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Buckling analysis of a cylindrical panel under axial stress using perturbation technique

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In this paper, a combination of the series method and the *perturbation theory* has been proposed to calculate the elastic buckling stresses of panels with two edges simply supported subjected to the uniform axial stress by considering the *linear theory of Donnell* as the governing equation. *As a parametric study*, the effects of changing the radius, length, and sector angles of the panels on the buckling stresses have been investigated and the results have been compared with the finite elements method. Also, to find the buckling stress, a correction factor for the Lorenz formula has been proposed.

1 Introduction

The stability of panels under different loads, is the subject of many papers in recent years. Gal et al. [2] presented the buckling analysis of a laboratory tested composite panel under axial compression by means of a simple shell element. Magnucki and Mackiewicz [7] solved the linear equation of Donnell for buckling of panels subjected to axial stress with three edges simply supported and one edge free using the Galerkin method. Patel et al. [8] investigated the static and dynamic instability characteristics of stiffened shell panels subjected to uniform in-plane harmonic edge loading by finite elements (FE) method. Szyc et al. [10] considered the problem of elastic buckling of an open sandwich cylindrical thin-walled panel with three edges simply supported and one edge free under axial compression. The governing equations were solved with the Bubnov–Galerkin method. Hu and Yang [3] calculated the critical buckling loads of composite cylindrical panels by the bifurcation buckling analysis implemented in the ABAQUS FE program. Wilde et al. [11] provided analytical and numerical approaches for determining the buckling load of a panel with three edges simply supported and one edge free using the linear theory of Donnell. They chose an appropriate function for deflection and derived an eight order differential equation which can be solved analytically. Also, they used ABAQUS FE package for verifying the results. Jiang et al. [5] used differential quadrature element method for buckling analysis of stiffened circular cylindrical panels subjected to axial uniform compressive stresses. Sofiyev et al. [9] presented an analytical procedure to study the free vibration and stability characteristics of homogeneous and non-homogeneous orthotropic conical shells with clamped edges under uniform external pressures. The governing equations according to the Donnell's theory were solved by Galerkin's method and critical hydrostatic and lateral pressures and fundamental natural frequencies have been found analytically. Joniak et al. [6] studied the elastic buckling and limit load of open circular cylindrical thin shells in pure bending with two edges supported and two edges free. They used a simple analytical description, numerical analysis (FE), and laboratory tests. The results have been compared and presented in figures. Also, they presented an analytical formula of critical stresses for these panels in pure bending state.

The most researches on the stability of panels are based on the *numerical method* like the Galerkin or FE codes. In this paper, the buckling stresses of cylindrical panels subjected to axial stress with the two opposite sides simply supported and the two other edges free or simply supported, have been calculated analytically by using the *perturbation method*. The governing equation is the *linear equation of Donnell*. As a parametric study, the effects of changing the radius, length, and panel sector angle on the buckling stresses have been investigated. The results have been compared with the FE method for some special cases. Also, to find the buckling stress, a correction factor which can be applied to the Lorenz formula, has been proposed.

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2 Analytical method

A thin cylindrical panel subjected to axial compressive stress σ_x has been considered (Fig. 1).

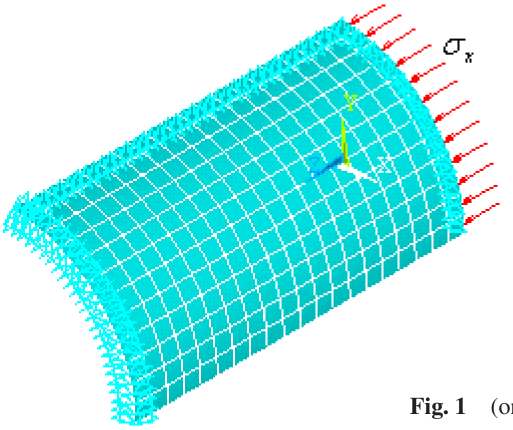

Fig. 1 (online colour at: www.zamm-journal.org) A typical panel subjected to axial stress.

The governing equations for the radial displacement are [7]:

$$\frac{D}{t}\nabla^4 w + \frac{1}{R}\frac{\partial^2 F}{\partial x^2} + \sigma_x \frac{\partial^2 w}{\partial x^2} = 0\,, \quad \nabla^4 F - \frac{E}{R}\frac{\partial^2 w}{\partial x^2} = 0 \tag{1}$$

where w radial displacement, R radius, t thickness, L length, x axial coordinate, ϕ angle coordinate $(0 \le \phi \le \beta)$, $y = R\phi$, β sector angle of a panel, $E =$ Young's modulus, ν Poisson's ratio, $D = Et^3/12\,(1-\nu^2)$ bending stiffness, and F Airy force function. By introducing the dimensionless parameters:

$$s_0 = \sigma_x/E\,, \quad \varepsilon = t/L\,, \quad F^* = F/EL^2\varepsilon\,, \quad R^* = R/L\,, \quad x^* = x/L\,, \quad w^* = w/t\,. \tag{2}$$

The non-dimensional forms of the governing equations are:

$$\frac{\varepsilon^2}{12(1-\nu^2)}\nabla^{*4} w^* + \frac{1}{R^*}\frac{\partial^2 F^*}{\partial x^{*2}} + s_0 \frac{\partial^2 w^*}{\partial x^{*2}} = 0\,, \quad \nabla^{*4} F^* - \frac{1}{R^*}\frac{\partial^2 w^*}{\partial x^{*2}} = 0\,, \tag{3}$$

$$\text{where} \quad \nabla^{*4} = \frac{\partial^4}{\partial x^{*4}} + \frac{2}{R^{*2}}\frac{\partial^4}{\partial x^{*2}\partial \phi^2} + \frac{1}{R^{*4}}\frac{\partial^4}{\partial \phi^4}\,.$$

The Boundary Conditions (BCs) as simply supported at $x^* = 0\,,1$ provide the trial solutions:

$$w^* = \sum_{m=1} w_m(\phi)\sin m\pi x^*\,; \quad F^* = \sum_{m=1} f_m(\phi)\sin m\pi x^* \tag{4}$$

w_m and f_m are unknown functions of ϕ and m is the number of longitudinal half-waves. By substituting Eqs. (4) in Eqs. (3), one can obtain:

$$[A_4]\{X\}^{(iv)} + [A_2]\{X\}'' + [A_0]\{X\} = \{0\}_{2*2}\,, \tag{5}$$

$$\text{where:} \quad [A_4] = \frac{1}{R^{*4}}\begin{bmatrix} t^* & 0 \\ 0 & 1 \end{bmatrix}, \quad [A_2] = -2(\frac{m\pi}{R^*})^2\begin{bmatrix} t^* & 0 \\ 0 & 1 \end{bmatrix}, \quad t^* = \frac{\varepsilon^2}{12(1-\nu^2)}$$

$$[A_0] = (m\pi)^2\begin{bmatrix} t^*(m\pi)^2 - s_0 & -\frac{1}{R^*} \\ \frac{1}{R^*} & (m\pi)^2 \end{bmatrix}, \quad \{X\} = \begin{Bmatrix} w_m \\ f_m \end{Bmatrix}, \quad \{\}^{(iv)} = \frac{d^4}{d\phi^4}\{\}\,, \quad \{\}'' = \frac{d^2}{d\phi^2}\{\}\,.$$

Eqs. (5) are a system of homogenous differential equations with constant coefficients. Its general solution is: $\{X\} = \{X_0\}\,\mathrm{e}^{\lambda\phi}$, where λ eigenvalue, $\{X_0\}$ eigenvector. Hence, Eqs. (5), give:

$$\left([A_4]\,\lambda^4 + [A_2]\,\lambda^2 + [A_0]\right)\{X_0\}\,\mathrm{e}^{\lambda\phi} = 0\,. \tag{6}$$

The eigenvalues are the roots of $\det\left[\left([A_4]\,\lambda^4 + [A_2]\,\lambda^2 + [A_0]\right)\right]_{2*2} = 0$ which is an algebraic equation in the form of:

$$b_4 q^4 + b_3 q^3 + b_2 q^2 + b_1 q + b_0 + s_0\left(d_2 q^2 + d_1 q + d_0\right) = 0\,, \tag{7}$$

where b_i, d_j constants, $i = 0 \dots 4$, $j = 0 \dots 2$, and $q = \lambda^2$. It is possible to find the exact solution of Eq. (7). Maple software can present this solution. This solution is a complicated function of s_0 useless in engineering calculations because it needs a very powerful computer for mathematical operations. Instead, one can use the perturbation theory for solving Eq. (7), because s_0 *is a small parameter*. Using this technique, the solution of Eq. (7) can be considered as a uniform series of s_0 i.e.:

$$q = q_0 + s_0 q_1 + s_0^2 q_2 + \dots \,. \tag{8}$$

By substituting Eq. (8) in Eq. (7) and considering the terms with the same Order of s_0, result:

$$O(s_0^0): \; b_4 q_0^4 + b_3 q_0^3 + b_2 q_0^2 + b_1 q_1 + b_0 = 0\,, \tag{9a}$$

$$O(s_0): \; g_1(q_0) q_1 + g_2(q_0) = 0\,, \tag{9b}$$

$$O(s_0^2): \; g_3(q_0) q_2 + g_4(q_0, q_1) = 0\,, \tag{9c}$$

where g_1, g_2, g_3 are functions of q_0 and g_4 is a function of q_0, q_1. Eqs. (9) give q_0, q_1, q_2 and from Eqs. (6), (8), the eigenvalues and the eigenvectors can be calculated. So, the general solution is:

$$\{X\} = \sum_{i=1}^{8} c_i \{X_0\}_i \, e^{\lambda_i \phi} \quad \text{and} \quad \left\{ \begin{array}{c} w^* \\ F^* \end{array} \right\} = \{X\} \sin m\pi x^*\,, \tag{10}$$

c_i calculated constants from the BCs.

Wilde et al. [11] assumed $w(x, y) = t.w(y)\sin(m\pi x/L)$ and $F(x, y) = -Et^2 R\beta F(y)\sin(m\pi x/L)$. After substituting in Eqs. (1) and combing the equations, they found an eight order ordinary differential equation (ODE). They solved the characteristic equation which is similar to Eq. (7), exactly and presented $w(y)$ as summation of products of trigonometric and hyperbolic functions (like $\sin(\alpha_1\phi)\sinh(\alpha_2\phi)$). It means that the roots of the characteristic equation is complex. They did not explain the reason of this conclusion. Eqs. (4) and the Wilde [11] solutions can be considered as an analogy of the Levy's method for rectangular plates [12].

2.1 Boundary conditions (Magnucki and Mackiewicz [7])

a. Free boundary condition at $\phi = 0$ or $\phi = \beta$

$$N_{yy}^* = 0\,, \quad \text{where} \quad N_{yy}^* = \frac{N_{yy}}{E} = \varepsilon \frac{\partial^2 F^*}{\partial x^{*2}}\,,$$

$$N_{xy}^* = 0\,, \quad \text{where} \quad N_{xy}^* = \frac{N_{xy}}{E} = -\varepsilon \frac{\partial^2 F^*}{\partial x^* \partial \phi}\,,$$

$$V_y^* = 0 \quad \text{where} \quad V_y^* = \frac{V_y}{Et} = -\frac{t^*\varepsilon}{R^*}\frac{\partial}{\partial \phi}\left(\frac{\partial^2 w^*}{\partial x^{*2}} + \frac{1}{R^{*2}}\frac{\partial^2 w^*}{\partial \phi^2}\right),$$

$$M_{yy}^* = 0 \quad \text{where} \quad M_{yy}^* = \frac{M_{yy}}{Et^2} = t^*\left(\frac{1}{R^{*2}}\frac{\partial^2 w^*}{\partial \phi^2} + \nu \frac{\partial^2 w^*}{\partial x^{*2}}\right).$$

b. Simply supported at $\phi = 0$ or $\phi = \beta$

$$w^*(x^*, \phi) = 0\,; \quad M_{yy}^* = 0\,, \quad N_{yy}^* = 0\,, \quad N_{xy}^* = 0\,.$$

c. Simply supported at $x^* = 0, 1$ are satisfied by the trial solutions Eqs. (4).

Applying the BCs to Eq. (10) gives a system of homogenous algebraic equations as $[a]\{c\} = \{0\}_{8*8}$. The elements of the vector $\{c\}$ are the constants of Eq. (10) and the coefficients matrix ($[a]$) contains the functions of s_0. The non-trivial solution can be obtained by setting $\det[a] = 0$ where "det" stands for determinant of matrix. This is a complicated algebraic equation. One can use the Taylor series to expand it in the form of a polynomial. The least positive real root of this equation is the buckling stress (s_0). The use of exact solution of Eq. (7) precludes an analytical solution of the equation $\det[a] = 0$. Wilde et al. [11] solved this equation numerically. By using the perturbation method, the results can be obtained analytically.

3 Analytical results

As the case studies, two categories of panels have been studied.
3.1 The first one contains the panels with simply supported at $x^* = 0, 1$, $\phi = 0$, and free at $\phi = \beta$ (*sssf*). The effects of changing the sector angle, radius and length on critical stresses of panels have been shown on Figs. 2–7. All the computations have been performed using Maple 11 package. Fig. 2 shows that the sector angle ($\phi > 90$) does not significantly affect the buckling stress.

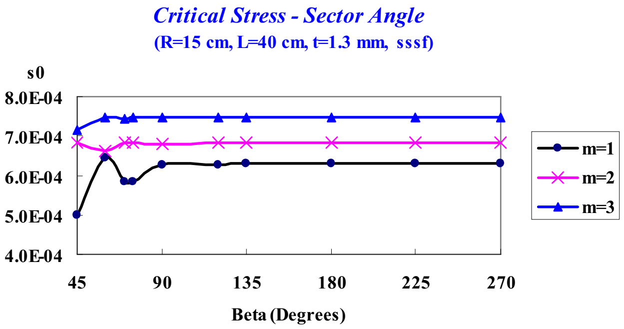

Fig. 2 (online colour at: www.zamm-journal.org) Critical stresses in terms of sector angle for different modes.

According to Figs. 3 and 4, enlarging the radius or increasing the length, can decrease the buckling stress. The variation of the buckling stress with the length is less important than the radius or the variation of the radius is more significant. Also, the buckling stress is more sensitive to the length changes for small length values.

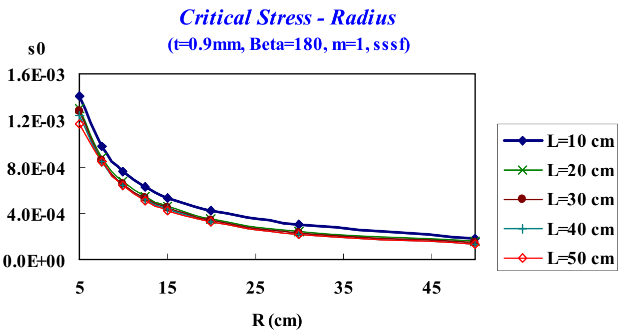

Fig. 3 (online colour at: www.zamm-journal.org) Critical stresses in terms of radius for different lengths.

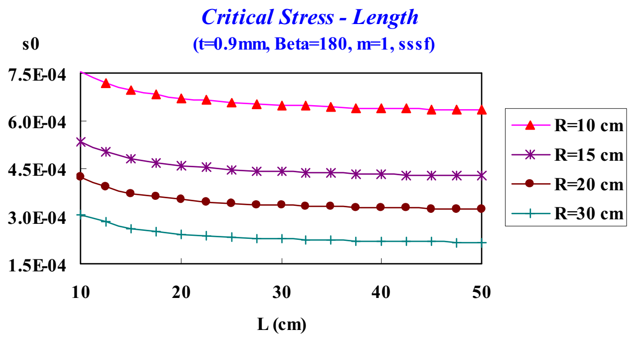

Fig. 4 (online colour at: www.zamm-journal.org) Critical stresses in terms of length for different radiuses.

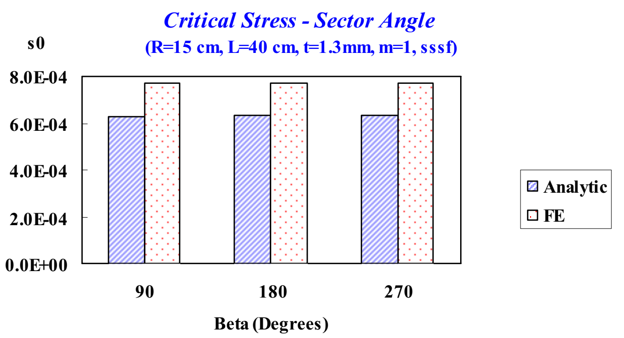

Fig. 5 (online colour at: www.zamm-journal.org) Critical stresses for different sector angles, comparison with FE (mode 1).

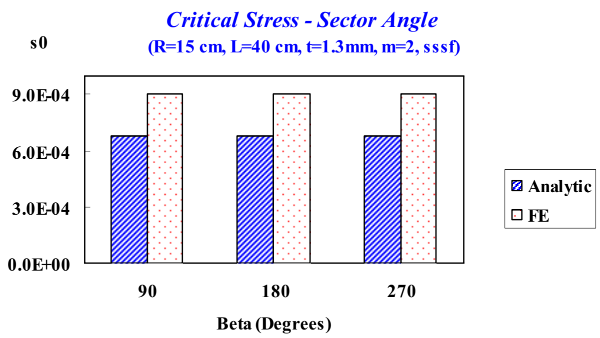

Fig. 6 (online colour at: www.zamm-journal.org) Critical stresses for different sector angles, comparison with FE (mode 2).

3.1.1 Results comparison. For verifying the results, *Ansys 11.0* package was used for buckling analysis of panels. *SHELL93* element which is particularly well suited to model curved shells was applied for meshing of the models. This element has six degrees-of-freedom at each node. Its shape is quadratic in both in-plane directions and it is defined by eight nodes. The

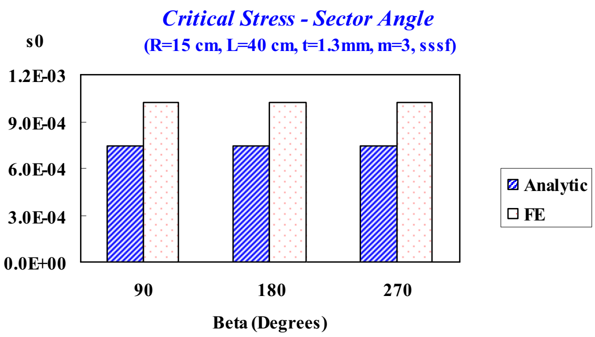

Fig. 7 (online colour at: www.zamm-journal.org) Critical stresses for different sector angles, comparison with FE (mode 3).

element has stress stiffening and large deflection capabilities. Fig. 1 shows a typical FE model. The mesh size has been selected with trial and error. Figs. 5–7 compare the buckling stresses calculated analytically (Eq. 10) and numerically (FE) for modes 1, 2, 3 of some panels. FE analysis approves the results of Fig. 2, i.e. the buckling stress is not sensitive with respect to the panel sector angle.

Table 1 compares the buckling stresses calculated by the analytical method, FE (Ansys), Magnucki [7] and Wilde [11]. Difference percentage has been defined as $\left|\frac{s_{0_{\mathrm{Method}}} - s_{0_{\mathrm{FE}}}}{s_{0_{\mathrm{FE}}}}\right| \times 100$. Fig. 8 shows the first buckling mode of this panel.

Table 1 Buckling stresses ($R = 500$ mm, $L = 1000$ mm, $t = 1.65$ mm, Beta $= 90$, $m = 1$, $E = 2.05\mathrm{e}5$ MPa, sssf).

Method	Critical stress (MPa)	s_0	Difference percentage with respect to FE
Analytical [Eq. (10)]	48.57	2.37 e-4	16.5%
FE (Ansys)	58.3	2.84 e-4	0
Magnucki [7]	-	1.98 e-5	93%
		1.57 e-4	44.7%
Wilde [11]-analytical	31.54	1.54 e-4	45.8%
Wilde [11]-numerical	31.90	1.56 e-4	45.1%

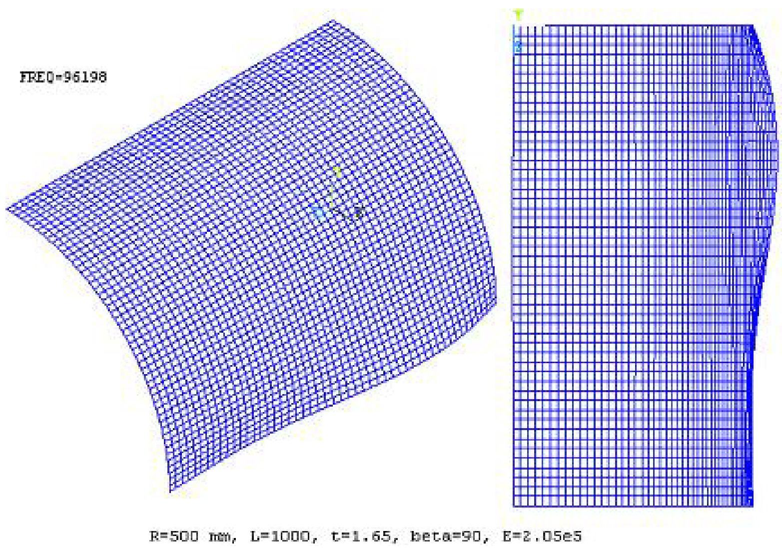

Fig. 8 (online colour at: www.zamm-journal.org) First buckling mode of a panel, sssf (Ansys).

3.2 In the second category, the BCs are simple at $x^* = 0, 1$ and free at $\phi = 0$, $\phi = \beta$ $(ssff)$. The similar results for this case can be found. The buckling stress is smaller than the case *sssf*. Figs. 9 and 10 show these results.

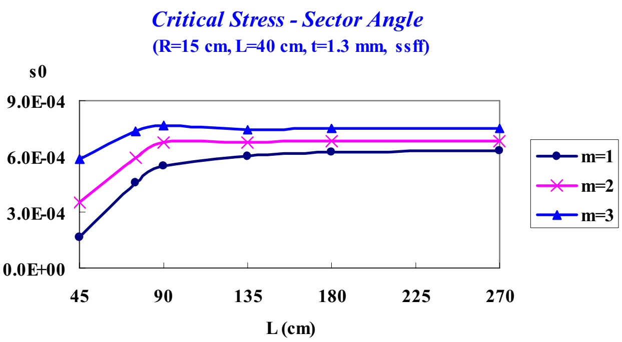

Fig. 9 (online colour at: www.zamm-journal.org) Critical stresses in terms of length for different modes (ssff).

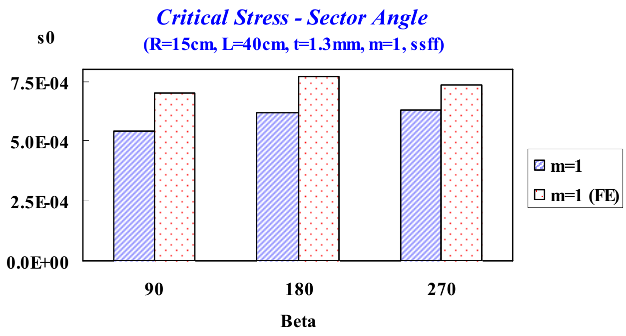

Fig. 10 (online colour at: www.zamm-journal.org) Critical stresses for different sector angles, comparison with FE (mode 1, ssff).

4 Correction factor

From combining Eqs. (1), one can find an eigth order partial differential equation as the governing equation. By assuming the radial deflection as $w = \sum_m \sum_n W_{mn} \sin m\pi x/L \sin n\phi$, the lowest buckling stress of a cylindrical shell (Lorenz formula) for $\nu = 0.3$ is as follows:

$$s_0 = \sigma_x/E = 0.605t/R\,. \tag{11}$$

This value poorly agrees with the values found in the tests and the discrepancies increase for small thickness of the shell in most cases. In some tests, the buckling stress was only 10% or 15% of Eq. (11) (Donnell [1]). In order to relate the theoretical value resulting from Eq. (11) to actual test data, it is necessary to correct the Lorenz formula by incorporating an empirical factor K_0, in Eq. (11). Ugral [12] proposed the following formula for cylinders having $L/R < 5$:

$$s_0 = 0.605K_0 t/R\,, \quad K_0 = 1 - 0.901(1 - \mathrm{e}^{-\psi})\,, \quad \psi = 1/16\sqrt{R/t}\,. \tag{12}$$

According to the analytical results i.e. Eq. (10), a correction factor has been proposed for Eq. (11) in terms of parameter $Z = \frac{L^2}{Rt}$. Jawad [4] defined $\frac{L^2}{Rt}\sqrt{1-v^2}$ as the curvature parameter. Fig. 11 shows the variations of K_0 with respect to Z by considering the first buckling mode, $sssf$ BCs and $\beta = 180°$. The results have been computed for the range of $10\ \text{cm} \le L \le 50\ \text{cm}$, $5\ \text{cm} \le R \le 50\ \text{cm}$. This graph (Fig. 11) can be approximated using a suitable model:

$$s_0 = 0.605K_0 t/R\,, \quad K_0 = \frac{1}{a + b\,Z^c} \tag{13}$$

where: $a = -149.24560$, $b = 153.46956$, $c = 3.84221$ e-3.

The curve fitting process was performed using Harris model in *Curve expert3.1* software.

Eq. (13) which is a corrected version of the Lorenz formula, can be used to estimate the buckling stress of a panel in the defined ranges.

5 Discussion of results

The analytical results are based on the *linear theory of Donnell*. Therefore, one has to know the restrictions or validity range of this theory. According to Figs. 5–7, there is about 20% difference between the FE and presented method (Eq. (10)). This difference may be due to the following reasons:

The first one is the mathematical simplifications and rounding error especially for $ssff$ BCs. For example, after applying the BCs, the Taylor expansion has been used for estimating the buckling stress from the determinant of the matrix coefficients which is a mathematical simplification.

The second one is that the governing equation in the FE formulation differs from the linear theory of Donnell.

In comparison with the similar works, the differences of our results with the Wilde [11] and Magnucki [7] results are noticeable (Table 1). Wilde [11] performed the FE analysis by ABAQUS package and reported a good agreement between the analytical and FE results, although one can expect *more discrepancy* in the results because the linear theory of Donnell

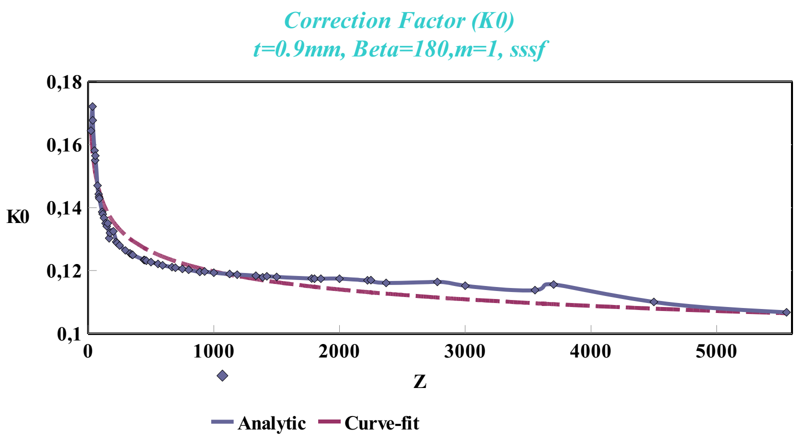

Fig. 11 (online colour at: www.zamm-journal.org) Correction factor in terms of Z for first mode of panels with *sssf* BCs.

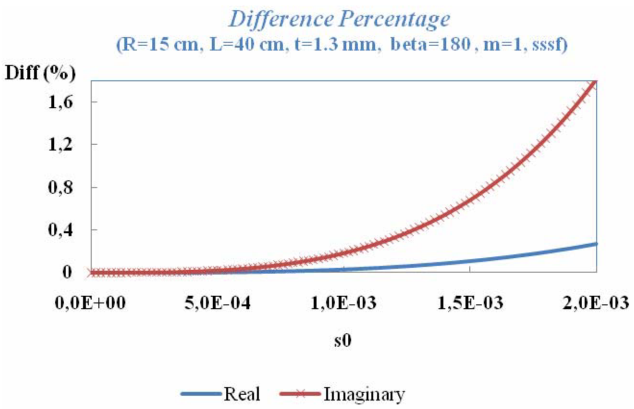

Fig. 12 Difference percentage for real and imaginary parts of an eigenvalue.

neglects some terms of the equilibrium equations but the FE uses a more general theory in formulation. The discrepancies of the presented method with the Magnucki [7] results are much larger. Magnucki [7] introduced a *trigonometric function* as the radial deflection and they used the orthogonality property to solve the problem. In our opinion, the orthogonality property is not correct for all values of β. Magnucki [7] calculated two eigenvalues for $m = 1$ which is valid for a special case study. The difference in case of the first eigenvalue is more.

The presented results as the correction factor (Fig. 11 and Eq. (13)) are based on the Eq. (1) in defined ranges of R and L for *sssf* BCs. A special value for the parameter Z can be produced by different values of R, K, or t. Although the results have been reported for $t = 0.9$ mm, but the calculations for 0.4 mm $\leq t \leq 2$ mm approved the Fig. 11, or K_0 depends on Z no the values of R, K, t. The results of Fig. 11 are for the case $\beta = 180°$ but in Figs. 2 and 5, it has been shown that the lowest buckling stress for $\phi > 90°$ is not sensitive to sector angle. So, one can expect the validity of Eq. (13) for a wide range of sector angles. The perturbation results were based on three terms of series Eq. (8). To check the solution convergence, a modified solution as

$$q = q_0 + s_0 q_1 + s_0^2 q_2 + s_0^3 q_3 \tag{14}$$

has been considered for Eq. (7). Fig. 12 shows the difference percentage for real and imaginary parts of an eigenvalue which have been computed using Eq. (8) and Eq. (14). It is seen that for $s_0 < 0.002$, this difference for real and imaginary parts are not more than 0.3% and 1.7%, respectively. So, Eq. (8) (with three terms) has a sufficient convergence and it is not necessary to consider more terms in expansion. The conversion range of s_0 for each problem can be found easily. $s_0 < 0.002$ was an acceptable range in the presented case studies.

6 Conclusion

An analytical method based on the perturbation theory, has been proposed to find the buckling axial stress of a panel. The presented procedure can be prepared as a simple program in Maple environment for studying the effects of different geometrical and mechanical characteristics of panels on the buckling stresses. So the results are obtained easier and faster than the *FE* method because it is not necessary to create and solve a *FE* model. According to the results, a correction factor in terms of Z, has been proposed to the Lorenz formula to find the buckling stress for a wide range of sector angles. The substantial error occurs in calculations due to use of the *linear theory of Donnell* as the governing equation.

References

[1] L. H. Donnell, Beam Plates and Shells (McGraw-Hill, New York, 1976).

[2] E. Gal, R. Levy, H. Abramovich, and P. Pavsner, Buckling analysis of composite panels, Compos. Struct. **73**, 179–185 (2006).

[3] H. Hu and J. Yang, Buckling optimization of laminated cylindrical panels subjected to axial compressive load, Compos. Struct. **81**, 374–385 (2007).

[4] M. Jawad and St. Louis, Design of Plate and Shell Structures (ASME Press, 2003).

[5] L. Jiang, Y. Wang, and X. Wang, Buckling analysis of stiffened circular cylindrical panels using differential quadrature element method, Thin-Walled Struct. **46**, 390–398 (2008).

[6] S. Joniak, K. Magnucki, and W. Szyc, Buckling study of steel Open circular cylindrical shells in pure bending, Strain (UK), Strain, An Int. J. for Exp. Mech., doi/10.1111/j.1475-1305.2009.00669.x (2010).

[7] K. Magnucki and M. Mackiewicz, Elastic buckling of an axially compressed cylindrical panel with three edges simply supported and one edge free, Thin-Walled Struct. **44**, 387–392 (2006).

[8] S. N. Patel, P. K. Datta, and A. H. Sheikh, Buckling and dynamic instability analysis of stiffened shell panels, Thin-Walled Struct. **44**, 321–333 (2006).

[9] A. H. Sofiyev, N. Kuruoglu, and H. M. Halilov, The vibration and stability of non-homogeneous orthotropic conical shells with clamped edges subjected to uniform external pressures, Appl. Math. Model., available online, doi:10.1016/j.apm.2009.09.025 (2009).

[10] W. Szyc, Z. Laszczyk, and K. Magnucki, Elastic buckling of an axially compressed sandwich cylindrical panel with three edges simply supported and one edge free, Thin-Walled Struct. **44**, 910–918 (2006).

[11] R. Wilde, P. Zawodny, and K. Magnucki, Critical state of an axially compressed cylindrical panel with three edges simply supported and one egde free, Thin-Walled Struct. **45**, 955–959 (2007).

[12] A. C. Ugral, Stresses in Plates and Shells (McGraw-Hill, New York, 1981).