A new method to measure the distance between interval-valued fuzzy numbers

Mohammad Reza Rabiee¹,*, Naser Reza Arghami¹,♣ and Bahram Sadeghpour¹,2,♠

¹Department of Statistics, Ferdowsi University of Mashhad, Mashhad, 91775 Iran,
²Department of Statistics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, 47419 Iran,
* Corresponding author. Email: rabie1354@yahoo.com
♣ Email: arghami_nr@yahoo.com
♠ Email: sadeghpour@umz.ac.ir

Abstract
This paper gives a new kind of distance between interval-valued fuzzy sets defined on real line R, denoted by \(D'_{ja}\). The applicability of the proposed method is investigated by a numerical data set and this distance is compared with two other distances by an example.

Keywords: Interval-valued fuzzy number, distance, Hausdorff metric.

1 Introduction
Since fuzzy set theory was introduced by Zadeh [18], many new approaches and theories treating imprecision and uncertainty have been proposed. Specially, the intuitionistic fuzzy set theory pioneered by Atanassov [1], and the interval-valued fuzzy set theory suggested by Grozafczany [6] and Turksen [11] are two well-known generalizations of the fuzzy set theory. In fact, it is pointed out that there is a strong connection between Atanassov’s intuitionistic fuzzy sets and the interval-valued fuzzy sets [3, 4, 12]. Over the last decades, the theory of interval-valued fuzzy set has been developed in different directions. For the proposes of this article, we briefly review some works on this topic.


The structure of this paper is as follows. Section 2 shows the preliminaries, it includes notations and basic concepts which will be used in the following section. Section 3 shows the new kind of distance and discusses some Proposition and Theorem. By using a numerical example, we compare this distance with two other distances in Section 4. A brief conclusion is given in the last section.

2 Preliminaries
In this section, we review some elementary definitions and a well-known result of the interval-valued fuzzy sets and interval-valued fuzzy numbers, biased on Wang and Li [13], Hong and Lee [8], and Zhixn and Hongmei [19]. Let \(I = [0,1]\) and \([I] = \{[a,b] | a \leq b, a,b \in I\}\). For any \(a \in I\), define \(\bar{a} = [a,a]\).
Definition 2.1 If \( a_t \in I, t \in T \), then we define \( \vee_{t \in T} a_t = \sup \{a_t : t \in T\} \) and \( \wedge_{t \in T} a_t = \inf \{a_t : t \in T\} \). We also define for \( [a_t^1, b_t^1] \in [I], t \in T \),

\[
\begin{aligned}
\vee_{t \in T} [a_t^1, b_t^1] &= \vee_{t \in T} a_t \vee_{t \in T} b_t \quad \text{and} \quad \wedge_{t \in T} [a_t^1, b_t^1] = \wedge_{t \in T} a_t \wedge_{t \in T} b_t, \\
[a_1, b_1] &= [a_2, b_2] \text{ iff } a_1 = a_2, b_1 = b_2, \\
[a_1, b_1] &\leq [a_2, b_2] \text{ iff } a_1 \leq a_2, b_1 \leq b_2, \\
[a_1, b_1] &< [a_2, b_2] \text{ but } [a_1, b_1] \neq [a_2, b_2].
\end{aligned}
\]

Definition 2.2 Let \( X \) be an ordinary nonempty set. Then:

- The mapping \( A : X \rightarrow [I] \) is called an interval-valued fuzzy set (IVFS) on \( X \). The set of all IVFS on \( X \) is denoted by \( \text{IF}(X) \).
- For \( A \in \text{IF}(X) \), let \( (x, X) = (A^-, x, \lambda^+, x) \), for all \( x \in X \). Then two fuzzy sets \( A^- \) and \( A^+ \) are called lower fuzzy set and upper fuzzy set of \( A \), respectively.
- The value of \( \Pi A(x) = A^+(x) - A^-(x) \) is called the degree of non-determinancy of the element \( x \in X \) to the IVFS \( A \).

Definition 2.3 Let \( A \in \text{IF}(X) \) and \( [\lambda_1^1, \lambda_2^1] \subseteq [I] \). We call \( A \left[ \lambda_1^1, \lambda_2^1 \right] = \{ x \in X : A^-(x) \geq \lambda_1^1, A^+(x) \geq \lambda_2^1 \} \) and \( A \left( \lambda_1^1, \lambda_2^1 \right) = \{ x \in X : A^-(x) > \lambda_1^1, A^+(x) > \lambda_2^1 \} \) the \( [\lambda_1^1, \lambda_2^1] \)-level set of \( A \) and the \( (\lambda_1^1, \lambda_2^1) \)-level set of \( A \), respectively.

Definition 2.4 Let \( A \in \text{IF}(R) \), where \( R \) is the real line. Assume the following conditions are satisfied:

- \( A \) is normal, i.e., there exists \( x_0 \in R \), such that \( A(x_0) = \bar{I} \),
- For arbitrary \( [\lambda_1^1, \lambda_2^1] \subseteq [I]^+ = [I] - \{0\}, A \left[ \lambda_1^1, \lambda_2^1 \right] \) is a closed bounded interval.

Then we call \( A \) an interval-valued fuzzy number (IVFN). We denote the set of all IVFNs by \( \text{IF}^*(R) \).

Definition 2.5 Let \( A, B \in \text{IF}(R) \) and \( \bullet \in \{+, -, \vee, \wedge\} \). We define the extended operations by \( (A \bullet B)(z) = \vee_{x \in X, y \in Y} \{A(x) \bullet B(y)\} \). For each \( [\lambda_1^1, \lambda_2^1] \subseteq [I]^+ \), we write \( A \left[ \lambda_1^1, \lambda_2^1 \right] \bullet B \left[ \lambda_1^1, \lambda_2^1 \right] = \{ x \bullet y \in A \left[ \lambda_1^1, \lambda_2^1 \right], y \in B \left[ \lambda_1^1, \lambda_2^1 \right] \} \).

Definition 2.6 A triangular IVFN is represented as \( A = [A^-, A^+] = [(a_1^-, a_1^+), (a_2^-, a_2^+)] \), where \( A^- \) and \( A^+ \) denote the lower and upper triangular fuzzy numbers of \( A \), \( A^- \subset A^+ \). Also, \( A \) is denoted by \( A = [A^-, A^+] = [(a_1^-, a_1^+), (a_2^-, a_2^+)] \).
3 A new distance between interval-valued fuzzy numbers

Based on definitions given in [9] and [16], we propose the following definition of distance between IVFNs.

Definition 3.1 Let \( A, B \in IF^\ast (R) \). The \( D^\ast_{p,f} \) distance between \( A \) and \( B \) is defined as

\[
D^\ast_{p,f}(A,B) = \max \{D^\ast_{p,f}(A^-,B^-), D^\ast_{p,f}(A^+,B^+)\}
\]

where

\[
D^\ast_{p,f}(A^\bullet_\lambda,B^\bullet_\lambda) = \left( \int_0^1 f(\lambda)d^p(A^\bullet_\lambda,B^\bullet_\lambda)d\lambda \right)^{1/p},
\]

where \( \bullet \in \{-,+,\} \) and

\[
d^p(A^\bullet_\lambda,B^\bullet_\lambda) = |a_1(\lambda) - b_1(\lambda)|^p + |a_2(\lambda) - b_2(\lambda)|^p,
\]

\[
A^\bullet_\lambda = [a_1(\lambda), a_2(\lambda)], \quad B^\bullet_\lambda = [b_1(\lambda), b_2(\lambda)]
\]

and \( f(\lambda) \) is an increasing function on \([0,1]\) with \( f(0) = 0 \) and \( \int_0^1 f(\lambda)d\lambda = \frac{1}{2} \).

Specially, for \( p = 2 \), we have:

\[
d^2(A^\bullet_\lambda,B^\bullet_\lambda) = (a_1(\lambda) - b_1(\lambda))^2 + (a_2(\lambda) - b_2(\lambda))^2.
\]

Note. [16] Clearly, \( d^p(A_\lambda,B_\lambda) \) is a distance of the \( \lambda \)-level set of fuzzy numbers \( A \) and \( B \). It reflects the degree of closeness between \( A_\lambda \) and \( B_\lambda \). Function \( f(\lambda) \) can be understood as the weight of \( d^\lambda(A_\lambda,B_\lambda) \), and the property of monotone increasingness of \( f(\lambda) \) means that the higher the membership of the level set, the more important it is in determining the distance between \( A \) and \( B \). The conditions \( f(0) = 0 \) and \( \int_0^1 f(\lambda)d\lambda = \frac{1}{2} \) ensure that the distance defined here is the extension of ordinary distance in \( R \) defined by an absolute value. That is, this distance becomes an ordinary one in \( R \) when the fuzzy numbers become decadent to crisp. In actual applications, function \( f(\lambda) \) can be chosen according to the actual situation. In the following, we put \( f(\lambda) = \lambda \) and we denote \( D_{p,\lambda} \) and \( D'_{p,\lambda} \) by \( D_\lambda \) and \( D'_\lambda \), respectively.

In the following, we prove that \( D^\ast_{p,f} \) is a metric on the space of IVFNs. At first, we need to express the following lemma.
Lemma 3.1 If \( a, b, c \) and \( d \) are real numbers, then
\[
\max\{a + b, c + d\} \leq \max\{a, c\} + \max\{b, d\}
\]
(9)

Proof: We have 24 possible permutations of \( a, b, c \) and \( d \). We prove (9) for two cases.

1) Let \( a \leq b \leq c \leq d \). Then \( a + b \leq c + d \), and therefore \( \max\{a + b, c + d\} = c + d \), \( \max\{a, c\} = c \) and \( \max\{b, d\} = d \).

Hence, relation (9) is satisfied.

2) Let \( b \leq c \leq d \leq a \). Then \( \max\{a, c\} = a \) and \( \max\{b, d\} = d \). If \( \max\{a + b, c + d\} = c + d \), then
\[
c \leq a \Rightarrow c + d \leq a + d \quad \text{i.e.} \quad \max\{a + b, c + d\} \leq \max\{a, c\} + \max\{b, d\}
\]
and so, relation (9) is held. And if \( \max\{a + b, c + d\} = a + b \), then
\[
b \leq d \Rightarrow a + b \leq a + d \quad \text{i.e.} \quad \max\{a + b, c + d\} \leq \max\{a, c\} + \max\{b, d\}
\]
and hence, relation (9) is satisfied.

Similarly, the remained 22 other cases can be proved. ■

Theorem 3.2 \( D_{p,f}^r \) is a metric on \( IF^r(R) \).

Proof: Suppose that \( A, B, C \in IF^r(R) \).

- \( D_{p,f}^r(A, B) \geq 0 \) is obviously held.
- If \( A = B \), then \( D_{p,f}^r(A, B) = 0 \). Conversely, if \( D_{p,f}^r(A, B) = 0 \), then \( D_{p,f}(A', B') = 0 \). Therefore, \( \forall x \in R, A'(x) = B'(x) \) and \( A'(x) = B'(x) \), and so we conclude \( A = B \).
- Symmetry property i.e. \( D_{p,f}^r(A, B) = D_{p,f}^r(B, A) \) is clearly held.
- Triangular inequality: Since \( D_{p,f}(A', B') \) and \( D_{p,f}(A'', B'') \) are metrics on the space of \( F(R) \) [17, 16], if \( A', B', C', A'', B'', C'' \) are fuzzy numbers, then
\[
D_{p,f}(A', B') \leq D_{p,f}(A', C') + D_{p,f}(C', B'),
\]
\[
D_{p,f}(A'', B'') \leq D_{p,f}(A'', C'') + D_{p,f}(C'', B'').
\]

Therefore, we have
\[
\max\{D_{p,f}(A', B'), D_{p,f}(A'', B'')\} \leq \max\{D_{p,f}(A', C') + D_{p,f}(C', B'), D_{p,f}(A'', C'') + D_{p,f}(C'', B'')\}.
\]
(10)

By using relation (10) and Lemma 3.1, we have
\[
D_{p,f}(A, B) = \max\{D_{p,f}(A', B'), D_{p,f}(A'', B'')\}
\]
\[
\leq \max\{D_{p,f}(A', C') + D_{p,f}(C', B'), D_{p,f}(A'', C'') + D_{p,f}(C'', B'')\}
\]
\[
\leq \max\{D_{p,f}(A', C'), D_{p,f}(A'', C'')\} + \max\{D_{p,f}(C', B'), D_{p,f}(C'', B'')\}
\]
\[
= D_{p,f}(A, C) + D_{p,f}(C, B). \quad \blacksquare
\]

In the following, the \( D_{p,f}^r \) distance will be used in triangular interval-valued fuzzy numbers and with a numerical example, we compare the distance with some other distances.

Proposition 3.3 Let \( A = (a_1, a_2, a_3) \) and \( B = (b_1, b_2, b_3) \) be two triangular fuzzy numbers. Then
\[
D_{p,f}(A, B) = \frac{(a - b)^2}{2} + \frac{1}{12}[(a_2 - b_2)^2 + (a_1 - b_1)^2] + \frac{1}{6}(a - b)[(a_2 - b_2) + (a_3 - b_3)].
\]
(11)
Proof: The level sets of triangular fuzzy numbers A and B can be expressed as

\[ A_\lambda = [a_\lambda + \lambda(a - a), a_\lambda, a_\lambda - \lambda(a - a)] \quad \text{and} \quad B_\lambda = [b_\lambda + \lambda(b - b), b_\lambda, b_\lambda - \lambda(b - b)] \]  \tag{12}

According to Eq. (6), we have

\[ D^*_2(A, B) = \int_0^1 \lambda[(a_\lambda - b_\lambda)]^2 d\lambda + \int_0^1 \lambda[(b_\lambda - a_\lambda)]^2 d\lambda \]

\[ = \frac{(a - b)^2}{2} + \frac{1}{12}[(a_\lambda - b_\lambda)^2 + (a_\lambda - b_\lambda)^2] + \frac{1}{6}(a - b)[(a_\lambda - b_\lambda) + (a_\lambda - b_\lambda)], \]  \tag{13}

and the proof is complete. \[ \blacksquare \]

**Theorem 3.4** Let \( A = ((a_i^*, a_i^*), (a_i^*, a_i^*), (a_i^*, a_i^*)) \) and \( B = ((b_i^*, b_i^*), (b_i^*, b_i^*), (b_i^*, b_i^*)) \) be two triangular IVFNs. Then

\[ D^*_2(A, B) = \frac{(a - b)^2}{2} + \max \left\{ \frac{1}{12}[(a_\lambda - b_\lambda)^2 + (a_\lambda - b_\lambda)^2] + \frac{1}{6}(a - b)[(a_\lambda - b_\lambda) + (a_\lambda - b_\lambda)] \right\}, \]

\[ = \frac{1}{12}[(a_\lambda - b_\lambda)^2 + (a_\lambda - b_\lambda)^2] + \frac{1}{6}[(a_\lambda - b_\lambda)^2 + (a_\lambda - b_\lambda)^2]] \]  \tag{13}

**Proof:** The proof is straightforward in view of Eq. (5). \[ \blacksquare \]

**Definition 3.2** The mean distance between \( A_i \) and \( B_i, i = 1, \ldots, m \) is defined by

\[ MD^*_i = \frac{1}{m} \sum_{j=1}^m D^*_i(A_j, B_j). \]  \tag{14}

### 4 Comparison with two other distances

In the following, we introduce two distances between interval-valued fuzzy numbers based on Hausdorff metric for evaluating the goodness of fit of an IVF regression model. Let \( u = [u_1, u_2] \) and \( v = [v_1, v_2] \) be two closed intervals. The Hausdorff metric between \( u \) and \( v \) is defined by [9]

\[ d_u(u, v) = \max\{|u_1 - v_1|, |u_2 - v_2|\}. \]  \tag{15}

**Definition 4.1** [9] Let \( A, B \in IF^*(R) \). The \( D^*_p \) distance between A and B is defined as

\[ D^*_p(A, B) = \max\{D_p(A, B^*), D_p(A^*, B)\} \]  \tag{16}

where

\[ D_p(A^*, B^*) = \left( \int_0^1 d^*_p(A^*_\lambda, B^*_\lambda) d\lambda \right)^{1/p}. \]  \tag{17}

Since \( A' \) and \( B' \) are fuzzy numbers, so for each \( \lambda \in (0, 1) \), \( A'_\lambda \) and \( B'_\lambda \) are bounded closed intervals, i.e. \( A'_\lambda = [a(\lambda), a(\lambda)], B'_\lambda = [b(\lambda), b(\lambda)] \). Therefore, from Eq. (15), we have

\[ d^*_p(A'_\lambda, B'_\lambda) \]  \tag{18}

where \( \bullet \in \{-, +\}. \)

**Theorem 4.1** [9] \( D^*_p \) is a metric on \( IF^*(R) \).

**Proposition 4.2** Let \( A = ((a_i^*, a_i^*), (a_i^*, a_i^*), (a_i^*, a_i^*)) \) and \( B = ((b_i^*, b_i^*), (b_i^*, b_i^*), (b_i^*, b_i^*)) \) be two triangular IVF numbers. Then, by Eq. (12) and Eq. (18), \( D^*_p(A, B) \) is obtained as
\[ D'_j(A, B) = \max \{ D'_j(A', B''), D'_j(A'', B') \}, \]  

where

\[ D'_j(A', B'') = \int_0^1 \max \{|(1-\lambda)(a'_i - b'') + \lambda(a-b)\|, |(1-\lambda)(a''_i - b'') + \lambda(a-b)\|\} d\lambda, \]

\[ D'_j(A', B') = \int_0^1 \max \{|(1-\lambda)(a'_i - b'') + \lambda(a-b)\|, |(1-\lambda)(a''_i - b'') + \lambda(a-b)\|\} d\lambda. \]

**Definition 4.2** The mean distance between \( A_i \) and \( B_i, i = 1, \ldots, m \) is defined by

\[ MD'_p = \frac{1}{m} \sum_{i=1}^{m} D'_j(A_i, B_i). \]  

**Definition 4.3** [9] Let \( A, B \in IF^+(R) \). The \( D'_+ \) distance between \( A \) and \( B \) is defined as

\[ D'_+(A, B) = \max \{ D'_+(A', B''), D'_+(A'', B') \} \]  

where

\[ D'_+(A', B'') = \sup_{\lambda \in [0,1]} d_{\lambda}(A'_i, B''_i), \]  

for \( \lambda \in \{-,+,\} \) and \( d_{\lambda}(A'_i, B''_i) \) can be obtained by Eq. (18).

**Theorem 4.3** [9] \( D'_+ \) is a metric on \( IF^+(R) \).

**Proposition 4.4** Let \( A = (a'_i, a'_i), a_i(a'_i, a'_i) \) and \( B = (b'_i, b'_i), b_i(b'_i, b'_i) \) be two triangular IVF numbers. Then, by Eq. (12) and Eq. (18), \( D'_+(A, B) \) is obtained as

\[ D'_+(A, B) = \max \{ D'_+(A', B''), D'_+(A'', B') \}, \]  

where

\[ D'_+(A', B'') = \sup_{\lambda \in [0,1]} \max \{|(1-\lambda)(a'_i - b'') + \lambda(a-b)\|, |(1-\lambda)(a''_i - b'') + \lambda(a-b)\|\}, \]

\[ D'_+(A', B') = \sup_{\lambda \in [0,1]} \max \{|(1-\lambda)(a'_i - b'') + \lambda(a-b)\|, |(1-\lambda)(a''_i - b'') + \lambda(a-b)\|\}. \]

**Definition 4.4** The mean distance between \( A_i \) and \( B_i, i = 1, \ldots, m \) is defined by

\[ MD'_+ = \frac{1}{m} \sum_{i=1}^{m} D'_+(A_i, B_i). \]  

**Example 4.1** Table 1 shows the triangular IVF values of \( A \) and \( B \) and their distances between them. The indices \( MD'_{+,p}, MD'_p \) and \( MD'_+ \) between \( A \) and \( B \) values are shown in this table. As we see, the \( MD'_p \) and \( MD'_+ \) are 3.00 and 3.23, respectively, which are very close to 2.78, i.e. the \( MD'_{+,p} \).
5 Conclusion
In this work, we proposed a new distance between two triangular IVFNs. The applicability of the proposed method was investigated by using a numerical data set.

6 References
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