. In this optimization criterion, the area under the frequency response curve of the system is minimized.

Some recent research work on the optimum tuning of DVA for sdof systems can be found in the reports of Asami et al. [4]. They derived the analytical solutions to $H_2$ and $H_{\infty}$ optimization problems of DVAs attached to damped linear systems. Different authors used varied approaches to find optimum parameters of DVAs on multi-degree-of-freedom (mdof) or continuous systems, Hadi and Arfiaid [5] used genetic algorithm to solve numerically $H_2$ optimization for mdof systems, Rice [6] reported the use of SIMPLEX nonlinear optimization to determine the $H_{\infty}$ optimum tuning of a vibration absorber applied for suppressing the vibration of a beam, Dayou [7] and Wong [8] also used fixed point theory which is the most common approach.

A variant design of the damped dynamic vibration absorber as shown in “Figure 1a” was proposed by Ren [9], and Liu and Liu [10] recently.

Figure 1: (a) Non-traditional DVA, (b) Traditional DVA

When a damper is too massive to be attached like traditional DVA, this variant design offers a solution, moreover; this arrangement would be useful in some applications such as energy harvesting [11]. Analytical derivation of the optimum parameters for minimizing the resonant vibration of sdof systems under various excitations such as force [9-11] or ground motion [12] was studied for this case. In 2011, Cheung and Wong [13] suggested new procedure for the $H_{\infty}$ optimization
and new optimum parameters derived and better result in lower maximum amplitude responses were found. Furthermore, they proposed H₂ optimization of this kind of DVAs for vibration control of sdof system [14].

In this article, the H₂ and H∞ optimum parameters of a damped dynamic vibration absorber of non-traditional form have been derived for minimizing the mean square motion of beam structures under a random force excitation. To the author’s knowledge, there is no research report found in literature on this topic.

**Theory**

The beam structure is considered as an mdof or continuous vibrating system. All discussions are based on these assumptions:

- The beam is assumed to be an Euler-Bernoulli beam so the dominant modes are always in the lower modes and the effects of shear deformation and rotational inertia are not large, the dynamic response of the beam is due to the dominant mode only, and the modes can be well separated.

- Referring to "Figure 2", consider the motion of a cantilever beam due to the distributed force. A DVA is attached at x = x₀. The Euler-Bernoulli equation can be written as

\[
\rho A \frac{\partial^2 w}{\partial t^2} + EI \frac{\partial^2 w}{\partial x^2} = p(t)g(x) + F(t)\delta(x - x₀)
\]

(1)

the length of the beam is L, mass per unit length ρA, with bending stiffness EI. The boundary conditions are any combination of pinned, clamped or free supports; here it has been assumed that the externally applied forcing function can be expressed as p(t)g(x), where p(t) is a stationary random function of time and g(x) is a deterministic function of x and F(t) is force transmitted from DVA.

![Figure 2: Cantilever beam with a DVA under external force](image)

The solution to equation (1) can be expanded in a Fourier series written as

\[
w(x,t) = \sum_{n=1}^{\infty} q_n(t)\phi_n(x)
\]

(2)

where \( \phi_n(x) \) is the eigenfunction of the beam without the DVA. Similarly the spatial part of the forcing function and the Dirac delta functions can also be expanded as

\[
g(x) = \sum_{n=1}^{\infty} a_n \phi_n(x), \delta(x-x₀) = \sum_{n=1}^{\infty} b_n \phi_n(x)
\]

(3)

where Fourier coefficients \( a_n \) and \( b_n \) are respectively

\[
a_n = \frac{1}{L} \int_0^L g(x)\phi_n(x)dx, b_n = \frac{\phi_n(x₀)}{L}
\]

(4)

If the equations (2) and (3) substituted into equation (1) and Laplace transformation is taken on the resulting equation with respect to time, the transformed result is a set of algebraic equations, if solved for the generalized co-ordinates \( Qₙ(s) \) and substituted in Laplace transformation of equation (2), the result may be written

\[
W(x,s) = \sum_{n=1}^{\infty} \frac{a_nP(s) + b_nF(s)}{\rho Aω_n^2 + EI\phi_n^2} \phi_n(x)
\]

(5)

For a damped DVA, the transfer function between the motion at the attachment point with the DVA, \( w(x₀,s) \), and the force transmitted to the beam, \( F(S) \), is

\[
F(s) = \frac{W(x₀,s)}{W(x,s)} = \frac{-k(cs + ma^2)}{ms^2 + cs + k}
\]

(6)

by substituting equation (6) into equation (5) and putting \( x = x₀, w(x₀,s) \) will find out, replacing \( \omega = \omega₂ \) for the steady-state response of the beam and rewrite the resulting equation in a non-dimensional form

\[
W(x,\lambda) = \frac{1}{\rho Aω₃} \sum_{n=1}^{\infty} \frac{a_n b_n \phi_n(x₀)}{\gamma (2\gamma ω₃ - \lambda - \omega₃ - \lambda)} H(λ)
\]

(7)

Now it is appropriate to define the following non-dimensional parameters: \( \mu = m/M \) is the mass ratio between the masses of the absorber and the beam; \( \zeta₃ = c/2nmk \) damping ratio of the absorber; \( \gamma = \omega₃/\omega₉ \) is the ratio between the absorber frequency and a reference natural frequency of the plate; \( \gamma = \omega₃/\omega₉ \) is the non-dimensional natural frequency of the beam referred to \( \omega₃ \) and \( \lambda = \omega₉/\omega₉ \) is the normalized frequency.

For a beam structure with well-separated natural frequencies, the modal displacement response in the vicinity of the \( n^{th} \) natural frequency may be approximated by considering \( n \) and ignoring other modes in the equation (7), so

\[
W(x,\lambda) = \frac{1}{\rho Aω₃} \sum_{n=1}^{\infty} \frac{a_n b_n \phi_n(x₀)}{\gamma (2\gamma ω₃ - \lambda - \omega₃ - \lambda)} H(λ)
\]

(8)

Equation (8) can be simplified as

\[
\frac{W(x,\lambda)}{P(λ)} = \frac{a_n b_n \phi_n(x₀)}{\rho Aω₃} H(λ)
\]

(9)

Where

\[
H(λ) = \frac{(\gamma - \lambda)(1 + \gamma - \lambda)}{(1 - \lambda + 2\gamma ω₃)(1 - \lambda + 2\gamma ω₃ - \lambda)}
\]

(10)

**H₂ optimization**

In considering \( H₂ \) optimization, the objective is to minimize the maximum vibration amplitude response of the primary system at point x.

\[
\max \left( \frac{W(x,\lambda,\gamma₃,\zeta₃)}{P(λ)} \right) = \min \left( \max \left( \frac{W(x,\lambda,\gamma₃,\zeta₃)}{P(λ)} \right) \right)
\]

(11)

it is noted that only the function \( H(λ) \) is required to be considered in the optimization because \( \frac{xₚ}{xₚ} \) is a constant term, the objective function of the optimization may be written as

\[
\max \left( \frac{W(x,\lambda,\gamma₃,\zeta₃)}{P(λ)} \right) = \frac{a_n b_n \phi_n(x₀)}{\rho Aω₃} \max \left| H(λ,\gamma₃,\zeta₃) \right|
\]

(12)

the expression \( H(λ) \) is equivalent to the amplitude ratio as derived by Ren [9] in the sdof system attached with a
non-traditional DVA if the term $\varepsilon$ is replaced by the mass ratio $\mu$. $\varepsilon$ may therefore be considered as the equivalent mass ratio for applying a vibration absorber to control vibrations in beam structures. $H_\infty$ optimization can be derived based on the fixed-points theory in the same way as in case of the sdf system. Optimum parameters are listed in "Table 1". $\xi_{H_\infty}$ is damping ratio and $\gamma_{H_\infty}$ is tuning frequency.

### Table 1: The $H_\infty$ optimum tuning at a point $x$ in the beam

<table>
<thead>
<tr>
<th>$\gamma_{H_\infty}$</th>
<th>$\xi_{H_\infty}$</th>
<th>Height of fixed points</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{\sqrt{1-\varepsilon}}$</td>
<td>$\frac{3\varepsilon}{\sqrt{1-0.5\varepsilon}}$</td>
<td>$\frac{a\varphi_0(x)}{\rho A_0^2 [1-\varepsilon]} \frac{P}{\varepsilon}$</td>
</tr>
</tbody>
</table>

**H2 optimization**

In considering $H_2$ optimization, the objective is to minimize the total vibration energy of the mass at point $x$ of the beam of all frequencies in the system and the performance index can be defined as

$$\min_{\gamma} \left[ E \left[ w^2(x,t) \right] \right]$$

(13)

The mean square motion, $E \left[ w^2(x,t) \right]$, of the stationary response can be obtained when the spectral density, $S_w(\omega)$, of the response is known, according to the following formulae,

$$E \left[ w^2(x,t) \right] = \int S_w(\omega) d\omega$$

(14)

If the forcing function $p(t)$ has power spectral density $S_p(\omega)$, the spectral density of the vibration response of the point $x$ on the beam may be written as

$$S_w(\omega) = \frac{W(x,\omega)}{P(\omega)} S_p(\omega)$$

(15)

So, the mean square motion can be written in terms of the input mean square spectral density as,

$$E \left[ w^2(x,t) \right] = \int \frac{W(x,\omega)}{P(\omega)} S_p(\omega) d\omega$$

(16)

If the input spectrum assumed to be ideally white, i.e., $S_p(\omega) = S_0$, a constant for all frequencies, the integral of equation (16) can be reduced to

$$E \left[ w^2(x,t) \right] = S_0 \int \frac{W(x,\omega)}{P(\omega)} \omega d\omega$$

(17)

So, the non-dimensional mean square motion can be defined as

$$E \left[ \frac{w^2(x,t)}{S_0} \right] = \frac{a\varphi_0(x)}{2\pi} \int \frac{a\varphi_0(x)}{\rho A_0^2} H(\lambda) d\lambda$$

(18)

As $\frac{a\varphi_0(x)}{2\pi} H(\lambda)$ is a constant term, the term $\frac{a\varphi_0(x)}{2\pi} H(\lambda)$, need to be considered in optimization, which is equal to mean square motion of variant design of DVA attached to sdf system [14]. Therefore the optimum DVA parameters can be found in the same way as in the case of the sdf system if the term $\varepsilon$ replaced by the mass ratio $\mu$. No global optimum tuning frequency exists in the proposed absorber and it is recommended that a high tuning frequency ratio be used if possible; furthermore, using of equivalent mass ratio, less than 0.11, the frequency ratio more than $0.5 \sqrt{(6-3\varepsilon)-(6-3\varepsilon)} - 32$ needed, otherwise, contrary results will be achieved.

The best value of damping ratio after selecting tuning frequency ratio is

$$\xi_{opt} = \sqrt{\frac{\gamma^4 + (\varepsilon-2)^2 + 1}{4\gamma^2}}$$

(19)

So, the mean square motion at point $x$ of the beam defined as:

$$E \left[ \frac{w^2(x,t)}{S_0} \right] = \frac{a\varphi_0(x)}{4\varepsilon\lambda_0^2} \left[ 1 + \frac{\varepsilon + 4\varepsilon^2 - 2}{\gamma^2} \right]$$

(20)

**Simulation results and discussion**

To show the efficiency of these optimization procedures of non-traditional DVA on beam structures in vibration suppression, the numerical case studies are presented in this section. An Euler beam with simply support boundary condition and attached with non-traditional DVA is considered, "Figure 3". The material of beam supposed to be aluminum of Young’s modulus and density of 207Gpa and 7870 kg/m$^3$, respectively, other properties are $A=2.42*10^4$ m$^2$, $I=8.13*10^{-10}$ m$^4$, $L=1$m. Absorber attached in $x_j=0.5L$ and harmonic force applied at $x_j=0.1L$ and The mass ratio $\mu$ is 0.2.

![Figure 3: schematic model which used in numerical test](image)

The eigenfunction and natural frequencies of beam considered as

$$\varphi_i(x) = \sin \left( \frac{i\pi x}{L} \right), \omega_i^2 \left( \frac{i\pi^4}{L^4} \right) \frac{EI}{\rho A}, \quad i=1,2,3,...$$

(21)

According to equation (4), Fourier coefficients are:

$$a_i = \frac{1}{L} \int \cos \left( \frac{i\pi x}{L} \right), h_i = \frac{1}{L} \int \sin \left( \frac{i\pi x}{L} \right), \quad i=1,2,3,...$$

(22)

the model which introduced is used to show that non-traditional DVA has better outcomes in minimizing the mean square vibration and spectral density of vibration amplitude at a point of a beam than use of traditional DVA while one of the best optimization procedures which followed by Cheung and Wong [8] applied on it. For non-traditional DVA, it is recommended that a high tuning frequency ratio be used if possible in $H_2$ optimization, for comparison tuning frequency of $H_\infty$ optimization is considered here, other optimum parameters are listed in "Table 2".

### Table 2: Optimum parameters for traditional and non-traditional DVAs

<table>
<thead>
<tr>
<th>Type of DVA</th>
<th>$\gamma_{11}$</th>
<th>$\xi_{11}$</th>
<th>$\gamma_{22}$</th>
<th>$\xi_{22}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Traditional</td>
<td>0.833</td>
<td>0.25</td>
<td>0.874</td>
<td>0.209</td>
</tr>
<tr>
<td>Non-traditional</td>
<td>1.118</td>
<td>0.288</td>
<td>1.118</td>
<td>0.249</td>
</tr>
</tbody>
</table>

At first the spectral density of the vibration amplitude at point $x$ on beam is calculated for both traditional and non-traditional DVAs and is plotted for $H_\infty$ and $H_2$ optimization procedures in "Figure 4" and "Figure 5".

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respectively, it is obvious that for both optimization procedures, non-traditional DVAs minimize the spectral density of the vibration amplitude at point x better than traditional one about 8.2% for $H_{\infty}$ and 11% for $H_2$ optimization.

According to above analysis, using tuning frequency as high as possible in $H_2$ optimization results in better mean square minimization, so, percentage reduction of the mean square motion of the proposed absorber relative to the traditional absorber at different equivalent mass ratios by use of $H_2$ optimization in both cases is calculated and plotted in "Figure 7".

Outcomes show, for a constant equivalent mass ratio, the percentage reduction of the mean square motion raises by increasing tuning frequency ratio, furthermore, for the fixed value of frequency ratio, increasing equivalent mass ratio results in more percentage reduction of the mean square, although this growth is not significant for frequency ratios less than 1.2. As mentioned before for the equivalent mass ratio less than 1.1, appropriate frequency ratio needed to choose, if it is not possible, using other optimization procedure is suggested.

Furthermore, the spectral density of the vibration amplitude is calculated for both optimizations in order to define better procedure for non-traditional DVA. As shown in "Figure 6", $H_2$ optimization is more effective that the other one in suppressing vibration of the beam under random force excitation due to 11.4% reduction in spectral density.

In the following, effect of two optimizations, different Dynamic Vibration Absorbers and equivalent mass ratios, $\epsilon$, in minimizing mean square motion is studied. Mean square motion is calculated for non-traditional and traditional DVA according to equation (20) and which proposed by Cheung and Wong [8], in this calculation optimum parameters of non-traditional DVA considered as formulae which proposed in Table 1 for $H$-infinity and equation (19), $\gamma=1.5$ for $H_2$ optimization. For tuning frequency and damping ratio of traditional DVA formulae which obtained by Cheung and Wong [8] is used.
Outcomes show that using $H$-infinity optimization and traditional DVA has the worst result in minimizing root mean square motion among others, so its percentage reduction of proposed absorber and two optimization procedures relative to the traditional DVA and $H$-infinity optimization is calculated for different $\varepsilon$ and plotted in "Figure 8". It is obvious, in a particular range of equivalent mass ratio, $H_2$ optimization and non-traditional DVA has the best result in minimizing mean square motion.

**Conclusion**

$H_2$ and $H$-infinity optimization procedures of a variant design of dynamic vibration absorbers for suppressing random vibration in beam structures under a random force excitation were derived. A set of optimum parameters proposed for $H$-infinity optimization, no global optimum tuning condition exists in the proposed absorber when $H_2$ optimization is applied, but it is recommended that a high tuning frequency ratio be used if is possible. The best value of damping ratio after selecting the tuning frequency ratio is derived too. To the author’s knowledge, there is no research report found in the literature on this topic. It has been shown that the performance of the variant DVA can be better than the ordinary one if the frequency and damping ratios of the DVA are chosen properly, i.e., for a practical range of equivalent mass ratio, $\varepsilon=0.2$, the proposed DVA can provide 18% and 33% or more reduction of mean square motion of beam at a point by applying $H$-infinity optimization and use of $H_2$ optimization with $\gamma=1.5$, respectively. Comparison of two optimization procedure for non-traditional DVAs showed the mean square motion at a point of the beam with the proposed absorber and $H_2$ optimization was 15% smaller than $H$-infinity optimization.

**References**


