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A new Bernoulli matrix method for solving second order linear partial differential equations with the convergence analysis ☆

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ABSTRACT

In this paper, a new matrix approach for solving second order linear partial differential equations (PDEs) under given initial conditions has been proposed. The basic idea includes integrating from the considered PDEs and transforming them to the associated integro-differential equations with partial derivatives. Therefore, Bernoulli operational matrices of differentiation and integration together with the completeness of Bernoulli polynomials can be used for transforming integro-differential equations to the corresponding algebraic equations. A rigorous error analysis in the infinity norm is given provided that the known functions and the exact solution are sufficiently smooth and bounded. A numerical example is included to demonstrate the validity and the applicability of the technique. The results confirm the theoretical prediction.

1. Introduction

The theory of second-order partial differential equations (PDEs) has found extensive applications in the study of problems in fluid mechanics, flow in porous media, heat conduction in solids, diffusive transport of chemicals in porous media, wave propagation in strings and membranes, and in mechanics of solids [33]. The usual numerical methods for PDEs are weighted residual techniques [18], finite element methods (FEMs) and boundary element methods (BEMs) [22]. Moreover, a huge size of research works are related to the finite difference methods (FDMs) [2,11,19]. For instance, in [11] several explicit difference schemes are discussed for the numerical solution of the linear hyperbolic equations subject to initial and Dirichlet boundary conditions.

In recent years several new approaches have been proposed for solving PDEs such as differential transform method (DTM) [1,33], homotopy analysis method [14] and Adomian decomposition method (ADM) [4,13]. Also, one can refer to the methods that are based on radial basis functions (RBFs). In [21], Power and Barraco present a complete numerical comparison between unsymmetric and symmetric radial basis function collocation methods for the numerical solution of boundary value problems for PDEs. However some classical ideas such as cubic B-spline scaling functions find their application for solving linear hyperbolic PDEs [10].

Operational matrices of differentiation and integration have become increasingly important in the field of numerical solution of PDEs. As a primary research work, one can refer to [26]. In [26], a double Walsh series is introduced to represent approximately functions of two independent variables, and is then applied to analyse single as well as simultaneous first order PDEs. The basic idea of this work is based on the Walsh operational matrix of integration. Kesan in [15,16] proposed

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two numerical techniques for solving linear PDEs by using Chebyshev and Taylor operational matrices of differentiation, respectively. In [7], Dascioglu presents a new chebyshev operational matrix of differentiation for solving high order linear PDEs with complicated initial and boundary conditions. After the work of [7], Bulbul and Sezer [5] propose a similar idea for solving second order PDEs by the aid of Taylor operational matrix of differentiation. In [34], Bernstein operational matrices for integration, differentiation and the product are introduced and are utilized to reduce the solution of the parabolic PDEs to the solution of algebraic equations. Nevertheless, no theoretical results are provided to justify the high accuracy numerically obtained. In this paper, in the light of the above-mentioned methods we present a new matrix method for solving second order PDEs in the following form

$$\alpha\frac{\partial^2 u}{\partial x^2}+\beta\frac{\partial^2 u}{\partial t\partial x}+\gamma\frac{\partial^2 u}{\partial t^2}+\delta\frac{\partial u}{\partial x}+\eta\frac{\partial u}{\partial t}+\theta u=G(x,t),\quad (x,t)\in[0,1]\times[0,1], \tag{1}$$

subject to the initial conditions for variable t

$$\begin{cases} u(x,0)=f(x), & x\in[0,1],\\ \frac{\partial u(x,0)}{\partial t}=m(x), & x\in[0,1],\end{cases} \tag{2}$$

together with the initial conditions for variable x

$$\begin{cases} u(0,t)=h(t), & 0<t\leqslant 1,\\ \frac{\partial u(0,t)}{\partial x}=k(t), & 0<t\leqslant 1.\end{cases} \tag{3}$$

It should be noted that we develop a new matrix approach, which was previously examined in [3,27–32], for solving second order linear PDEs. Integrating from (1) with respect to x and t, enables us to impose the initial conditions (2),(3). Thus, completeness of Bernoulli polynomials together with the Bernoulli operational matrices of differentiation and integration can be used to reduce the main problem to the associated system of algebraic equations. Actually this is the first operational matrix approach for which the high accuracy can be justified both theoretically and numerically.

This paper is divided into the following sections. The properties of Bernoulli polynomials are presented in the next Section. The numerical scheme for the solution of (1)–(3) is described in Section 3. A rigorous error analysis in the infinity norm is given provided that the known functions and the exact solution are sufficiently smooth and bounded. The results of numerical experiments are given in Section 5. Section 6 consists of a brief conclusion.

2. The properties of Bernoulli polynomials

Bernoulli polynomials (see, for instance [3,20]) and also Bernoulli functions [23], have received considerable attention in numerical analysis. They appear in the integral representation of the differentiable periodic functions, since they are employed for approximating such functions in terms of polynomials. They are also used for representing the remainder term of the composite Euler-Maclaurin quadrature rule [23]. Bernoulli polynomials can be defined in many ways such as the following form

$$\begin{cases} B_n'(x)=nB_{n-1}(x), & \forall n\geqslant 1,\\ \int_0^1 B_n(x)dx=0, & \forall n\geqslant 1,\\ B_0(x)=1. \end{cases} \tag{4}$$

By using the following classical Corollary, one can expand an enough smooth function $g(x)$ in terms of linear combination of Bernoulli polynomials.

Corollary 1. [17] *Assume that $g\in H=L^2[0,1]$ be an enough smooth function and also is approximated by the Bernoulli serie $\sum_{n=0}^{\infty}g_nB_n(x)$, then the coefficients g_n for all $n=0,1,\ldots,\infty$ can be calculated from the following relation*

$$g_n=\frac{1}{n!}\int_0^1 g^{(n)}(x)dx. \tag{5}$$

Corollary 2. *Assume that $K(x,t)\in H\times H=L^2[0,1]\times L^2[0,1]$ be an enough smooth function and also is approximated by the two variable Bernoulli series $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}k_{m,n}B_m(x)B_n(t)$, then the coefficients $k_{m,n}$ for all $m,n=0,1,\ldots,\infty$ can be calculated from the following relation*

$$k_{m,n}=\frac{1}{m!n!}\int_0^1\int_0^1\frac{\partial^{m+n}K(x,t)}{\partial x^m\partial t^n}dxdt. \tag{6}$$

Proof. By applying a similar procedure in two variables we can conclude the desired result. □

In the following lines, we will introduce Bernoulli operational matrices of differentiation and integration. If we assume that $B(x) = [B_0(x),\ B_1(x), \ldots,\ B_N(x)]$, then we have

$$\underbrace{[B_0(x),\ B_1(x), \ldots,\ B_N(x)]'}_{B'(x)} = \underbrace{[B_0(x),\ B_1(x), \ldots,\ B_N(x)]}_{B(x)} \underbrace{\begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & N \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}}_{M}, \tag{7}$$

where M is the Bernoulli operational matrix of differentiation. Similarly we have

$$\int_0^x \underbrace{[B_0(x'),\ B_1(x'), \ldots,\ B_N(x')]}_{B(x')} dx' \approx \underbrace{[B_0(x),\ B_1(x), \ldots,\ B_N(x)]}_{B(x)} \underbrace{\begin{bmatrix} -B_1(0) & \frac{-B_2(0)}{2} & \cdots & \frac{-B_N(0)}{N} & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \frac{1}{N} & 0 \end{bmatrix}}_{P}, \tag{8}$$

where P is the Bernoulli operational matrix of integration. It should be noted that according to (4) one can write

$$\int_0^x \underbrace{[B_0(x'),\ B_1(x'), \ldots,\ B_N(x')]}_{B(x')} dx' = B(x)P + \frac{B_{N+1}(x) - B_{N+1}(0)}{N+1} e_{N+1}^T,$$

where e_{N+1}^T denotes the unit vector of dimension $N+1$. In our next computations we need to Kronecker multiplication of matrices. Hence, we recall the definition of Kronecker multiplication of matrices and also an important property which is related to Kronecker multiplications.

Remark 1. Suppose that A and B are two matrices of dimensions $m \times n$ and $p \times q$, respectively, then the Kronecker multiplication of A and B is denoted by $A \otimes B = kron(A, B)$ and is defined in the following form

$$A_{m\times n} \otimes B_{p\times q} = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}_{mp\times nq}. \tag{9}$$

Moreover the following interesting property is satisfied for matrices A, B, C and D with appropriate dimensions

$$(AB) \otimes (CD) = (A \otimes C)(B \otimes D). \tag{10}$$

Now suppose that

$$B(x,t) = [B_0(x,t)\ B_1(x,t) \ldots\ B_N(x,t)]_{1\times(N+1)^2}, \tag{11}$$

where $B_i(x,t) = [B_{i,0}(x,t)\ B_{i,1}(x,t) \ldots\ B_{i,N}(x,t)]$ for all $i = 0, 1, \ldots, N$ and $B_{m,n}(x,t) = B_m(x)B_n(t)$ for all $m, n = 0, 1, \ldots, N$. Evidently $B(x,t) = B(x) \otimes B(t)$.

For clarity of presentation in our next computations, we assume that $\overline{M} = M \otimes I_{N+1}$ and $\widehat{M} = I_{N+1} \otimes M$, where I_{N+1} denots the identity matrix of dimension $(N+1)$. Trivially $\frac{\partial B(x,t)}{\partial x} = B_x(x,t) = B(x,t)\overline{M}$, because

$$B_x(x,t) = (B(x) \otimes B(t))_x = B'(x) \otimes B(t) = (B(x)M) \otimes (B(t)I_{N+1}) = \overbrace{(B(x) \otimes B(t))}^{B(x,t)}\overbrace{(M \otimes I_{N+1})}^{\overline{M}}. \tag{12}$$

By a similar way, we have $\frac{\partial B(x,t)}{\partial t} = B_t(x,t) = B(x,t)\hat{M}$.

It should be noted that, such these differentiation processes are exact relations, in other words in the above relations the equality symbol can be seen obviously. However in similar integration processes the equality symbol do not exists and the approximation symbol is replaced.

Again we assume that $\overline{P} = P \otimes I_{N+1}$ and $\widehat{P} = I_{N+1} \otimes P$. Obviously $\int_0^x B(x',t)dx' \approx B(x,t)\overline{P}$, because

$$\int_0^x B(x',t)dx' = \int_0^x (B(x') \otimes B(t))dx' = \left(\int_0^x B(x')dx'\right) \otimes B(t) \approx (B(x)P) \otimes (B(t)I_{N+1}) = \overbrace{(B(x) \otimes B(t))}^{B(x,t)}\overbrace{(P \otimes I_{N+1})}^{\overline{P}}. \tag{13}$$

By a similar way, we have $\int_0^t B(x,t')dt' \approx B(x,t)\widehat{P}$.

In our computations, we also need to approximate $\int_0^x \int_0^t B(x',t')dt'dx'$ by Bernoulli operational matrices of integration. For this reason we can use both of the above-mentioned formulae as follows

$$\int_0^x \int_0^t B(x',t')dt'dx' \approx \int_0^x B(x',t)\hat{P}dx' \approx B(x,t)\overline{P}\hat{P}. \tag{14}$$

According to (8), the associated errors from using Bernoulli operational matrices of integration can be illustrated in the following Corollary.

Corollary 3. *We have*

(i) $\int_0^t B(t')dt' = B(t)P + ES_{N,t}(B(t))$, where $ES_{N,t}(B(t)) = \frac{B_{N+1}(t)-B_{N+1}(0)}{N+1}e_{N+1}^T$.
(ii) $\int_0^t B(x,t')dt' = B(x,t)\widehat{P} + ES_{N,t}(B(x,t))$, where $ES_{N,t}(B(x,t)) = B(x)\otimes ES_{N,t}(B(t))$.
(iii) $\int_0^x B(x',t)dx' = B(x,t)\overline{P} + ES_{N,x}(B(x,t))$, where $ES_{N,x}(B(x,t)) = ES_{N,x}(B(x))\otimes B(t)$.
(iv) $\int_0^x \int_0^t B(x',t')dt'dx' = B(x,t)\overline{P}\hat{P} + ES_{N,x,t}(B(x,t))$, where

$$ES_{N,x,t}(B(x,t)) = ES_{N,x}(B(x,t))\hat{P} + ES_{N,t}(B(x,t))\overline{P} + ES_{N,x}(B(x))\otimes ES_{N,t}(B(t)).$$

Proof. Part (i) is proved, using the definition of Bernoulli operational matrix of integration:

$$\int_0^t B(t')dt' = B(t)P + \frac{B_{N+1}(t)-B_{N+1}(0)}{N+1}e_{N+1}.$$

Part (ii) follows from

$$\int_0^t B(x,t')dt' = \int_0^t (B(x)\otimes B(t'))dt' = B(x)\otimes\left(\int_0^t B(t')dt'\right)$$
$$= (B(x)I_{N+1})\otimes(B(t)P + ES_{N,t}(B(t))) = B(x,t)\widehat{P} + B(x)\otimes ES_{N,t}(B(t)).$$

The proof of part (iii) is similar to that of part (ii).

Part (iv) follows from $(i), (ii)$ and (iii):

$$\int_0^x \left(\int_0^t B(x',t')dt'\right)dx' = \int_0^x \left(B(x',t)\widehat{P} + B(x')\otimes ES_{N,t}(B(t))\right)dx'$$
$$= \left(\int_0^x B(x',t)dx'\right)\widehat{P} + \left(\int_0^x B(x')dx'\right)\otimes ES_{N,t}(B(t))$$
$$= \left(B(x,t)\overline{P} + ES_{N,x}(B(x,t))\right)\widehat{P} + (B(x)P + ES_{N,x}(B(x)))\otimes ES_{N,t}(B(t))$$
$$= B(x,t)\overline{P}\widehat{P} + ES_{N,x}(B(x,t))\widehat{P} + ES_{N,t}(B(x,t))\overline{P} + ES_{N,x}(B(x))\otimes ES_{N,t}(B(t)). \qquad \square$$

In the next section, we will transform the basic PDEs into their associted system of linear algebraic equations.

3. The basic idea

In this section we want to convert the main problem (1) to an equivalent integro-differential equation which includes initial conditions (2),(3) using a technique which can be generalized to equations in higher dimensions. Integrating both sides of (1) with respect to t and x, respectively and also imposing the initial conditions (2),(3) yields

$$\alpha\int_0^t [u_x(x,t') - \overbrace{u_x(0,t')}^{k(t')}]dt' + \beta[u(x,t) - \overbrace{u(x,0)}^{f(x)} - \overbrace{u(0,t)}^{h(t)} + \overbrace{u(0,0)}^{f(0)}]$$
$$+\gamma\int_0^x [u_t(x',t) - \overbrace{u_t(x',0)}^{m(x')}]dx' + \delta\int_0^t [u(x,t') - \overbrace{u(0,t')}^{h(t')}]dt'$$
$$+\eta\int_0^x [u(x',t) - \overbrace{u(x',0)}^{f(x')}]dx' + \theta\int_0^x \int_0^t u(x',t')dx'dt' = g(x,t), \tag{15}$$

where $g(x,t) = \int_0^x \int_0^t G(x',t')dt'dx'$. Without loss of generality, we can assume that $\beta \neq 0$. If $\beta = 0$ in (1), by linear transformations in spatial x and time t variables, one may apear β in (1) and take it to be nonzero. With the assumption of $\beta \neq 0$, Eq. (15) can be rewritten in the following form

$$u(x,t) + \frac{\alpha}{\beta}\int_0^t u_x(x,t')dt' + \frac{\gamma}{\beta}\int_0^x u_t(x',t)dx' + \frac{\delta}{\beta}\int_0^t u(x,t')dt' + \frac{\eta}{\beta}\int_0^x u(x',t)dx' + \frac{\theta}{\beta}\int_0^x \int_0^t u(x',t')dt'dx' = Q(x,t), \tag{16}$$

where

$$Q(x,t) = f(x) - f(0) + h(t) + \frac{1}{\beta}\int_0^x (\gamma m(x') + \eta f(x'))dx' + \frac{1}{\beta}\int_0^t (\alpha k(t') + \delta h(t'))dt' + \frac{1}{\beta}g(x,t).$$

We now suppose that $u(x,t)$ can be approximated in terms of linear combination of Bernoulli polynomials as follows

$$u(x,t) \approx u_N(x,t) = \sum_{m=0}^{N}\sum_{n=0}^{N} u_{m,n}B_m(x)B_n(t) = B(x,t)U, \tag{17}$$

where

$$U = [u_{0,0}\ u_{0,1} \ldots\ u_{0,N}\ u_{1,0}\ u_{1,1} \ldots\ u_{1,N} \ldots\ u_{N,0}\ u_{N,1} \ldots\ u_{N,N}]^T.$$

Our aim is to determine all the components of U. For this purpose we should approximate $Q(x,t)$ with the aid of Bernoulli polynomials in the following form

$$Q(x,t) \approx Q_N(x,t) = \sum_{m=0}^{N}\sum_{n=0}^{N} Q_{m,n}B_m(x)B_n(t) = B(x,t)Q. \tag{18}$$

We note that all components of Q can be obtained from (6) where

$$Q = [Q_{0,0}\ Q_{0,1} \ldots\ Q_{0,N}\ Q_{1,0}\ Q_{1,1} \ldots\ Q_{1,N} \ldots\ Q_{N,0}\ Q_{N,1} \ldots\ Q_{N,N}]^T.$$

By substituting $u_N(x,t)$ and $Q_N(x,t)$ (as approximations of $u(x,t)$ and $Q(x,t)$, respectively) in (16) we have

$$\begin{aligned} &u_N(x,t) + \frac{\alpha}{\beta}\int_0^t u_{N,x}(x,t')dt' + \frac{\gamma}{\beta}\int_0^x u_{N,t}(x',t)dx' + \frac{\delta}{\beta}\int_0^t u_N(x,t')dt' \\ &+\frac{\eta}{\beta}\int_0^x u_N(x',t)dx' + \frac{\theta}{\beta}\int_0^x\int_0^t u_N(x',t')dt'dx' = Q_N(x,t). \end{aligned} \tag{19}$$

By using (12)–(14), the integral terms in Eq. (19) can be approximated and we have

$$B(x,t)U + \frac{\alpha}{\beta}B(x,t)\widehat{P}\overline{M}U + \frac{\gamma}{\beta}B(x,t)\overline{P}\widehat{M}U + \frac{\delta}{\beta}B(x,t)\widehat{P}U + \frac{\eta}{\beta}B(x,t)\overline{P}U + \frac{\theta}{\beta}B(x,t)\overline{P}\widehat{P}U \approx B(x,t)Q. \tag{20}$$

Since Bernoulli polynomials form a complete basis, the above equation can be simplified and hence

$$\left(I_{(N+1)^2} + \frac{\alpha}{\beta}\widehat{P}\overline{M} + \frac{\gamma}{\beta}\overline{P}\widehat{M} + \frac{\delta}{\beta}\widehat{P} + \frac{\eta}{\beta}\overline{P} + \frac{\theta}{\beta}\overline{P}\widehat{P}\right)\widetilde{U} = Q, \tag{21}$$

where $\widetilde{U}$ is an approximation of U. The above equation is a linear system including $(N+1)^2$ equations and $(N+1)^2$ unknowns. Eq. (21) can be written in the form $W\widetilde{U} = Q$, where

$$W = I_{(N+1)^2} + \frac{\alpha}{\beta}\widehat{P}\overline{M} + \frac{\gamma}{\beta}\overline{P}\widehat{M} + \frac{\delta}{\beta}\widehat{P} + \frac{\eta}{\beta}\overline{P} + \frac{\theta}{\beta}\overline{P}\widehat{P}.$$

It should be mentioned that, the linear system $W\widetilde{U} = Q$ is a sparse one. Because, all of the matrices $\overline{M}, \widehat{M}, \overline{P}$ and $\widehat{P}$ are sparse and have the following forms

$$\overline{M} = M \otimes I = \begin{bmatrix} 0_{N+1} & I_{N+1} & 0_{N+1} & \cdots & 0_{N+1} \\ 0_{N+1} & 0_{N+1} & 2I_{N+1} & \cdots & 0_{N+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_{N+1} & 0_{N+1} & 0_{N+1} & \cdots & NI_{N+1} \\ 0_{N+1} & 0_{N+1} & 0_{N+1} & \cdots & 0_{N+1} \end{bmatrix}, \quad \widehat{M} = I \otimes M = \begin{bmatrix} M_{N+1} & 0_{N+1} & \cdots & 0_{N+1} \\ 0_{N+1} & M_{N+1} & \cdots & 0_{N+1} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{N+1} & 0_{N+1} & \cdots & M_{N+1} \end{bmatrix},$$

$$\overline{P} = P \otimes I = \begin{bmatrix} -B_1 I_{N+1} & \frac{-B_2}{2} I_{N+1} & \cdots & \frac{-B_N}{N} I_{N+1} & 0_{N+1} \\ I_{N+1} & 0_{N+1} & \cdots & 0_{N+1} & 0_{N+1} \\ 0_{N+1} & \frac{1}{2} I_{N+1} & \cdots & 0_{N+1} & 0_{N+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_{N+1} & 0_{N+1} & \cdots & \frac{1}{N} I_{N+1} & 0_{N+1} \end{bmatrix}, \quad \widehat{P} = I \otimes P = \begin{bmatrix} P_{N+1} & 0_{N+1} & \cdots & 0_{N+1} \\ 0_{N+1} & P_{N+1} & \cdots & 0_{N+1} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{N+1} & 0_{N+1} & \cdots & P_{N+1} \end{bmatrix},$$

where I_{N+1} and 0_{N+1} denote the identity and zero matrices (of dimension $N+1$) respectively. Also, M_{N+1} and P_{N+1} are the same that were defined in Section 2. In order to establish the sparsity of the coefficients matrix W, we assume that all of the parameters $\alpha, \gamma, \delta, \eta$ and θ to be nonzero and depict the structure of matrix $W_{256\times256}$ (associated to $N = 15$) in Fig. 1. From this Figure, one can see that the coefficients matrix of system (21) is sparse. However in our numerical experiments we use

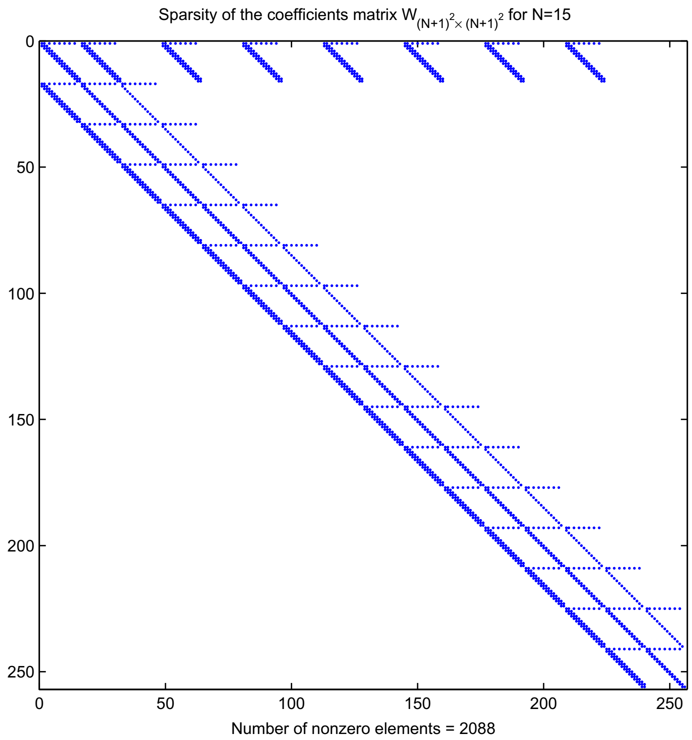

Fig. 1. Coefficients matrix ($W_{(N+1)^2\times(N+1)^2}$) sparsity of the linear system (21) for $N = 15$

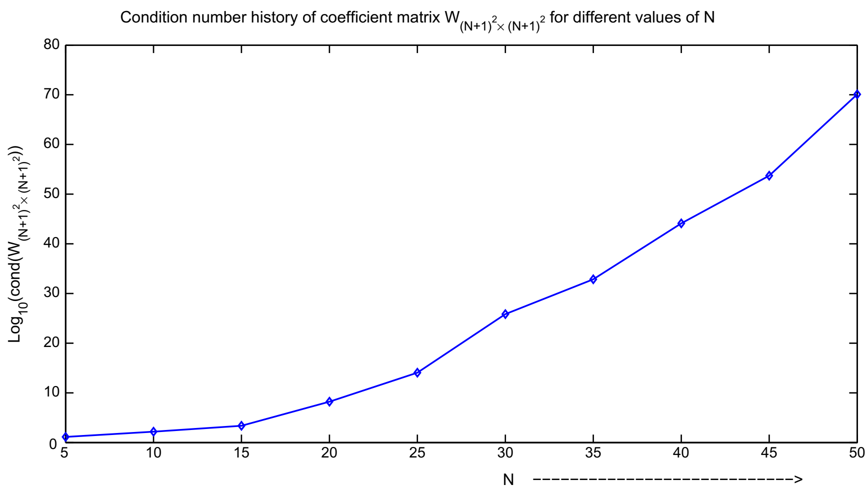

Fig. 2. Condition number history of the coefficients matrix ($W_{(N+1)^2\times(N+1)^2}$) associated to the system (21)

direct solvers, but one can use high accurate robust iterative solvers which are suitable for solving such these sparse systems [24]. Moreover, sometimes $W_{(N+1)^2\times(N+1)^2}$ is an ill-conditioned matrix for large values of N (see Fig. 2). Then, we need to precondition the system (21) with similar ideas in [9]. Since the basic aim of this research work is the polynomial approximation of some classes of PDEs and providing the convergence analysis, we do not focus on the concepts of preconditioning and sparse iterative solvers. We will consider such these topics in the future works.

4. Convergence analysis

In all parts of this section we assume that $\|g(x)\|_\infty = \sup_{x\in[0,1]}|g(x)|$ and $\|u(x,t)\|_\infty = \sup_{(x,t)\in D}|u(x,t)|$, where D is the unit sqaure in $\Re^2$.

Now by using Corollary 1, we shall provide the error of the associated approximation.

Lemma 1 [17]. *Suppose that $g(x)$ be an enough smooth function in $[0,1]$ and be approximated by Bernoulli polynomials as done in Corollary 1. With more details assume that $g_N(x)$ is the approximated polynomial of $g(x)$ in terms of linear combination of Bernoulli Polynomials and $E(g_N(x))$ is the remainder term. Then, the associated formulas are stated in the following forms*

$$g(x) = g_N(x) + E(g_N(x)), \quad x \in [0,1],$$

$$g_N(x) = \int_0^1 g(x)dx + \sum_{j=1}^{N} \frac{B_j(x)}{j!}\left(g^{(j-1)}(1) - g^{(j-1)}(0)\right),$$

$$E(g_N(x)) = -\frac{1}{N!}\int_0^1 B_N^*(x-t)g^{(N)}(t)dt,$$

where $B_N^*(x) = B_N(x-[x])$ and $[x]$ denotes the largest integer not greater than x.

Lemma 2. *Suppose $g(x) \in C^\infty[0,1]$ and $g_N(x)$ be its approximated polynomial using Bernoulli polynomials. Then the error bound would be obtained as follows*

$$\|E(g_N(x))\|_\infty \leqslant C\widehat{G}(2\pi)^{-N}, \quad x \in [0,1],$$

where $\widehat{G}$ denotes a bound for all the derivatives of function $g(x)$ (i.e., $\|g^{(i)}(x)\|_\infty \leqslant \widehat{G}$ for $i = 0,1,\ldots.$) and C is a positive constant.

Proof. By using Lemma 1, we have

$$|E(g_N(x))| = \left|\frac{1}{N!}\int_0^1 B_N^*(x-t)g^{(N)}(t)dt\right| \leqslant \frac{1}{N!}\widehat{G}\|B_N(x)\|_\infty.$$

According to [3] one can write

$$B_N(x) = \sum_{n=0}^{N}\binom{N}{n}B_n(0)x^{N-n} = \sum_{l=0}^{[\frac{N}{2}]}\binom{N}{2l}B_{2l}(0)x^{N-2l} - \frac{1}{2}\binom{N}{1}x^{N-1}, \qquad x \in [0,1].$$

Now we use the formula (1.1.5) in [17] for the even Bernoulli numbers as follows

$$|B_{2l}(0)| \le 2(2l)!(2\pi)^{-2l}.$$

Therefore

$$|B_N(x)| \leqslant 2N!\sum_{l=0}^{[\frac{N}{2}]}\frac{(2\pi)^{-2l}}{(N-2l)!} + \frac{N}{2} = 2N!(2\pi)^{-N}\sum_{l=0}^{[\frac{N}{2}]}\frac{(2\pi)^{N-2l}}{(N-2l)!} + \frac{N}{2} \leqslant 2N!(2\pi)^{-N}exp(2\pi) + \frac{N}{2}.$$

In other words $\|B_N(x)\|_\infty \leqslant CN!(2\pi)^{-N}$, wher C is a positive constant independent of N. This completes the proof. □

Lemma 3. *Under the assumptions of Lemma 1, we have*

$$\|g'(x) - g_N'(x)\|_\infty \leqslant \|g(x) - g_N(x)\|_\infty + (2\pi)^{-N}\widehat{C}\widehat{G}, \quad x \in [0,1],$$

where $\widehat{C} = (2\pi+1)C$ and C together with $\widehat{G}$ are defined in Lemma 2.

Proof. According to Lemma 1. one can write

$$g(x) - g_N(x) = -\frac{1}{N!}\int_0^1 B_N^*(x-t)g^{(N)}(t)dt,$$
$$g'(x) - g_N'(x) = -\frac{1}{(N-1)!}\int_0^1 B_{N-1}^*(x-t)g^{(N)}(t)dt.$$

Thus

$$g'(x) - g_N'(x) - (g(x) - g_N(x)) = -\frac{1}{(N-1)!}\int_0^1 B_{N-1}^*(x-t)g^{(N)}(t)dt + \frac{1}{N!}\int_0^1 B_N^*(x-t)g^{(N)}(t)dt$$
$$= \frac{1}{N!}\int_0^1 \left(B_N^*(x-t) - NB_{N-1}^*(x-t)\right)g^{(N)}(t)dt.$$

Now by using the fact that $\|NB_{N-1}(x)\|_\infty \leqslant (2\pi)^{1-N}C(N!)$ we have

$$\|g'(x) - g'_N(x)\|_\infty \leqslant \|g(x) - g_N(x)\|_\infty + (2\pi)^{-N}\widehat{C}\widehat{G}. \qquad \square$$

In [8], a generalization of Lemma 1, can be found. Therefore we just recall the error of the associated approximation in two dimensional functions.

Lemma 4. *Suppose that $u(x,t)$ be an enough smooth function and $u_N(x,t)$ be the approximated polynomial of $u(x,t)$ in terms of linear combination of Bernoulli polynomials by the aid of Corollary 2. Then the error bound would be obtained as follows*

$$||E(u_N(x,t))||_\infty := ||u(x,t) - u_N(x,t)||_\infty \leqslant C\Lambda N(2\pi)^{-N}, \quad (x,t) \in [0,1] \times [0,1],$$

where Λ is a positive constant independent of N and is a bound for all the partial derivatives of $u(x,t)$.

Remark 2. From Lemma 3, we have

$$\|u_x(x,t) - u_{N,x}(x,t)\|_\infty \leqslant \|u(x,t) - u_N(x,t)\|_\infty + N(2\pi)^{-N}\widehat{\widehat{C}}\Lambda, \tag{22}$$

$$\|u_t(x,t) - u_{N,t}(x,t)\|_\infty \leqslant \|u(x,t) - u_N(x,t)\|_\infty + N(2\pi)^{-N}\widehat{\widehat{C}}\Lambda,$$

where $\widehat{\widehat{C}}$ is a positive constant independent of N.

Lemma 5. *If we assume that the exact solution of (1) is sufficiently smooth and bounded, under the assumptions of Corollary 2 and 3, we have*

$$\lim_{N\to\infty} \|ES_{N,t}(u_N(x,t))\|_\infty := \lim_{N\to\infty} \| \int_0^t u_N(x,t')dt' - \overbrace{S_{N,t}(u_N(x,t))}^{B(x,t)\hat{P}U} \|_\infty = 0, \tag{23}$$

$$\lim_{N\to\infty} \|ES_{N,x}(u_N(x,t))\|_\infty := \lim_{N\to\infty} \| \int_0^x u_N(x',t)dx' - \overbrace{S_{N,x}(u_N(x,t))}^{B(x,t)\bar{P}U} \|_\infty = 0,$$

$$\lim_{N\to\infty} \|ES_{N,t}(u_{N,x}(x,t))\|_\infty := \lim_{N\to\infty} \| \int_0^t u_{N,x}(x,t')dt' - \overbrace{S_{N,t}(u_{N,x}(x,t))}^{B(x,t)\widehat{P}\overline{M}U} \|_\infty = 0,$$

$$\lim_{N\to\infty} \|ES_{N,x}(u_{N,t}(x,t))\|_\infty := \lim_{N\to\infty} \| \int_0^x u_{N,t}(x',t)dx' - \overbrace{S_{N,x}(u_{N,t}(x,t))}^{B(x,t)\overline{P}\widehat{M}U} \|_\infty = 0,$$

$$\lim_{N\to\infty} \|ES_{N,x,t}(u_N(x,t))\|_\infty := \lim_{N\to\infty} \| \int_0^x \int_0^t u_N(x',t')dt'dx' - \overbrace{S_{N,x,t}(u_N(x,t))}^{B(x,t)\overline{P}\widehat{P}U} \|_\infty = 0.$$

Proof. We just prove the first term and the proof of the other terms is similar. Trivially $ES_{N,t}(u_N(x,t)) = ES_{N,t}(B(x,t))U$, because

$$ES_{N,t}(u_N(x,t)) = \int_0^t u_N(x,t')dt' - S_{N,t}(u_N(x,t)) = \left[\int_0^t B(x,t')dt' - B(x,t)\widehat{P}\right]U = ES_{N,t}(B(x,t))U,$$

where $ES_{N,t}(B(x,t))$ was defined in Corollary 3. It should be noted that

$$ES_{N,t}(B(x,t)) = B(x) \otimes ES_{N,t}(B(t)) = \frac{B_{N+1}(t) - B_{N+1}(0)}{N+1}\left[B_0(x)e^T_{N+1},\ B_1(x)e^T_{N+1},\ \cdots,\ B_N(x)e^T_{N+1}\right].$$

According to Lemma 2, we have

$$|ES_{N,t}(B(x,t))| \leqslant 2C^2(N!)(2\pi)^{-(N+1)}\left[\overbrace{0,\ \cdots,\ 0}^{N\ times}, 0!(2\pi)^0, \cdots\cdots, \overbrace{0,\ \cdots,\ 0}^{N\ times}, N!(2\pi)^{-N}\right].$$

Now (6) implies that $|u_{i,j}| \leqslant \frac{\Lambda}{i!j!}$ for all $i,j \in \{0,1,\cdots\}$. Therefore

$$|U| \leqslant \Lambda\left[\frac{1}{0!0!}, \cdots \frac{1}{0!1!}, \frac{1}{0!N!}, \frac{1}{1!0!}, \cdots, \frac{1}{1!N!}, \cdots, \frac{1}{N!0!}, \cdots, \frac{1}{(N!)^2}\right]^T.$$

Thus

$$|ES_{N,t}(u_N(x,t))| \leqslant |ES_{N,t}(B(x,t))||U| \leqslant \frac{2C^2N!\Lambda}{(2\pi)^{N+1}}\left(\frac{1}{N!}+\frac{1}{2\pi N!}+\cdots+\frac{1}{(2\pi)^N N!}\right),$$
$$= \frac{2C^2\Lambda}{(2\pi)^{N+1}}\left(1+\frac{1}{2\pi}+\cdots+\frac{1}{(2\pi)^N}\right). \qquad \Box$$

Lemma 6. [25] *(Wendroff inequality) Let $u(x,t)$ (and also $\phi(x,t)$) be continuous and nonnegative functions on the unique square $D=[0,1]\times[0,1]$. If $u(x,t)$ and $\phi(x,t)$ satisfy in the following inequlity*

$$u(x,t) \leqslant \phi(x,t) + a\int_0^t u(x,t')dt' + b\int_0^x u(x',t)dx' + c\int_0^x\int_0^t u(x',t')dt'dx',$$

then

$$\|u(x,t)\|_\infty \leqslant \|\phi(x,t)\|_\infty exp[a+b+ab+c].$$

In the following lines the main Theorem of this section will be provided.

Theorem 1. *Assume that $u_N(x,t)=B(x,t)\widetilde{U}$ be the approximated solution of (16) where the unknown Bernoulli coefficient vector $\widetilde{U}$ is determined by solving the algebraic system of Eqs. (21). If $u(x,t)$ be the exact solution of (16), then we have $\lim_{N\to\infty}u_N(x,t)=u(x,t)$.*

Proof. Eq. (20) can be rewritten in the following form

$$u_N(x,t)+\frac{\alpha}{\beta}S_{N,t}(u_{N,x}(x,t))+\frac{\gamma}{\beta}S_{N,x}(u_{N,t}(x,t))+\frac{\delta}{\beta}S_{N,t}(u_N(x,t))$$
$$+\frac{\eta}{\beta}S_{N,x}(u_N(x,t))+\frac{\theta}{\beta}S_{N,x,t}(u_N(x,t)) = Q_N(x,t), \tag{24}$$

where $S_{N,t}(u_{N,x}(x,t))=B(x,t)\widehat{P}\overline{M}U$, $S_{N,x}(u_{N,t}(x,t))=B(x,t)\overline{P}\widehat{M}U$, $S_{N,t}(u_N(x,t))=B(x,t)\widehat{P}U$, $S_{N,x}(u_N(x,t))=B(x,t)\overline{P}U$, $S_{N,x,t}(u_N(x,t))=B(x,t)\overline{P}\widehat{P}U$.

According to the definitions of Lemma 5, one can deduce that

$$u_N(x,t)+\frac{\alpha}{\beta}\left(\int_0^t u_{N,x}(x,t')dt' - ES_{N,t}(u_{N,x}(x,t))\right)+\frac{\gamma}{\beta}\left(\int_0^x u_{N,t}(x',t)dx' - ES_{N,x}(u_{N,t}(x,t))\right)$$
$$+\frac{\delta}{\beta}\left(\int_0^t u_N(x,t')dt' - ES_{N,t}(u_N(x,t))\right)+\frac{\eta}{\beta}\left(\int_0^x u_N(x',t)dx' - ES_{N,x}(u_N(x,t))\right)$$
$$+\frac{\theta}{\beta}\left(\int_0^x\int_0^t u_N(x',t')dx'dt' - ES_{N,x,t}(u_N(x,t))\right) = Q_N(x,t). \tag{25}$$

Now by subtracking (25) from (16) we have

$$u(x,t)-u_N(x,t)+\frac{\alpha}{\beta}\int_0^t(u_x-u_{N,x})dt'+\frac{\gamma}{\beta}\int_0^x(u_t-u_{N,t})dx'+\frac{\delta}{\beta}\int_0^t(u-u_N)dt'$$
$$+\frac{\eta}{\beta}\int_0^x(u-u_N)dx'+\frac{\theta}{\beta}\int_0^x\int_0^t(u-u_N)dt'dx' = Q(x,t)-Q_N(x,t)+J_N(x,t), \tag{26}$$

where

$$J_N(x,t) = -\Bigg((\frac{\alpha}{\beta}ES_{N,t}(u_{N,x}(x,t))+\frac{\gamma}{\beta}ES_{N,x}(u_{N,t}(x,t))+\frac{\delta}{\beta}ES_{N,t}(u_N(x,t))$$
$$+\frac{\eta}{\beta}ES_{N,x}(u_N(x,t))+\frac{\theta}{\beta}ES_{N,x,t}(u_N(x,t))\Bigg).$$

It should be noted that $(x,t)\in[0,1]\times[0,1]$. Now by using Remark 2. one can write

$$|u(x,t)-u_N(x,t)| \leqslant \frac{|\alpha|}{|\beta|}\int_0^t |u(x,t')-u_N(x,t')|dt' + \frac{|\gamma|}{|\beta|}\int_0^x |u(x',t)-u_N(x',t)|dx'$$
$$+\frac{|\delta|}{|\beta|}\int_0^t |u(x,t')-u_N(x,t')|dt' + \frac{|\eta|}{|\beta|}\int_0^x |u(x',t)-u_N(x',t)|dx'$$
$$+\frac{|\theta|}{|\beta|}\int_0^x\int_0^t |u(x',t')-u_N(x',t')|dt'dx' + |Q(x,t)-Q_N(x,t)| + |J_N(x,t)| + 2\hat{\hat{C}}\Lambda N(2\pi)^{-N}, \quad (27)$$

where $\hat{\hat{C}}$ is a positive constant independent of N. Now we can use Wendroff inequality [25,6,12] (or Gronwal inequality in two dimensional functions) and hence

$$\|u(x,t)-u_N(x,t)\|_\infty \leqslant \tilde{C}(\|Q(x,t)-Q_N(x,t)\|_\infty + \|J_N(x,t)\|_\infty), \quad (28)$$

where $\tilde{C}$ is a positive constant independent of N. Lemma 4. implies that $\lim_{N\to\infty}\|Q(x,t)-Q_N(x,t)\|_\infty = 0$, and also Lemma 5. implies that $\lim_{N\to\infty}\|J_N(x,t)\|_\infty = 0$. These complete the proof. □

5. Numerical experiments

In this section a numerical example is considered to demonstrate the efficiency and accuracy of the proposed method. In this example, the linear algebraic systems are solved by using direct solvers in MATLAB 7.12.0 software with the Digits environment variable assigned to be 20. However, one can use several iterative krylov subspace methods and determine the vector $\hat{U}$ and hence the approximated solution $B(x,t)\hat{U}$ is obtained. For more information about iterative krylov subspace methods one can point out to the [24]. In this book, several iterative methods have been introduced for solving large sparse linear systems. All calculations are run on a Pentium 4 PC laptop with 2.70 GHz of CPU and 2 GB of RAM. One of the basic advantages of the proposed method is that, if the exact solution of the PDEs is a polynomial, one can find it by using sufficient values of N. Moreover, the proposed scheme obtain high order accuracy for dealing with PDEs which have exact solutions in the nonpolynomial forms. The readers can see the efficiency of the proposed method from the provided Figures and Table in the following Example.

Numerical example. As a typical numerical example, we consider the following second order linear PDE

$$u_{xx} - 3u_{xt} + u_{tt} = 3exp(-t)cos(x),$$

subject to the time initial conditions

$$\begin{cases} u(x,0) = sin(x), \\ u_t(x,0) = -sin(x), \end{cases}$$

together with the spatial initial conditions

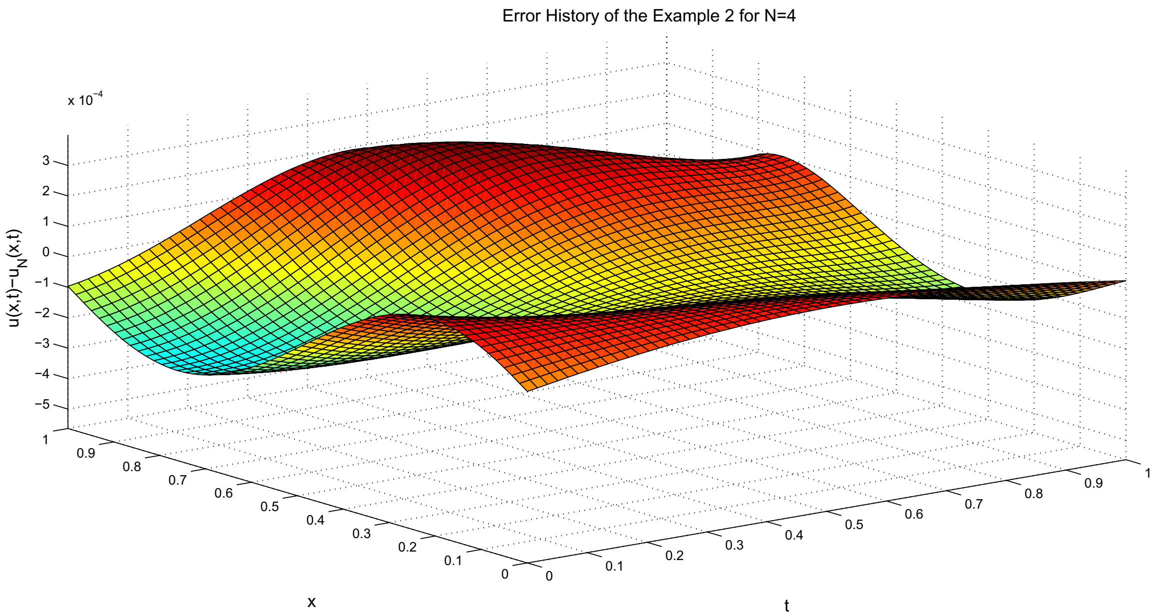

Fig. 3. The error history $e_N(x,t)$ (=$u(x,t)-u_N(x,t)$) of numerical example for $N=4$

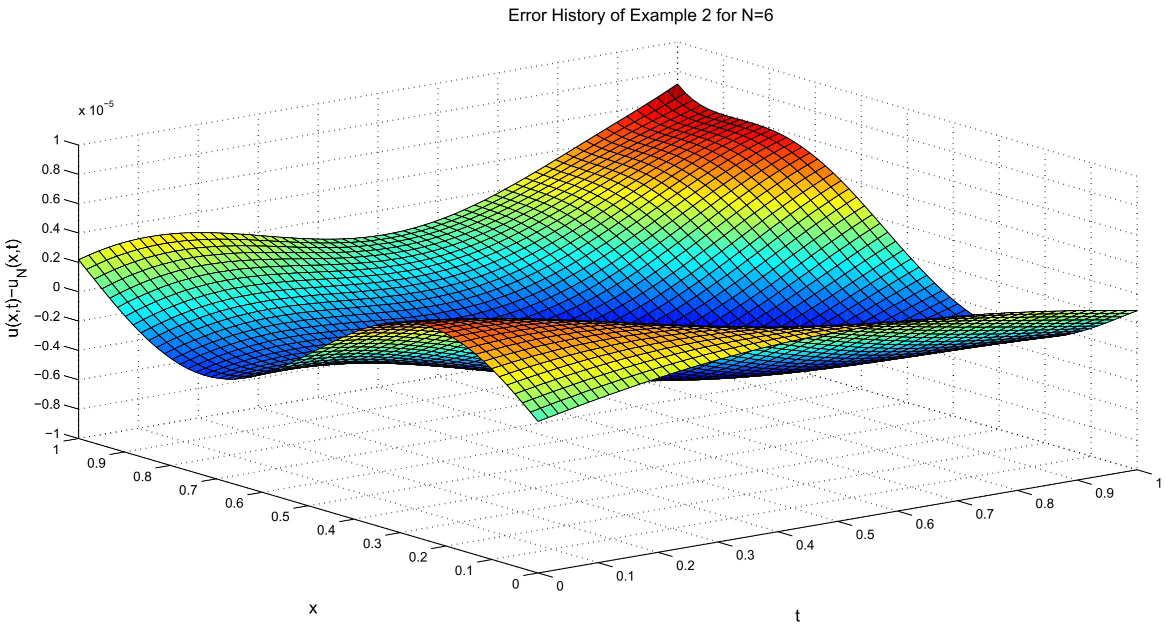

Fig. 4. The error history $e_N(x,t)$ $(=u(x,t)-u_N(x,t))$ of numerical example for $N=6$

Table 1
Absolute values of the error $|e_N(x,t)|$ $(=|u(x,t)-u_N(x,t)|)$ at the selected points of numerical example

(x,t)	$N=8$	$N=10$	$N=12$	$N=14$	$N=16$	$N=18$
(0, 0)	4.500e − 010	2.060e − 010	2.900e − 013	1.310e − 013	6.700e − 016	8.200e − 017
(0.1, 0.1)	6.560e − 008	2.192e − 009	4.201e − 011	1.405e − 012	2.665e − 014	7.510e − 016
(0.2, 0.2)	6.135e − 008	2.556e − 009	3.944e − 011	1.640e − 012	2.531e − 014	9.070e − 016
(0.3, 0.3)	2.030e − 009	9.950e − 010	1.370e − 012	6.370e − 013	2.440e − 015	2.640e − 016
(0.4, 0.4)	5.748e − 008	1.469e − 009	3.699e − 011	9.430e − 013	2.576e − 014	7.700e − 016
(0.5, 0.5)	6.627e − 008	3.178e − 009	4.255e − 011	2.040e − 012	2.753e − 014	1.503e − 015
(0.6, 0.6)	1.514e − 008	2.959e − 009	9.590e − 012	1.900e − 012	4.880e − 015	1.451e − 015
(0.7, 0.7)	5.665e − 008	9.250e − 010	3.633e − 011	5.930e − 013	2.354e − 014	5.150e − 016
(0.8, 0.8)	9.087e − 008	1.591e − 009	5.816e − 011	1.019e − 012	4.441e − 014	4.800e − 016
(0.9, 0.9)	5.613e − 008	2.947e − 009	3.623e − 011	1.892e − 012	2.132e − 014	1.023e − 015
(1.0, 1.0)	3.232e − 008	2.331e − 009	1.566e − 011	1.483e − 012	1.221e − 014	7.690e − 016

$$\begin{cases} u(0,t)=0, \\ u_x(0,t)=exp(-t). \end{cases}$$

The exact solution of this PDE is $u(x,t)=exp(-t)sin(x)$. Integrating from the above PDE with respect to x and t and imposing the initial conditions yield

$$\int_0^t u_x(x,t')dt' - \int_0^t u_x(0,t')dt' - 3[u(x,t)-sin(x)]$$
$$+\left[\int_0^x u_t(x',t)dx' - \int_0^x u_t(x',0)dx'\right]$$
$$=3\int_0^x\int_0^t exp(-t)cos(x')dx'dt'.$$

In other words

$$\int_0^t u_x(x,t')dt' - 3u(x,t) + \int_0^x u_t(x',t)dx' = Q(x,t),$$

where

$$Q(x,t)=\int_0^t exp(-t')dt' - 3sin(x) - \int_0^x sin(x')dx' + 3\int_0^x\int_0^t exp(-t)cos(x')dx'dt'.$$

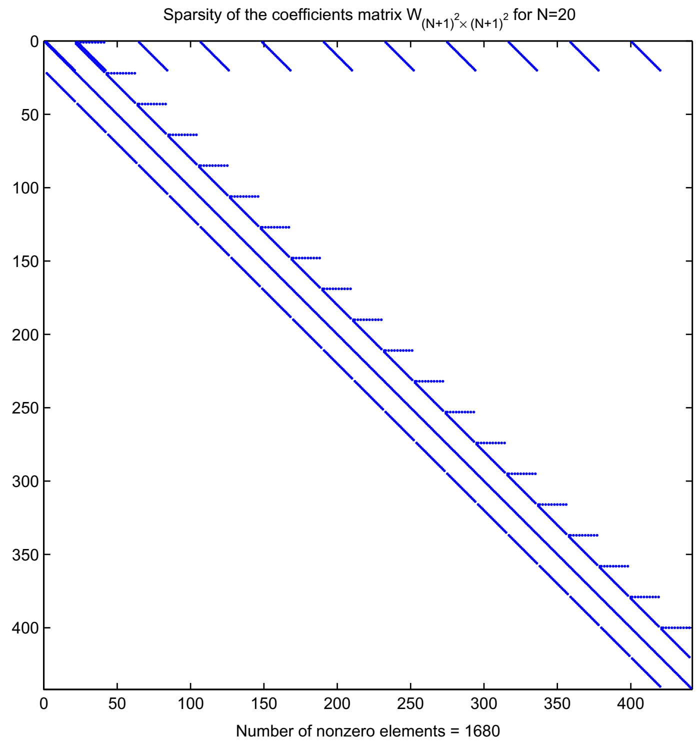

Fig. 5. Coefficients matrix ($W_{(N+1)^2\times(N+1)^2}$) sparsity of the linear system (29) for $N = 20$

Eq. (21) implies that

$$\left(\widehat{P}\overline{M} - 3I_{(N+1)^2} + \overline{P}\widehat{M}\right)\widetilde{U} = Q, \tag{29}$$

where the vector $\widetilde{U}$ should be determined after solving the above system of linear algebraic equations.

We solve the above system by taking $N = 4,\ 6,\ 8,\ 10,\ 12,\ 14,\ 16$ and 18. The error histories.

$e_N(x,t)$=$u(x,t) - u_N(x,t)$ for $N = 4$ and 6 are depicted in Figs. 3 and 4. Moreover, absolute values of the error function $|e_N(x,t)|$=$|u(x,t) - u_N(x,t)|$ at the points $(x,t) = (\frac{i}{10}, \frac{i}{10})$ where $i = 0, 1, \ldots, 10$ are provided in Table 1 by taking $N = 8,\ 10,\ 12,\ 14,\ 16$ and 18. From this table one can see high order of accuracy of the presented method. The sparsity of the coefficients matrix $W_{(N+1)^2\times(N+1)^2} = \widehat{P}\overline{M} - 3I_{(N+1)^2} + \overline{P}\widehat{M}$ for $N = 20$ are depicted in Fig. 5.

6. Conclusions

This paper presents a new matrix approach by using double truncated Bernoulli series for solving second order linear PDEs subject to the given initial conditions. The technique is based upon integrating from the considered PDEs and then transforming them to the associated Volterra integro differential equations. After this main step, Bernoulli operational matrices of differentiation and integration together with the completeness of Bernoulli polynomials can be used for reducing equations to the corresponding systems of algebraic equations. A rigorous error analysis in the infinity norm is given provided that the known functions and the exact solution are sufficiently smooth and bounded. It should be noted that this is the first operational matrix approach for which the high accuracy can be justified both theoretically and numerically. A numerical example is provided to confirm high order accuracy of the proposed method. However we examine here one-dimensional problems only, it is straightforward to extend the method to more dimensions.

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