



ADAPTIVE NUMERICAL METHOD FOR MATHEMATICAL MODEL OF AVASCULAR TUMOR GROWTH

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ABSTRACT. Among all mathematical models for avascular tumor growth, the present model is one of the newest models released by Sherratt and Chaplain in 2001. This model is based on cell densities of proliferating, quiescent and necrotic cells in a one-dimensional domain in space and is formulated as a system of partial differential equations.

In this paper, adaptive moving mesh method is described briefly which in this method, to raise accuracy and efficiency, instead of using a uniform grid by a monitor function nodes are concentrated in areas where greater accuracy is required. The rest of the paper the adaptive moving mesh method is used for the numerical solution of the above avascular tumor growth model.

1. INTRODUCTION

When a normal cell is turned to cancer cell the rate of proliferation increases abnormally and causes a clump of tumor cells with fast growing. The growth of a tumor consists of three distinct steps; avascular(tumor without blood vessels), vascular(angiogenesis) and metastatic[1]. For each stage, there are many mathematical models. Two well-known models for avascular tumor growth are Ward- King model[2] and Sherrat- Chaplain [3].

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Improvement of numerical methods for approximate solution is very important to analyze the models. Our chosen model in this paper is the Sherratt-Chaplain[3], [4], model for avascular tumor growth and we use a dynamical moving mesh method for approximating the solution of this model[5]. in addition, we examine the several monitor functions to improve the numerical solutions.

2. THE MODEL

If the cell densities for proliferating, quiescent and necrotic cells are denoted by $p(x, t)$, $q(x, t)$ and $n(x, t)$, respectively then Sherratt-Chaplain model can be written by the following system of partial differential equations

$$\begin{aligned} \frac{dp}{dt} &= \frac{\partial}{\partial x} \left(\frac{p}{p+q} \frac{\partial(p+q)}{\partial x} \right) + g(c)p(1-p-q-n) - f(c)p, \\ \frac{dq}{dt} &= \frac{\partial}{\partial x} \left(\frac{q}{p+q} \frac{\partial(p+q)}{\partial x} \right) + f(c)p - h(c)q, \\ \frac{dn}{dt} &= h(c)q, \end{aligned}$$

where $c(x, t)$ is the local nutrient concentration and it is assumed that nutrients are distributed across the surface of the tumor into the inward via the intracellular space[1], $g(c)$ is the mitosis rate function of the proliferating cells and is an increasing function, proliferating cells become quiescent at rate $f(c)$ and the rate of turning quiescent cells to necrosis is denoted by $h(c)$. Both of f and h are decreasing functions and will be zero as c tend to $+\infty$, in addition $f(c) > h(c)$.

In Sherratt-Chaplain model $c = \frac{c_0\gamma}{\gamma+p}(1 - \alpha(p+q+n))$, where α and γ are dimensionless parameters.

For the initial conditions, we have $p(x, 0) = e^{-0.1x}$, $q(x, 0) = 0$, $n(x, 0) = 0$ and $c_0 = 1$. The boundary conditions are $\frac{\partial p}{\partial x} = 0$, $\frac{\partial q}{\partial x} = 0$ and $\frac{\partial c}{\partial x} = 0$, at $x = 0$ and as $x \rightarrow \infty$ but it is not feasible to tend $x \rightarrow \infty$, furthermore, $x = 210$ is chosen as sufficiently large value.

3. MOVING MESH METHOD AND NUMERICAL SOLUTION

Let x and ξ denote the physical and computational coordinates, respectively and suppose a one-to-one coordinate transformation between these coordinate as follows:

$$\begin{aligned} \Omega_c = [0, 1] &\rightarrow \Omega = [a, b], \\ x = x(\xi, t), \quad \xi &\in [0, 1], \\ x(0, t) = a, \quad x(1, t) &= b. \end{aligned}$$

Suppose that a uniform mesh on computational domain is given by $\xi_i = \frac{i}{n}$, $i = 0, 1, \dots, n$, where n is a positive integer and corresponding mesh in x is denoted by $\{a = x_0, x_1, \dots, x_n = b\}$ where $x_i = x(\xi_i, t)$, $i = 0, 1, \dots, n$.

The equidistribution principle (EP) is the foundation of moving mesh method

and the mesh in x , should be satisfied the EP for all values of time t . For a chosen monitor function $M(x, t)$, EP can be given in integral form as

$$\int_a^{x(\xi,t)} M(\tilde{x}, t) d\tilde{x} = \xi\theta(t)$$

where $\theta(t) = \int_a^b M(\tilde{x}, t) d\tilde{x}$. By twice differentiating above equation we can obtain

$$\frac{\partial}{\partial \xi} \left\{ M(x(\xi, t), t) \frac{\partial}{\partial \xi} x(\xi, t) \right\} = 0.$$

If a mesh satisfy in EP equation at the time $t + \tau$ ($0 < t \ll 1$) instead of t , according to the above we have

$$\frac{\partial}{\partial \xi} \left\{ M(x(\xi, t + \tau), t + \tau) \frac{\partial}{\partial \xi} x(\xi, t + \tau) \right\} = 0,$$

the parameter τ is called a relaxation time.

Using the expansions of $M(x(\xi, t + \tau), t + \tau)$ and $\frac{\partial}{\partial \xi} x(\xi, t + \tau)$ and dropping higher order term, various MMPDEs is obtained[5]. In this work, we use MMPDE4 for moving mesh method:

$$\frac{\partial}{\partial \xi} \left(M \frac{\partial \dot{x}}{\partial \xi} \right) = -\frac{1}{\tau} \frac{\partial}{\partial \xi} \left(M \frac{\partial x}{\partial \xi} \right).$$

The monitor function M is used to guide the mesh redistribution. The arc-length mesh monitor function $M = (1 + |u_x|^2)^{1/2}$ and the curvature mesh monitor function $M = (1 + |u_{xx}|^2)^{1/4}$ are two well-known monitor functions. We have used several monitor functions as follows:

$$M = \left(1 + \frac{\beta_1 |p_x|^2 + \beta_2 |q_x|^2 + \beta_3 |n_x|^2}{\beta_1 + \beta_2 + \beta_3} \right)^{1/2}$$

$$M = \frac{1}{3} (1 + |p| + |q| + |n|)$$

$$M = \frac{\beta_1 M_1 + \beta_2 M_2 + \beta_3 M_3}{\beta_1 + \beta_2 + \beta_3}$$

where M_1 , M_2 and M_3 are three suitable monitor functions based on p , q

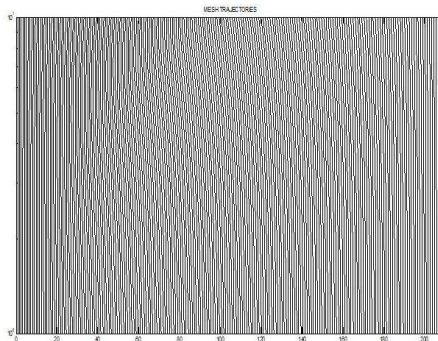


figure1: Mesh trajectories

and n , respectively.

We obtain numerical solution for p , q and n , for this model of avascular

tumor growth. The initial values and functions are defined $f(c) = (1 - \tanh(4c - 2))/2$, $h(c) = f(c)/2$ and $g(c) = \beta e^{\beta c}$ in addition $\alpha = 0.8$, $\beta = 0.5$ and $\gamma = 10$.

Mesh trajectories by using $M = \frac{\beta_1 M_1 + \beta_2 M_2 + \beta_3 M_3}{\beta_1 + \beta_2 + \beta_3}$, where M_1, M_2 and M_3 are the arc-length monitor functions of p, q and n , respectively with $\beta_1 = \beta_2 = \beta_3 = 1/3$, is presented in Figure 1.

Cell densities of proliferating, quiescent and necrotic cells plotted as a function of space in Figure 2.

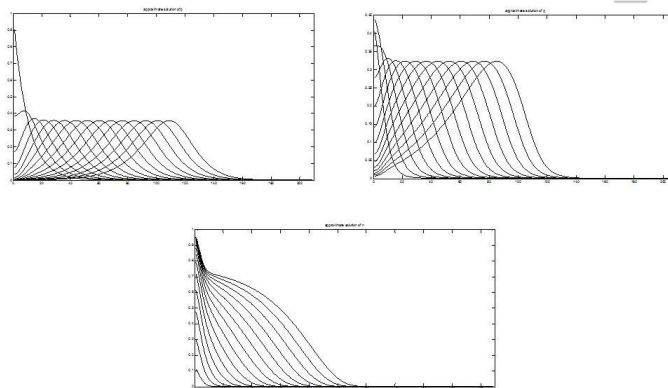


figure 2: Numerical solution for p, q and n , at times $t=0, 2, \dots, 14$.

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