Particle swarm optimization approach to portfolio fuzzy optimization

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ABSTRACT: This paper considers a heuristic method for solving portfolio selection problem based on the possibility theory, therefore, a particle swarm optimization is proposed to solve the corresponding optimization problem. Random fuzzy portfolio optimization are formulated as nonlinear programming. An example, which utilizes the data from Tehran exchange corporation, illustrate the whole idea on fuzzy portfolio optimization problem.

Keywords: Portfolio optimization, possibility theory, particle swarm optimization

INTRODUCTION

The basic assumption for using Markowitz’s mean-variance model in that the situation of assets in the future can be correctly reflected by asset data in the past, that is to say, the means, variances and covariances in future are similar to those in the past. However, since the security market is so complex and the occurrence of new security is so quick., in many cases security returns cannot be accurately predicted by historical data. In this case, fuzzy set theory proposed by zadeh(1965). Since then, researchers began to employ fuzzy set theory to solve many problem s including financial risk investment. Inuiguchi and Tanino (2000) discussed portfolio selection using fuzzy decision theory. Tanaka and Guo (2000) proposed two kinds of portfolio selection models based on fuzzy probabilities and exponential possibility distribution. Other than Markowitz’ model, Mansini and Speranza (1997) proposed their respective portfolio selection models based on Konno et al.(1993) mean absolute deviation model. There were also many research works that minimize the probability of a bad outcome.

Since in reality people cannot always have access to well – defined and precise information, scholars began to use fuzzy models to solve optimization problems. In the field of fuzzy portfolio selection, researchers mainly followed Mrkowitz’s mean-variance idea in different ways. For example, Tanaka and Guo (2000) quantified mean and variance of a portfolio through fuzzy probability and possibility distributions. Parra et al.(2002)introduced vague goals for return risk and liquidity based on expected intervals. Carlson et al. (2001) found the optimum portfolio by use of their own definition of mean and variance of fuzzy numbers. The particle swarm optimization (PSO) approach is a heuristic technique introduced comparatively recently by Kennedy and Eberhart (1995). There are few studies on PSO in the literature, and almost no one of them deals with portfolio optimization (PO). Although the task of yielding minimum risk and maximum return looks simple, there is more than one way of establishing an optimum portfolio. Markowitz (1952) formulated the fundamental theorem of a mean-variance portfolio framework, which explains the trade-off between mean and variance, representing expected returns and risk of a portfolio, respectively. An advanced model was introduced by Konno and Yamazaki (1991) in which a mean –absolute deviation(MAD) model and absolute deviation are utilized as a measure of risk. However, it was insensitive to some extremes, which could be the source of serious error, contrary to the suggestion that the MAD model is suitable under all circumstances (2006). Some researchers have investigated the multi-period PO case, in which investors invest continuously rather than at intervals or only once. Celikyurt and Ozekici (2007) accomplished this, assuming that there are some economic, social, political and other factors affecting the asset returns.

In this paper, we will discuss the portfolio selection problems based on possibilistic theory, and design an effective heuristic algorithm- particle swarm optimization to solve the corresponding optimization problem.
Liu(1987) introduces a random fuzzy variables with definitions that we refer them as follows. Definition. Let $\xi_1, \xi_2, \ldots, \xi_n$ be random fuzzy variables, then, $f(\xi_1, \xi_2, \ldots, \xi_n)$ is a random fuzzy variable on the product possibility space $(\Theta, P(\Theta), Pos)$. 

$$\eta = \sup_{\eta \in R_{1,2,\ldots,n}} \{\min_{1 \leq j \leq n} \mu(\eta_j) | \eta = f(\eta_1, \eta_2, \ldots, \eta_n) \}$$

The previous studies on random and fuzzy portfolio selection problems often have considered mean variance model but safety first model introducing probability or fuzzy chance constraints based on a standard asset allocation problem. Therefore, in this paper, we deal with the following portfolio selection problem, Problem involving the random fuzzy variable based on the standard asset allocation problem to maximize the total future return:

Maximize $\sum_{j=1}^{n} \tilde{r}_j x_j$

subject to $\sum_{j=1}^{n} a_j x_j \leq b$, $0 \leq x_j \leq \tilde{b}_j$, $j = 1, 2, \ldots, n$

$X_j$: Budgeting allocation to the jth financial asset

$\tilde{r}_j$: Future return of the jth financial asset assumed to be a random fuzzy variable

$a_j$: Cost of investing the jth financial asset

$b$: Limited upper value with respect to fund budgeting

$b_j$: Limited upper value of each budgeting to the jth financial asset

$n$: Total number of assets.

In portfolio selection problems, each future return is generally considered as a random variable distributed according to the normal distribution. However, for the lack of efficient information from the real market, we assume the case where the expected return includes an ambiguity and the probability function of each future return is represented with the following from based on the introduction obtained by Hasuike and Katagiri (2009):

$$f(z) = \frac{1}{\sqrt{2\pi} \sigma} \exp \left( - \frac{(z - \bar{M})^2}{2\sigma^2} \right)$$

When the future return $r$ is a random fuzzy variable, Each membership function value is expressed as a degree of possibility that $\tilde{r}_j$ is equal to $\gamma_j$. Then the objective function is defined as a random fuzzy variable characterized by the following membership function on fixed the parameters $x_j$:

$$\mu_{\tilde{r}}(\gamma_j) = \sup_{\tilde{r}} \left\{ \min_{1 \leq j \leq n} \mu_{\tilde{r}_j}(\gamma_j) \right\}, \forall \tilde{r} \in Y$$

$$\mu_{\tilde{u}}(\bar{u}) = \sup_{\tilde{u}} \left\{ \min_{1 \leq j \leq n} \mu_{\tilde{u}_j}(\bar{u}_j) \right\}, \forall \tilde{u} \in Y$$

$$\mu_{\tilde{b}}(\gamma) = \sup_{\tilde{b}} \left\{ \min_{1 \leq j \leq n} \mu_{\tilde{b}_j}(\gamma_j) \right\}, \forall \tilde{b} \in Y$$

Furthermore, we discuss the probability

$$p = \sup \min \left\{ \mu_{\tilde{b}}(s_j) \right\} = \Pr\{\omega|\tilde{u}\omega \geq \tilde{f}\}, \tilde{u} \sim N\left(\sum_{j=1}^{n} s_j \tilde{u}_j, \sum_{j=1}^{n} \sum_{i=1}^{n} \sigma_{ij} \tilde{x}_i \tilde{x}_j \right)$$

here, since problem is not a well defined problem due to including fuzzy variable returns, we need to set a criterion with respect to probability and possibility of future returns, we need to set a criterion with respect to probability and possibility of future returns for the deterministic optimization. In general decision cases with respect to investment, an investor usually focused on maximizing either the goal of the total profit. In previous research,
mean variance models for portfolio selection problems based on Markowitz model. Hasuike et al., (2009) introduce the expected return maximization model for random fuzzy portfolio selection model as follows:

Maximize \( \hat{E}(Z) \)

subject to \( \sum_{j=1}^{n} \sigma_{ij} x_j \leq b_i \), \( 0 \leq x_j \leq \bar{b}_j \), \( j = 1, 2, \ldots, n \)

In this problem, \( \mu_{\hat{E}(Z)} \) means an expected value derived from the following expression:

\[
\mu_{\hat{E}(Z)}(\eta) = \sup_{\eta} \left\{ \min_{s \in \mathcal{S}} \mu_{q}(s) \right\} \gamma \sim \mathcal{N}(\eta) \sigma^2, \eta = \mathbb{E} \left( \sum_{j=1}^{n} \tilde{y}_j \right)
\]

\( \mu_{\hat{E}(Z)} \) is a fuzzy optimization problem for portfolio selection problems and is solved by using results of previous studies on fuzzy portfolio selection models. And means a variance express as following problem:

Minimize \( \sum_{j=1}^{n} \sigma_{ij} x_j \)

subject to \( \hat{E}(Z) \geq \gamma \)

\( \sum_{j=1}^{n} \sigma_{ij} x_j \leq b_i \), \( 0 \leq x_j \leq \bar{b}_j \), \( j = 1, 2, \ldots, n \)

In the case that a decision maker sets the target values of probability fractile level \( b \) and possibility fractile level \( \beta \) using the chance constraint, maximizing the target future return \( f \) is mainly considered. Therefore,

Maximize \( f \)

subject to \( \mu_{\beta}(p) \geq b \), \( p \geq \beta \)

\( \sum_{j=1}^{n} \sigma_{ij} x_j \leq b_i \), \( 0 \leq x_j \leq \bar{b}_j \), \( j = 1, 2, \ldots, n \)

In this problem, the constrained is transformed into the following form based on the results obtained by Hassuike and Katagiri:

\[
\sup_{s} \min_{\mu_{k}(s) \geq h} \left\{ \mu_{k}(s) \right\} \geq f \geq \beta, \quad \tilde{u} \sim \mathcal{N} \left( \sum_{j=1}^{n} \tilde{y}_j, \sum_{j=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_j \right) > h
\]

Where \( R(X) \) is a pseudo-inverse function of \( R(X) \) in membership function. Then, we transform problem into the following problem:

\[
\text{maximise } f
\]

s.t. \( \text{Pr} \{ \omega | \tilde{u}(\omega) \geq f \} \geq \beta \)

\( \tilde{u} \sim \mathcal{N} \left( \sum_{j=1}^{n} (m_j + R'(h)(a_j)) x_j, \sum_{j=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_j \right) \)

Next, we consider the transformation of probability chance constraint \( \text{Pr} \{ \omega | \tilde{u}(\omega) \geq f \} \geq \beta \)

\[
\tilde{u} - \sum_{j=1}^{n} (m_j + R'(h)(a_j)) x_j \geq \frac{f - \beta}{\sqrt{\sum_{j=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_j}}
\]

Is a random variable with the standard normal distribution. Therefore, we obtain the following transformation of probability chance constraint:

\[
\text{Pr} \{ \omega | \tilde{u}(\omega) \geq f \} \geq \beta \iff \text{Pr} \left\{ \frac{\tilde{u} - \sum_{j=1}^{n} (m_j + R'(h)(a_j)) x_j}{\sqrt{\sum_{j=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_j}} \geq \frac{f - \beta}{\sqrt{\sum_{j=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_j}} \right\} \geq \beta
\]

\[
\iff \sum_{j=1}^{n} (m_j + R'(h)(a_j)) x_j - f \geq K\beta \sum_{j=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_j
\]
Where \( F(y) \) is the distribution function of the standard normal distribution and \( K=F \). In this paper, we consider \( b>1 \) due to the following assumptions:

(a) In the practical decision making, almost all decision makers do not select a portfolio whose achievement probability for the goal of total return is less than half.

(b) In this problem, we find the decision variable \( f \) is involved only in first constraint and minimizing \( f \) is transformed into the following problem:

\[
\begin{align*}
\min & : \ - \sum_{j=1}^{n} (m_j + R^*(h)\alpha_j)x_j + K\beta \sum_{j=1}^{n} \sum_{i=1}^{n} \sigma_{ij}x_ix_j \\
\text{s.t.} & \quad \prod_{i=1}^{n} G_i \geq h, \quad \mu_c(f) \geq h;
\end{align*}
\]

This problem is equivalent to a convex programming problem, its global optimal solution surely exists. However, it is difficult to solve it directly because problem includes the square root term. Hasuike et al. (2009), transformed. First, for simplicity of the following discussion, we do transformations of variables as follows since a symmetric variance-covariance matrix is a positive definite matrix:

\[
K\beta \sum_{j=1}^{n} \sum_{i=1}^{n} \sigma_{ij}x_ix_j = K\beta x^T\Lambda x = y^Ty
\]

\( \Lambda = \begin{pmatrix} 
\lambda_1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \lambda_n 
\end{pmatrix} \),

\( \lambda_i : \) eigenvalue of \( v \), \( \sqrt{\Lambda} = \begin{pmatrix} 
\sqrt{\lambda_1} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \sqrt{\lambda_n} 
\end{pmatrix} \)

\[
m' = \frac{1}{\sqrt{K\beta}} (m_j + R^*(h)\alpha_j)(\sqrt{\Lambda})^{-1} Q, \quad a' = \frac{1}{\sqrt{K\beta}} a(\sqrt{\Lambda})^{-1} Q
\]

\[
b' = \frac{1}{\sqrt{K\beta}} (\sqrt{\Lambda})^{-1} Qb^* = \frac{1}{\sqrt{K\beta}} (\sqrt{\Lambda})^{-1} Qb^*
\]

However, since the objective function includes random fuzzy variables, it is not a well-defined problem. Therefore, introducing the possibility chance constraint to the objective function, we set the target level \( h \) of possibility and introduce the following problem:

Maximise \( \beta \)

\[
\begin{align*}
\text{s.t.:} & \quad \mu_p(P) \geq h, \quad p \geq \beta
\end{align*}
\]

In practical situations, the relation between the target future \( f \) and probability \( \beta \) is ambivalent, and a decision maker considers increasing the goal of total profit and that of probability, simultaneously. Furthermore, considering many real decision cases and taking account of the vagueness of human judgment and flexibility for the execution of a plan, a decision maker often has subjective and ambiguous goals with respect to \( \beta \) and \( f \) such as the achievement probability is hopefully more than \( \beta \), and total future return is approximately larger than \( f1 \). Hasuike et al. (2009) represent these subjective goals with respect to \( \beta \) and \( f \) as fuzzy goals characterized. Where \( (\tilde{a}_p)(w) \) is a strictly increasing continuous function. Furthermore, using a concept of possibility measure, we introduce the degree of possibility as follows:

Maximize \( \min \left\{ \prod_{p} (G_p), \mu_c(f) \right\} \)

Maximize \( h \);

\[
\begin{align*}
\text{s.t.:} & \quad \prod_{p} (G_p) \geq h, \quad \mu_c(f) \geq h;
\end{align*}
\]

\( \Rightarrow \omega: \sup_s \min_{\gamma \in \gamma_n} \left\{ \mu_{\tilde{G}}(s) \right\} = \Pr(\omega|\tilde{u}(\omega) \geq g_{P^{-1}}(h)), \tilde{u} \sim N \left( \sum_{j=1}^{n} \sum_{i=1}^{n} \sigma_{ij}x_ix_j \right) \geq h
\]

Maximize \( h \);
s.t. \[ \sum_{j=1}^{n} [n_j + R^*(h)a_j]x_j - Kg_{p-1}(h) \leq \sum_{j=1}^{n} \sum_{i=1}^{n} o_{ij}x_ix_j \geq f \]

**Pso Approach**

This study for solving a portfolio optimization, introduces a pso heuristic method, which is one of the latest evolutionary optimization methods and is based on the metaphor of social interaction and communication such as bird flocking and fish schooling. The swarm in PSO consists of a population and each member of the population is called a particle, which represents a portfolio in this study. This follows the gbest neighborhood topology described by Kennedy et al. [1995], according to which, each particle remembers its best previous position and the best previous position visited by any particle in the whole swarm. In other words, a particle moves towards its best previous position and towards the best particle.

**Fitness function**

Kennedy and Ebehart (1995) suggested a fitness value associated with each particle. Thus, a particle moves solution space with respect to its previous position where it has met the best fitness value. In this study, the fitness function is defined:

\[ f_p = \sum_{i=1}^{N} \sum_{j=1}^{N} z_{ij} x_{pi} z_{ij} x_{pj} o_{ij} \left( \sum_{i=1}^{N} z_{ij} x_{pi} \right) \]

Where \( f_p \) is the fitness value of particle \( p \).

At each one of the iterations, a particle's personal best position and the best neighbor in the swarm are updated if an improvement in any of the best fitness values is observed.

**Moving particle**

We have mentioned that a particle moves towards its personal best position and towards the best particle of the swarm at each one of the iterations. Indeed, this movement depends on its current velocity, which is defined as:

\[ v_{z_{pi}}^{t+1} = v_{z_{pi}}^k + c_1(G_{z_{bi}}^k - z_{pi}^k) + c_2(G_{z_{bi}}^k - z_{pi}^k) \]

\[ v_{x_{pi}}^{t+1} = v_{x_{pi}}^k + c_1(G_{x_{bi}}^k - x_{pi}^k) + c_2(G_{x_{bi}}^k - x_{pi}^k) \text{ if } z_{pi}^{t+1} = 1. \]

Where both \( c_1 \) and \( c_2 \) denote uniform random numbers between 0 and 2, \( t \) and \( b \) denote the iteration number and the best particle in the swarm respectively, \( v_{z_{pi}}^{t+1} \) denotes the velocity of particle \( p \) on dimension \( x_i \), As seen in Eq. \( v_{x_{pi}}^{t+1} \) will be updated if asset \( i \) is selected by particle (or portfolio) \( p \) at iteration \( t+1 \), which means \( z_{pi}^{t+1} = 1 \) and \( c_{z_{pi}}^k \) denotes the best previous position of particle \( p \) on dimension \( x_i \). Thus, particle \( p \) moves at iteration \( t+1 \) as follows:

\[ z_{pi}^{t+1} = \text{Round} \left( \frac{1}{1 + e^{-\alpha x_i}} \right) \]

For a given asset and optimization portfolio, we use the Carlson and fuller method.

Carlsson and fuller (2001), defined notations of the lower and upper possibilistic mean values and variance of as:

\[ M_* (A) = \frac{\int_0^1 a_1(y) \text{ Pos} [A \leq a_1(y)] dy}{\gamma_1 a_1(\gamma)} = \frac{2}{\int_0^1 \gamma_1 a_1(\gamma) dy} \]

\[ M^* (A) = \frac{\int_0^1 a_2(y) \text{ Pos} [A \leq a_2(y)] dy}{\gamma_2 a_2(\gamma)} = \frac{2}{\int_0^1 \gamma_2 a_2(\gamma) dy} \]

Then the following lemma can directly be proved using the definition of interval-valued possibilistic mean. Lemma 1. Let \( A_1, A_2, ..., A_n \) be \( n \) fuzzy numbers, and let \( a \) is a real number. Then:

\[ M_* \left( \sum_{i=1}^{n} A_i \right) = \sum_{i=1}^{n} M_* (A_i) \]

\[ M^* \left( \sum_{i=1}^{n} A_i \right) = \sum_{i=1}^{n} M^* (A_i) \]
The following theorem obviously holds by lemma 1.

**Theorem 1:**
Carlsson and Fuller also defined the crisp possibilistic mean value of A as:

\[ M(\lambda A) = \frac{\lambda M^* + M(\lambda A)}{2} \]

**Theorem 2.** Let A1, A2, \ldots, An be n fuzzy numbers, and let. Then:

\[ M\left( \lambda_0 + \sum_{i=1}^{n} \lambda_i A_i \right) = \lambda_0 + \sum_{i=1}^{n} \lambda_i M(A_i) \]

Let A with \( A^\gamma = [a_1(\gamma), a_2(\gamma)] \) and B with \( B^\gamma = [b_1(\gamma), b_2(\gamma)] \) be two fuzzy numbers, Carlsson and Fuller also introduced possibilistic variance and covariance of fuzzy numbers as:

\[ \text{Var}(A) = \int_0^1 (a_2(\gamma) - a_1(\gamma))^2 \, d\gamma \]

\[ \text{Cov}(A, B) = \int_0^1 \left[ (a_2(\gamma) - a_1(\gamma)) (b_2(\gamma) - b_1(\gamma)) \right] \, d\gamma \]

Optimization portfolio problem with n risky assets and a random variable with expected return \( r_j \)

Using the definitions of the lower and upper possibilistic mean, and crisp possibilistic mean of fuzzy numbers, Chen et al., obtained:

\[ M_*(r) = \int_0^1 \gamma \, a - \alpha L^{-1}(\gamma) \, d\gamma = \alpha_0 - 2 \alpha L E_L \]

\[ M^*(r) = \int_0^1 \gamma \, b - \beta R^{-1}(\gamma) \, d\gamma = b_0 + \beta R E_R \]

\[ M(r) = \frac{a_0 + b_0}{2} - \alpha L E_L + \beta R E_R \]

\[ E_L = \int_0^1 \gamma L^{-1}(\gamma) \, d\gamma \]

\[ E_R = \int_0^1 \gamma R^{-1}(\gamma) \, d\gamma \]

Furthermore, using the definitions of the possibilistic variance and covariance of fuzzy numbers,

\[ \text{Var}(r) = \frac{1}{2} \int_0^1 \gamma (b - \beta R^{-1}(\gamma) - a) - \alpha L^{-1}(\gamma) \, d\gamma \]

\[ = \frac{1}{2} \left( \beta^2 F_{RR} + 2 \alpha \beta F_{RL} + \alpha^2 F_{LL} \right) + \]

\[ \text{Cov}(r_j, r_i) = \frac{1}{2} \int_0^1 \gamma \left[ b_i - \beta_i R^{-1}(\gamma) - a_i - \alpha_i L^{-1}(\gamma) \right] \, d\gamma \]

where

\[ F_{RR} = \int_0^1 \gamma \, R^{-1}(\gamma) \, d\gamma \]

\[ F_{LL} = \int_0^1 \gamma \, L^{-1}(\gamma) \, d\gamma \]

Chen et al., described the possibilistic mean-variance mean model as following equation

\[ \min \frac{1}{2} F_{LL} \left[ \sum_{i=1}^{n} (\alpha_i + \beta_i) x_i \right] \]

s.t. \( \sum_{i=1}^{n} \left[ \frac{a_i + b_i}{2} + (\beta_i - \alpha_i) E_L \right] - \sum_{i=1}^{n} \beta_i |x_i - x_0| \geq 0 \]

Particle swarm optimization for portfolio selection model consider with fuzzy number. Our experiments on various restricted portfolios clearly demonstrate the efficiency of particle swarm optimization technique in solving high dimensional constrained optimization problems. In order to examine the particle swarm optimization algorithm more closely, we choose the experiments on 20-asset portfolio as an example.
RESULTS

The PSO experiments for the portfolio fuzzy optimization has been performed on unrestricted risky portfolio cases. Table 2 shows the results of these 20 portfolios using PSO approach. In the experiments, the PSO Solver has been developed using Matlab as software development tool. The composition of the optimal risky portfolios developed by PSO for 20 portfolios are shown in the Table 2. The updating process of the PSO solver for optimizing the 20 portfolios and with the termination condition 1000 steps was done. In order to compare our proposed models with other models for portfolio selection problems, let us consider an example shown in Table 1 based on data introduced by Markowitz. These data have been used in previous many studies concerning portfolio problems. From the optimal portfolio, we find that the optimal portfolio for our model selects higher return. Then it is similar to the optimal portfolio for Carlsson et al. model. This means that some rates of asset allocations for our model are similar to those of Vercher et al. model.

As discussed in, the weighting function is responsible for the randomness of particles movement,

CONCLUSION

In this paper, a portfolio selection model with fuzzy random returns is proposed after quantifying the return, risk and liquidity of a portfolio. To avoid the difficulty of evaluating a large set of efficient solutions to ensure that the best solution is selected, a fuzzy approach-based particle swarm optimization was designed to resolve this model. In addition, a numerical example was presented to illustrate this modeling concept and to demonstrate the effectiveness of the proposed algorithm. The key of learning mechanisms in the PSO algorithm are driven by a metaphor of social behavior that good solutions uncovered by one member of a population are observed.

REFERENCES


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<th>returns</th>
<th>Sample mean</th>
<th>SD</th>
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<tr>
<td>1</td>
<td>0.026</td>
<td>0.238</td>
<td>0.0, 0.066</td>
</tr>
<tr>
<td>2</td>
<td>0.062</td>
<td>0.125</td>
<td>(0.02,0.062)</td>
</tr>
<tr>
<td>3</td>
<td>0.146</td>
<td>0.131</td>
<td>(0.02, 0.146)</td>
</tr>
<tr>
<td>4</td>
<td>0.173</td>
<td>0.318</td>
<td>(0.08,0.173)</td>
</tr>
<tr>
<td>5</td>
<td>0.198</td>
<td>0.368</td>
<td>(0.01,0.198)</td>
</tr>
<tr>
<td>6</td>
<td>0.055</td>
<td>0.209</td>
<td>(0.01,0.55)</td>
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<tr>
<td>7</td>
<td>0.128</td>
<td>0.175</td>
<td>(0.05,0.128)</td>
</tr>
<tr>
<td>8</td>
<td>0.118</td>
<td>0.286</td>
<td>(0.08,0.118)</td>
</tr>
<tr>
<td>9</td>
<td>0.116</td>
<td>0.29</td>
<td>(0.06,0.116)</td>
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Table 2. Sample data from Tehran stock exchange

<table>
<thead>
<tr>
<th>Fuzzy number</th>
<th>M%</th>
<th>σ%</th>
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<th>stock</th>
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<tr>
<td>(0.02, 0.062, 0.402, 0.467)</td>
<td>0.034</td>
<td>0.006</td>
<td>40061</td>
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<tr>
<td>(0.06, 0.146, 0.217, 0.345)</td>
<td>0.9</td>
<td>0.0012</td>
<td>12436</td>
<td>2</td>
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<tr>
<td>(0.08, 0.173, 0.324, 0.672)</td>
<td>0.56</td>
<td>0.008</td>
<td>46788</td>
<td>3</td>
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<tr>
<td>(0.04, 0.198, 0.404, 0.868)</td>
<td>0.67</td>
<td>0.3</td>
<td>67332</td>
<td>4</td>
</tr>
<tr>
<td>(0.01, 0.055, 0.245, 0.329)</td>
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<td>0.004</td>
<td>84512</td>
<td>5</td>
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<tr>
<td>(0.002, 0.128, 0.322, 0.417)</td>
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<td>0.0024</td>
<td>27654</td>
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<tr>
<td>(0.005, 0.118, 0.193, 0.348)</td>
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<td>0.067</td>
<td>76456</td>
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<tr>
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<td>0.031</td>
<td>14564</td>
<td>8</td>
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<tr>
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<td>0.68</td>
<td>0.0009</td>
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<td>9</td>
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<td>(0.02, 0.062, 0.234, 0.453)</td>
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<td>0.0024</td>
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<td>10</td>
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<tr>
<td>(0.0, 0.146, 0.19, 0.364)</td>
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<td>0.00016</td>
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<td>11</td>
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<td>(0.08, 0.173, 0.274, 0.478)</td>
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<td>0.0</td>
<td>67898</td>
<td>12</td>
</tr>
<tr>
<td>(0.02, 0.198, 0.356, 0.678)</td>
<td>0.87</td>
<td>0.0009</td>
<td>23656</td>
<td>13</td>
</tr>
<tr>
<td>(0.01, 0.055, 0.356, 0.734)</td>
<td>0.056</td>
<td>0.012</td>
<td>16547</td>
<td>14</td>
</tr>
<tr>
<td>(0.0, 0.128, 0.203, 0.426)</td>
<td>0.26</td>
<td>0.0015</td>
<td>26845</td>
<td>15</td>
</tr>
<tr>
<td>(0.0, 0.128, 0.463, 0.862)</td>
<td>0.038</td>
<td>0.043</td>
<td>32445</td>
<td>16</td>
</tr>
<tr>
<td>(0.004, 0.116, 0.354, 0.576)</td>
<td>0.073</td>
<td>0.00028</td>
<td>11237</td>
<td>17</td>
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<tr>
<td>(0.03, 0.056, 0.345, 0.674)</td>
<td>0.006</td>
<td>0.0005</td>
<td>95677</td>
<td>18</td>
</tr>
<tr>
<td>(0.02, 0.062, 0.257, 0.531)</td>
<td>0.09</td>
<td>0.001</td>
<td>45455</td>
<td>19</td>
</tr>
<tr>
<td>(0.09, 0.146, 0.354, 0.632)</td>
<td>0.54</td>
<td>0.00065</td>
<td>16584</td>
<td>20</td>
</tr>
</tbody>
</table>