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On almost sure convergence rates for the Kernel estimation of a covariance operator under negative association

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Abstract

Let $\{X_n,\ n \geq 1\}$ be a strictly stationary sequence of negatively associated random variables, with common continuous and bounded distribution function F. In this paper, we consider the estimation of the two-dimensional distribution function of (X_1, X_{k+1}) based on Kernel type estimators as well as the estimation of the covariance function of the limit empirical process induced by the sequence $\{X_n,\ n \geq 1\}$. Then, we derive uniform strong convergence rates for the Kernel estimator of two-dimensional distribution function of (X_1, X_{k+1}) which were not found already and do not need any conditions on the covariance structure of the variables. Finally, assuming a convenient decrease rate of the covariances $Cov(X_1, X_{n+1}),\ n \geq 1$, we introduce uniform strong convergence rate for covariance function of the limit empirical process based on Kernel type estimators.

Key Words: Almost sure convergence rate, Bivariate distribution function, Empirical process, Kernel estimation.

1 Introduction

Approximation of distribution functions of random pairs (two-dimensional distribution functions) has been always a subject of interest of many statisticians. The case of independent underlying random variables was studied by Donsker [2]. The case of nonindependent random variables had been studied, too. One case is negatively associated random variables. A finite family of random variables $\{X_i,\ 1 \leq i \leq n\}$ is said to be negatively associated (NA) if for every pair of disjoint subsets A and B of $\{1, 2, ..., n\}$,

$$Cov(f_1(X_i, i \in A), f_2(X_j, j \in B)) \leq 0$$

whenever f_1 and f_2 are coordinatewise increasing and such that the covariance exists. An infinite family of random variables is NA if every finite subfamily is NA. We refer to Jabbari and Azarnoosh [5] and Jabbari [6] for knowing some of the most important studies have been performed on different aspects of NA random variables.
The mentioned comments above motivated the interest on the estimation of the bivariate distribution function under negative association. A natural estimator of $F_k(x, y) = P(X_1 \leq x, X_{k+1} \leq y)$ with k fixed, is defined by

$$\hat{\varphi}_n(x, y) = \frac{1}{n-k} \sum_{i=1}^{n-k} \{1_{(-\infty,x]}(X_i) 1_{(-\infty,y]}(X_{k+i})\}. \tag{1}$$

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The asymptotic behavior of this estimator was studied by Jabbari and Azarnoosh [5]. For dependent sequences, under certain conditions (see Newman [10], Theorem 17 and the first remark of p. 137), the limit of the uniform empirical process still is a centered Gaussian process, but the covariance function changes to

$$\Gamma(r,s) \;=\; \varphi_k(r,s)+\sum_{k=1}^{\infty}\varphi_k(r,s)+\sum_{k=1}^{\infty}\varphi_k(s,r), \tag{2}$$

where $\varphi_k(r,s)=F_k(r,s)-F(r)F(s)$. Jabbari and Azarnoosh [5] and Henriques and Oliveira [3] and [4] drove a uniform strong convergence rate of $n^{-1/2}$ for two-dimensional empirical distribution function of (X_1,X_{k+1}) and covariance function of the limit empirical process assuming a convenient decrease rate of the covariance. Jabbari [6] and Azevedo and Oliveira [1] considered the Kernel estimator of F_k, defined by

$$\hat{F}_k(r,s) \;=\; \frac{1}{n-k}\sum_{i=1}^{n-k}U\Big(\frac{r-X_i}{h_n},\frac{s-X_{k+i}}{h_n}\Big). \tag{3}$$

where U is a given distribution function and $\{h_n, n\ge 1\}$ is a sequence of positive numbers converging to zero. They found the optimal bandwidth convergence rate of order n^{-1}. In this paper using $\hat{F}_k$ in (3), we define the Kernel estimator of $\varphi_k(r,s)$ and $\Gamma(r,s)$ as follow:

$$\hat{\varphi}_k(r,s)=\hat{F}_k(r,s)-\hat{F}(r)\hat{F}(s),\;\;\hat{\Gamma}(r,s)=\hat{F}_k(r,s)-\hat{F}(r)\hat{F}(s)+\sum_{k=1}^{n}(\hat{\varphi}_k(r,s)+\hat{\varphi}_k(s,r)) \tag{4}$$

and drive a uniform convergence rate of $h^2_{n-k}n^{-\gamma}$ for the above estimators, where

$$\hat{F}(r)=\frac{1}{n}\sum_{i=1}^{n}U\Big(\frac{r-X_i}{h_n}\Big)$$

and $0<\gamma<1/2$. For this convergence rate, we need no condition on the covariance structure of the variables. The above rate is flexible because of including the term h_n which can be optionally chosen. This flexibility makes us able to have a rate that tends to zero (as is necessary for a convergence rate) and on the other hand, can be a better rate than what was found by Jabbari and Azarnoosh [5]. It is noted that the proofs are almost similar to those of Jabbari and Azarnoosh [5]

In all sections of this paper suppose that C is a positive constant not depend on n. Also, we use the following general assumption throughout the article:
(A). $\{X_n,\ n\ge 1\}$ is a NA and strictly stationary sequence of random variables having density function bounded by M_0 and

$$\Big|U\Big(\frac{r-X_i}{h_n},\frac{s-X_{i+k}}{h_n}\Big)-EU\Big(\frac{r-X_i}{h_n},\frac{s-X_{i+k}}{h_n}\Big)\Big|\le Ch_n^2, \qquad a.s. \tag{5}$$

for any $1\le i\le n$ and fixed $r,s\in \mathbb{R}$.
In Section 2, we will present some auxiliary results needed to establish the above mentioned convergence rates. The moment inequality used for the proofs is presented in this section. The strong uniform convergence rates are proved in Sections 3 and 4.

2 Auxiliary results

In this section we used the following moment inequality for NA random variables and proved an inmportant inequality that are needed for proving our convergence rates.

Lemma 2.1 *(Su et al. [12] and Matula [8]) Let $(X_1, X_2, ..., X_n)$ be an NA random vector with $EX_j = 0$ and $E|X_j|^p < \infty$ for some $p \geq 2$ and all $j = 1, ..., n$. Then, there exists a constant $C = C(p) > 0$, such that*

$$E|\sum_{j=1}^{n} X_j|^p \leq C[\sum_{j=1}^{n} E|X_j|^p + (\sum_{j=1}^{n} EX_j^2)^{p/2}]. \qquad \square \tag{6}$$

Lemma 2.2 *Let $k \in I\!N_0$ be fixed and ε_n a sequence of positive numbers. Suppose (A) is satisfied. Then, there exists a constant C such that, for $r, s \in I\!R$ and $p \geq 2$,*

$$P_r(|\hat{F}_k(r,s) - F_k(r,s)| > \varepsilon_n) \leq \frac{Ch_{n-k}^{2p}}{\varepsilon_n^p (n-k)^p}. \tag{7}$$

Proof. For each $n \in I\!N$, $1 \leq i \leq n$ and fixed $r, s \in I\!R$ define

$$Z_{k,i} = U(\frac{r - X_i}{h_n}, \frac{s - X_{i+k}}{h_n}) - F_k(r,s),$$

and also

$$W_{k,i} = Z_{k,i} - E(Z_{k,i}).$$

So, we have

$$\begin{aligned} \hat{F}_k(r,s) - E(\hat{F}_k(r,s)) &= \frac{1}{n-k}\sum_{i=1}^{n-k} Z_{k,i} + F_k(r,s) - E(\hat{F}_k(r,s)) \\ &= \frac{1}{n-k}\sum_{i=1}^{n-k} W_{k,i} + \frac{1}{n-k}\sum_{i=1}^{n-k} E(Z_{k,i}) + F_k(r,s) - E(\hat{F}_k(r,s)). \end{aligned}$$

Regarding $\frac{1}{n-k}\sum_{i=1}^{n-k} E(Z_{k,i}) = E(\hat{F}_k(r,s)) - E(F_k(r,s))$, we will have

$$\hat{F}_k(r,s) - E(\hat{F}_k(r,s)) = \frac{1}{n-k}\sum_{i=1}^{n-k} W_{k,i}. \tag{8}$$

Since (A) is hold, it is clear that $W_{k,n}$ are decreasing functions of the variables X_n. So according to the properties of NA random variables (see for more information Joag-Dev and Proschan [7]), $\{W_{k,n},\ n \geq 1\}$ is NA and strictly stationary. Also, $|W_{k,n}| \leq Ch_n^2$ and $E(W_{k,n}) = 0$ then, $E|W_{k,n}|^p < \infty$, for each $n \geq 1$ and $p \geq 2$. We can apply Lemma 2.1 to the sequence $\{W_{k,n},\ n \geq 1\}$. So for all $n \geq 1$, we obtain

$$\begin{aligned} E|\sum_{i=1}^{n} W_{k,i}|^p &\leq C[\sum_{i=1}^{n} E|W_{k,i}|^p + (\sum_{i=1}^{n} EW_{k,i}^2)^{p/2}] \\ &\leq Ch_n^{2p}. \end{aligned} \tag{9}$$

Now for fixed $r, s \in I\!R$, we can write

$$\begin{aligned} P(|\hat{F}_k(r,s) - F_k(r,s)| > \varepsilon_n) &\leq P(|\hat{F}_k(r,s) - E(\hat{F}_k(r,s))| > \frac{\varepsilon_n}{2}) \\ &\quad + P(|F_k(r,s) - E(\hat{F}_k(r,s))| > \frac{\varepsilon_n}{2}). \end{aligned}$$

Since $0 < F_k(r,s), \hat{F}_k(r,s) < 1$ for fixed $\in I\!N_0$ and $r,s \in I\!R$, we conclude $P(|F_k(r,s) - E(\hat{F}_k(r,s))| > \frac{\varepsilon_n}{2}) \to 0$ as $n \to +\infty$. Now regarding this, using the Markov inequality and from (8) and (9) we find, for all $n > k$,

$$\begin{aligned} P(|\hat{F}_k(r,s) - F_k(r,s)| > \varepsilon_n) &\leq \frac{2^p}{\varepsilon_n^p (n-k)^p} E|\sum_{i=1}^{n-k} W_{k,n}|^p \\ &\leq \frac{C h_{n-k}^{2p}}{\varepsilon_n^p (n-k)^p}. \qquad \square \end{aligned} \tag{10}$$

To prove the next results, we should define the following notations as introduced in Jabbari and Azarnoosh [5]. Let t_n be a sequence of positive integers such that $t_n \to +\infty$. For each $n \in I\!N$ and each $i = 1, ..., t_n$, put $x_{n,i} = Q(i/t_n)$, where Q is the quantile function of F. Then for $n, k \in I\!N$, define

$$D_{n,k} = \sup_{r,s \in I\!R} |\hat{F}_k(r,s) - F_k(r,s)|,$$

and

$$D_{n,k}^* = \max_{i,j=1,\ldots,t_n} |\hat{F}_k(x_{n,i}, x_{n,j}) - F_k(x_{n,i}, x_{n,j})|.$$

Furthermore, we will need the following result as in Theorem 2 of Henriques and Oliveira [3] and Lemma 2.3 of Jabbari and Azarnoosh ([5]).

Lemma 2.3 *If the sequence $\{X_n,\ n \geq 1\}$ satisfies (A), then, for each $n \in I\!N$ and each $k \in I\!N_0$,*

$$D_{n,k} \leq D_{n,k}^* + \frac{2}{t_n} \qquad a.s.\ . \qquad \square \tag{11}$$

Lemma 2.4 *Let ε_n and t_n be two sequences of positive numbers such that $t_n \to +\infty$ and $\varepsilon_n t_n \to +\infty$, $p \geq 2$ and $k \in I\!N_0$ be fixed. Suppose (A) holds. Then, for any large enough n,*

$$P(\sup_{r,s \in I\!R} |\hat{F}_k(r,s) - F_k(r,s)| > \varepsilon_n) \leq \frac{C t_n^2}{\varepsilon_n^p (n-k)^p} h_{n-k}^{2p}. \tag{12}$$

Proof. Following the same steps in Lemma 2.4 of Jabbari and Azarnoosh [5] and applying Lemma 2.2 and Lemma 2.3 the result is concluded. $\square$

3 Uniform strong convergence rates of $\hat{F}_k$

In this section, we summarize the previous results to get uniform strong convergence rates of $\hat{F}_k$.

Lemma 3.1 *Let $k \in I\!N_0$ be fixed and suppose (A) holds. Then under the conditions of Lemma 2.4 and for every $0 < \delta < p-1$, we have*

$$\sup_{r,s \in I\!R} |\hat{F}_k(r,s) - F_k(r,s)| = O(h_{n-k}^{\frac{2p-2}{p+2}} n^{-\frac{p-1-\delta}{p+2}}) \qquad a.s.\ . \tag{13}$$

Proof. Put $t_n = \frac{1}{\varepsilon_n h_{n-k}}$ and let $0 < \delta < p-1$. Since $t_n \to \infty$ and $t_n \varepsilon_n \to \infty$ when $n \to \infty$, Then from Lemma 2.4 for $\varepsilon_n = h_{n-k}^{\frac{2p-2}{p+2}} n^{-\frac{p-1-\delta}{p+2}}$ and n large enough,

$$P(\sup_{r,s \in I\!R} |\hat{F}_k(r,s) - F_k(r,s)| > \varepsilon_n) \leq \frac{C}{\varepsilon_n^{p+2} h_{n-k}^{2-2p} (n-k)^p} \leq C n^{-(1+\delta)}. \tag{14}$$

The proof is complete using the Borel-Cantelli Lemma. Because for all $\delta > 0$, the sequence on the right-hand side above being summable. $\square$

If $p \to \infty$, $\varepsilon_n \to h_{n-k}^2 n^{-1}$. Since $h_{n-k}^2 \to 0$ when $n \to \infty$, the convergence rate of Lemma 3.1 remains reasonable for a large p. In the next theorem, we summarize the results of of this section.

Theorem 3.1 *Under the assumptions of Lemma 3.1 and for every $0 < \gamma < 1/2$, we have*

$$\sup_{r,s \in I\!R} |\hat{F}_k(r,s) - F_k(r,s)| = O(h_{n-k}^2 n^{-\gamma}) \qquad a.s. \ . \tag{15}$$

Proof. Using Lemma 3.1 and along the lines of Theorem 3.1 in Jabbari and Azarnoosh [5], we get the desired result. □

If $k = 0$ and $s = r$ the estimator $\hat{F}_k(r,s)$ becomes to the one-dimentional Kernel distribution function $\hat{F}(r)$. So, the results of Theorem 3.1 hold true for $\hat{F}$ and for every n large enough, we can write

$$\sup_{r \in I\!R} |\hat{F}(r) - F(r)| = O(h_n^2 n^{-\gamma}). \tag{16}$$

Remark 3.1 *From the results of Theorem 3.1, we understand that the convergence rate $h_{n-k}^2 n^{-\gamma}$ for every $0 < \gamma < 1/2$ and h_n is very faster than those obtained later by Jabbari and Azarnoosh [5] (i.e. $n^{-\gamma}$). So, the Kernel estimator of two-dimensional and one-dimentional distribution function F_k and F is better than empirical one.*

Now, we can obtain the convergence rate of the Kernel estimator of φ_k.

Theorem 3.2 *Under the assumptions of Theorem 3.1 and for every $0 < \gamma < 1/2$, we have*

$$\sup_{r,s \in I\!R} |\hat{\varphi}_k(r,s) - \varphi_k(r,s)| = O(h_{n-k}^2 n^{-\gamma}) \qquad a.s. \ . \tag{17}$$

Proof. The proof is similar to that of Theorem 3.2 in Jabbari and Azarnoosh [5]. So, we omit it.

4 Uniform strong convergence rates of $\hat{\Gamma}$

As Jabbari and Azarnoosh [5], we will introduce uniform strong convergence rates for the Kernel estimators of the sum $\sum_{k=1}^{\infty} \varphi_k(r,s)$ and the covariance function $\Gamma(r,s)$.
Regarding that the covariance structure of a sequence of NA random variables highly determines its approximate independence (see Newman [10]), it is common to have assumptions on the covariance structure of the random variables. For this, we use the same definition of Jabbari and Azarnoosh [5] as

$$u(n) = \sum_{j=n+1}^{\infty} |Cov^{1/3}(X_1, X_j)|. \tag{18}$$

In the following lemma, we prove the uniform strong convergence rate for the sum $\sum_{k=1}^{\infty} \hat{\varphi}_k(r,s)$ which is sufficeint to obtain the desired result for two-dimensional distribution function of (X_1, X_{k+1}).

Lemma 4.1 *Let (A) hold and suppose that $a_n = n^{\frac{p-1-\delta}{p^2+3p}}$ for some $p \geq 2$ and for each $0 < \delta < p - 1$. If*

$$u(a_n) \leq C h_{n-k}^{\theta + \frac{1-p^2}{p+2}} a_n^{-\theta} \tag{19}$$

for all $n \geq 1$, we have

$$\sup_{r,s\in I\!R} |\sum_{k=1}^{a_n} \hat{\varphi}_k(r,s) - \sum_{k=1}^{\infty} \varphi_k(r,s)| = O(h_{n-k}^{\frac{2p-2}{p+2}} n^{-\frac{(p-1)(p-1-\delta)}{p(p+2)}}) \qquad a.s. \, . \tag{20}$$

Proof. The idea is essentially same as the proof of Lemma 4.1 of Jabbari and Azarnoosh [5]. So, we repeat their proof using our required notations and definitions.
Take $\varepsilon_n = h_{n-k}^{\frac{2p-2}{p+2}} n^{-\frac{(p-1)(p-1-\delta)}{p(p+2)}}$ for each $0 < \delta < p-1$ and $t_n = \frac{a_n}{\varepsilon h_{n-k}}$. Now, we can write

$$P(\sup_{r,s\in I\!R} |\sum_{k=1}^{a_n} (\hat{F}_k(r,s) - F_k(r,s))| > \varepsilon_n) \leq \sum_{k=1}^{a_n} P(\sup_{r,s\in I\!R} |\hat{F}_k(r,s) - F_k(r,s)| > \frac{\varepsilon_n}{a_n}). \tag{21}$$

Since $0 < \delta < p-1$, $\frac{(p-1)(p-1-\delta)}{p(p+2)} > 0$ and $0 < \frac{p-1-\delta}{p^2+3p} < 1$, it is easy to see $\varepsilon_n \to 0$, $a_n \to +\infty$, $t_n \to +\infty$, $\frac{\varepsilon_n}{a_n} t_n \to +\infty$ and $\frac{a_n}{n} \to 0$ as $n \to +\infty$.
Using $\frac{\varepsilon_n}{a_n}$ in place of ε_n in Lemma 2.4, we obtain for all n large enough,

$$\begin{aligned} P(\sup_{r,s\in I\!R} |\sum_{k=1}^{a_n} (\hat{F}_k(r,s) - F_k(r,s))| > \varepsilon_n) &\leq \sum_{k=1}^{a_n} \frac{Ct_n^2 a_n^p}{\varepsilon_n^p (n-k)^p} h_{n-k}^{2p} \\ &\leq \frac{Ct_n^2 a_n^{p+1}}{\varepsilon_n^p (n-a_n)^p} h_{n-k}^{2p} \\ &= \frac{Ca_n^{p+3}}{\varepsilon_n^{p+2} (n-a_n)^p} h_{n-k}^{2p-2}. \end{aligned} \tag{22}$$

By elementary calculations, we may write $\varepsilon_n = h_{n-k}^{\frac{2p-2}{p+2}} a_n^{\frac{p+3}{p+2}} n^{-\frac{p-1-\delta}{p+2}}$. Inserting this on the right-hand side of (22) leads to summable upper bound. So, we have by Borel-Cantelli Lemma As $\frac{a_n}{n} \to 0$, we have

$$\sup_{r,s\in I\!R} |\sum_{k=1}^{a_n} (\hat{F}_k(r,s) - F_k(r,s))| = O(h_{n-k}^{\frac{2p-2}{p+2}} n^{-\frac{(p-1)(p-1-\delta)}{p(p+2)}}) \qquad a.s. \, . \tag{23}$$

Now, as Jabbari and Azarnoosh [5], we can write

$$\begin{aligned} \sup_{r,s\in I\!R} |\sum_{k=1}^{a_n} \hat{\varphi}_k(r,s) - \sum_{k=1}^{\infty} \varphi_k(r,s)| &\leq \sup_{r,s\in I\!R} |\sum_{k=1}^{a_n} (\hat{F}_k(r,s) - F_k(r,s))| \\ &+ 2a_n \sup_{s\in I\!R} |F(s) - \hat{F}(s)| \\ &+ \sup_{r,s\in I\!R} |\sum_{k=a_n+1}^{\infty} \varphi_k(r,s)|. \end{aligned} \tag{24}$$

For the first term on the right-hand side of (24), we use (23). Since $\frac{p+3}{p+2} > 1$ by using Lemma 3.1 for the second term, we have

$$\begin{aligned} a_n \sup_{s\in I\!R} |F(s) - \hat{F}(s)| &= O(a_n h_{n-k}^{\frac{2p-2}{p+2}} n^{-\frac{p-1-\delta}{p+2}}) \\ &= O(h_{n-k}^{\frac{2p-2}{p+2}} n^{-\frac{(p-1)(p-1-\delta)}{p(p+2)}}) \qquad a.s. \, . \end{aligned} \tag{25}$$

For the third term on the right-hand side of (24), we use Corollary of Theorem 1 in Sadikova [11] and relation (21) in Newman [9] as those applied in Jabbari and Azarnoosh [5]. So by

(19) for $\theta = \frac{(p-1)(p+3)}{p+2} > 0$ and $a_n = n^{\frac{p-1-\delta}{p^2+3p}}$, we obtain

$$\begin{aligned}
\sup_{r,s\in\mathbb{R}} |\sum_{k=a_n+1}^{\infty} \varphi_k(r,s)| &\leq C \sum_{k=a_n+1}^{\infty} |Cov^{1/3}(X_1, X_{k+1}) \\
&= Cu(a_n) \leq Ch_{n-k}^{\frac{(p-1)(p+3)}{p+2} - \frac{1-p^2}{p+2}} a_n^{-\frac{(p-1)(p+3)}{p+2}} \\
&= Ch_{n-k}^{\frac{2p-2}{p+2}} n^{-\frac{(p-1)(p-1-\delta)}{p(p+2)}}. \qquad (26)
\end{aligned}$$

Hence the proof is completed. □

We now summarize the above result in the following theorem.

Theorem 4.1 *Under the assumption Lemma 4.1 and condition (19) for all* $n \geq 1$, $\theta > 0$ *and* $0 < \gamma < 1/2$, *we have*

$$\sup_{r,s\in\mathbb{R}} |\sum_{k=1}^{a_n} \hat{\varphi}_k(r,s) - \sum_{k=1}^{\infty} \varphi_k(r,s)| = O(h_{n-k}^2 n^{-\gamma}) \qquad a.s. . \qquad (27)$$

Proof. As in proof of Theorem 4.1 of Jabbari and Azarnoosh [5], we apply the lines of proof of Theorem 3.1 and use Lemma 4.1 instead of Lemma 3.1. So, for $\delta > 0$ and $p \geq 2$ we have $\frac{(p-1)(p-1-\delta)}{p(p+2)} > \gamma$ and then the proof is concluded. □

Now, applying the lines of proof of Theorem 4.2 of Jabbari and Azarnoosh [5] and using Theorems 3.1 and 4.1, we can state the following theorem which summarizes the results for $\hat{\Gamma}$.

Theorem 4.2 *Suppose* (A) *holds. Under condition (19) for all* $n \geq 1$, $\theta > 0$, $p \geq 2$ *and* $0 < \gamma < 1/2$, *we have*

$$\sup_{r,s\in\mathbb{R}} |\hat{\Gamma}(r,s) - \Gamma(r,s)| = O(h_{n-k}^2 n^{-\gamma}) \qquad a.s. . \quad \square \qquad (28)$$

Remark 4.1 *As stated in Remark 3.1, our convergence rate* $h_{n-k}^2 n^{-\gamma}$ *for every* $0 < \gamma < 1/2$ *and* h_n *in Theorem 4.2 is very faster than those obtained later by Jabbari and Azarnoosh [5] (i.e.* $n^{-\gamma}$*). So, the Kernel estimator of* $\hat{\Gamma}$ *is better than empirical one.*

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