Research Article

Convergent Homotopy Analysis Method for Solving Linear Systems

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By using homotopy analysis method (HAM), we introduce an iterative method for solving linear systems. This method (HAM) can be used to accelerate the convergence of the basic iterative methods. We also show that by applying HAM to a divergent iterative scheme, it is possible to construct a convergent homotopy-series solution when the iteration matrix $G$ of the iterative scheme has particular properties such as being symmetric, having real eigenvalues. Numerical experiments are given to show the efficiency of the new method.

1. Introduction

Computational simulation of scientific and engineering problems often depend on solving linear system of equations. Such systems frequently arise from discrete approximation to partial differential equations. Systems of linear equations can be solved either by direct or by iterative methods. Iterative methods are ideally suited for solving large and sparse systems. For the numerical solution of a large nonsingular linear system,

$$Au = b,$$

where $A \in \mathbb{R}^{n \times n}$ is given, $b \in \mathbb{R}^n$ is known, and $u \in \mathbb{R}^n$ is unknown, one class of iterative methods is based on a splitting $(M, N)$ of the matrix $A$, that is,

$$A = M - N,$$

where $M$ is taken to be invertible and cheap to invert, which mean that a linear system with matrix coefficient $M$ is much more economical to solve than (1). Based on (2), (1) can be written in the fixed-point form

$$u = Gu + c,$$

which yields the following iterative scheme for the solution of (1):

$$u^{(k+1)} = Gu^{(k)} + c, \quad k = 0, 1, 2, \ldots,$$

$$u^{(0)} \in \mathbb{R}^n \text{ is arbitrary.}$$

A sufficient and necessary condition for (4) to converge to the solution of (1) is $\rho(G) < 1$, where $\rho(G)$ denotes spectral radius. Some effective splitting iterative methods and preconditioning methods were presented for solving the linear system of (1), see [1–9]. Recently, Keramati [10], Yusufoglu [11], and Liu [12] applied the homotopy perturbation method to obtain the solution of linear systems and deduced the conditions for checking the convergence of homotopy series. In this work, we show how the homotopy analysis method may be regarded as an acceleration procedure based on the iterative method (4). We observe that it is not necessary that the basic method (4) be convergent. When $\rho(G) > 1$, it is sufficient that the eigenvalues $\lambda_i = \text{Re}(\lambda_i) + i \text{Im}(\lambda_i), i = 1, \ldots, n$ of iteration matrix $G$ satisfy the relation $\text{Re}(\lambda_i) < 1, i = 1, \ldots, n,$ (or $\text{Re}(\lambda_i) > 1, i = 1, \ldots, n$). When $\rho(G) < 1$, by applying the homotopy analysis method to the basic iterative method (4), we can improve the rate of convergence of the iterative method (4). This paper is organized as follows. In Section 2, we introduce the basic concept of HAM, derive the
conditions for convergence of the homotopy series, and apply the homotopy analysis method to the Jacobi, Richardson, SSOR, and SAOR methods. In Section 3, some numerical examples are presented to show the efficiency of the method. Finally, we make some concluding remarks in Section 4.

2. Basic Idea of HAM

The homotopy analysis method (HAM) [13, 14] was first proposed by S. J. Liao in 1992. The HAM was further developed and improved by S. Liao for nonlinear problem in [15].

Here, we apply the homotopy analysis method (HAM) to the problem (3) for finding the solution of (1) when det(A) ≠ 0. Consider (3), where u is an unknown vector of (1) and G is the iteration matrix of an iterative method. Let \( u_0 \) denote an initial guess of exact solution \( u \), \( h \neq 0 \) an convergence control parameter. Then, we can apply the homotopy analysis method and define \( H(u, p) \) by

\[
H(u, 0) = F(u), \quad H(u, 1) = L(u)
\]

and a homotopy as follows:

\[
H(u, p) = (1 - p) F(u) - phL(u) = 0,
\]

where

\[
L(u) = (I - G) u - c, \quad F(u) = u - u_0,
\]

and \( p \) is an embedding parameter. Hence, it is obvious that

\[
H(u, 0) = u - u_0, \quad H(u, 1) = (I - G) u - c.
\]

And as the embedding parameter \( p \in [0, 1] \) increases from 0 to 1, the solution of \( H(u, p) \) varies continuously from the initial approximation \( u_0 \) to the exact solution \( u \) of the original equation \( L(u) = 0 \). The homotopy analysis method uses the parameter \( p \) as an expanding parameter (see [16–18]) to obtain

\[
u = u_0 + p u_1 + p^2 u_2 + \cdots,
\]

and it gives an approximation to the solution of (3) as

\[
\lim_{p \to 1} \left( u_0 + p u_1 + p^2 u_2 + \cdots \right) = \lim_{p \to 1} \sum_{k=0}^{\infty} p^k u_k = \sum_{k=0}^{\infty} u_k.
\]

By substituting (9) in (6) and equating the terms with identical powers of \( p \), we can obtain

\[
p^0: u_0 - u_0 = 0,
\]

\[
p^1: u_1 - h \left[ (I - G) u_0 - c \right] = 0,
\]

\[
p^2: u_i - u_{i-1} - h (I - G) u_{i-1} = 0, \quad i = 2, 3, \ldots
\]

This implies that

\[
u_0 = u_0,
\]

\[
u_1 = h \left[ (I - G) u_0 - c \right],
\]

\[
u_i = \left[ (h + 1) I - hG \right] u_{i-1}, \quad i = 2, 3, \ldots
\]

Taking

\[
G_h = (h + 1) I - hG
\]

yields that

\[
u_1 = h \left[ (I - G) u_0 - c \right],
\]

\[
u_i = G_h u_{i-1}, \quad i = 2, 3, \ldots
\]

where \( u_0 \) is an initial guess of exact solution \( u \). Therefore,

\[
v = \sum_{i=0}^{\infty} p^i G_h^{-i} u_1
\]

By setting \( p = 1 \), we obtain

\[
v = u_0 + \sum_{i=1}^{\infty} G_h^{-i} u_1
\]

It is obvious that if \( \rho(G_h) < 1 \), then the series \( \sum_{i=1}^{\infty} G_h^{-i} u_1 \), converges and we have

\[
v = u_0 + (I - G_h)^{-1} u_1
\]

\[
= u_0 + h^{-1} (I - G)^{-1} u_1
\]

\[
= (I - G)^{-1} c,
\]

which is the exact solution of (3). A series of vectors can be computed by (14), and our aim is to choose the convergence control parameter \( h \neq 0 \) so that \( \rho(G_h) < 1 \). For improving the rate of convergence of iterative method, we present the following theorem.

Theorem 1. Suppose that \( \rho(G) < 1 \), and let \( \lambda_i = Re(\lambda_i) + i \ Im(\lambda_i) \), and \( \mu_i \), \( i = 1, 2, \ldots, n \), be the eigenvalues of \( G \) and \( G_h \), respectively. Let \( \alpha_i = (1 - Re(\lambda_i))^2 + Im(\lambda_i)^2 \), \( \beta_i = 2(1 - Re(\lambda_i)) \), and let \( g_i(h) = |\mu_i|^2 - \rho(G)^2 = \alpha_i h^2 + \beta_i h + 1 - \rho(G)^2 \), \( i = 1, 2, \ldots, n \). If \( \alpha_i > 0 \) and \( \beta_i \geq 2\alpha_i \neq 0, i = 1, 2, \ldots, n \), then

(i) the quadratic equation \( g_i(h) = 0 \), \( i = 1, 2, \ldots, n \), has simple real roots \( \gamma_{i}^{(0)} < \gamma_{i}^{(1)} < 0 \),

(ii) \( h = -1 \) belongs to the interval \( \left[ \max_{i=1}^{n} \gamma_{i}^{(1)}, \min_{i=1}^{n} \gamma_{i}^{(1)} \right] \) and \( \rho(G_{-1}) = \rho(G) < 1 \),

(iii) for each \( h \in \left[ \max_{i=1}^{n} \gamma_{i}^{(1)}, \min_{i=1}^{n} \gamma_{i}^{(1)} \right] \) and \( h \neq -1 \), the relation \( \rho(G_h) < \rho(G) < 1 \) holds.

Proof. (i) We begin by defining two index sets \( N_1 = \{i \mid |\mu_i| = \rho(G) \} \) and \( N_2 = \{i \mid |\mu_i| \neq \rho(G) \} \). Since \( \alpha_i > 0 \) and \( \rho(G) < 1 \), for \( i \in N_2 \), we have

\[
sign g_i(-\infty) = \begin{cases} 1, & g_i(-1) > 0, \quad g_i(0) > 0 \end{cases}
\]

So, \( g_i(h), i \in N_2 \), has simple real roots

\[
\gamma_{i}^{(0)} < -1, \quad -1 < \gamma_{i}^{(1)} < 0.
\]

For \( i \in N_1 \), we have \( \alpha_i - \beta_i < 0 \) and

\[
g_i(-1) = 0, \quad g_i \left( \frac{\alpha_i - \beta_i}{\alpha_i} \right) = 0.
\]
Table 1: Test problem information.

<table>
<thead>
<tr>
<th>Matrix</th>
<th>Order</th>
<th>nnz</th>
<th>Symmetric</th>
<th>Positive definite</th>
<th>Condition number</th>
</tr>
</thead>
<tbody>
<tr>
<td>cage5</td>
<td>37</td>
<td>233</td>
<td>No</td>
<td>No</td>
<td>15.4166</td>
</tr>
<tr>
<td>pivtol</td>
<td>102</td>
<td>306</td>
<td>No</td>
<td>No</td>
<td>109.607</td>
</tr>
<tr>
<td>pde225</td>
<td>225</td>
<td>1065</td>
<td>No</td>
<td>No</td>
<td>39.0638</td>
</tr>
<tr>
<td>Si2</td>
<td>769</td>
<td>17801</td>
<td>Yes</td>
<td>No</td>
<td>170.848</td>
</tr>
<tr>
<td>bfwb782</td>
<td>782</td>
<td>5982</td>
<td>Yes</td>
<td>No</td>
<td>18.0724</td>
</tr>
</tbody>
</table>

The following theorem shows that by applying the HAM to a iterative scheme which is divergent, it is possible to construct a convergent homotopy-series vectors when the iteration matrix $G$ has particular properties.

**Theorem 2.** Let $\lambda_j = \Re(\lambda_j) + i \Im(\lambda_j)$, and $\mu_i$, $i = 1, 2, \ldots, n$, be the eigenvalues of $G$ and $G_h$, respectively. Let $\alpha_i = (1 - \Re(\lambda_i))^2 + \Im(\lambda_i)^2$ and $\beta_i = 2(1 - \Re(\lambda_i))$. Suppose that $\alpha_i \neq 0, i = 1, 2, \ldots, n$.

(i) If $\Re(\lambda_i) > 1$, for $i = 1, 2, \ldots, n$, and $h \in (0, \min_{1 \leq i \leq n} - (\beta_i/\alpha_i))$, then $\rho(G_h) = \rho(G) < 1$.

(ii) If $\Re(\lambda_i) < 1$, for $i = 1, 2, \ldots, n$, and $h \in (\max_{1 \leq i \leq n} - (\beta_i/\alpha_i), 0)$, then $\rho(G_h) < \rho(G) < 1$.

Proof. From (13), we have $\mu_i = 1 + h(1 - \Re(\lambda_i)) - ih \Im(\lambda_i)$.

So, it is sufficient to have

$$|\mu_i|^2 - 1 - \alpha_i h^2 + \beta_i h < 0, \quad \text{for } i = 1, 2, \ldots, n. \quad (22)$$

Under the hypothesis of part (i), we have $\beta_i < 0, i = 1, 2, \ldots, n$, and the relation (22) holds if $h \in (0, \min_{1 \leq i \leq n} - (\beta_i/\alpha_i))$, and the proof of (i) is complete. A similar argument holds if $\Re(\lambda_i) < 1$, for $i = 1, 2, \ldots, n$, and part (ii) follows.

When the assumption of part (i) (or part (ii)) of Theorem 2 does not hold, the following theorem shows that, for certain cases, instead of (3) we can consider the equivalent equation

$$u = Gu + \bar{e}, \quad \text{with } \bar{G} = (G - I)^2 + I, \quad \bar{e} = (G - I) e, \quad (23)$$

in which the iteration matrix $\bar{G}$ has the eigenvalues with the desired properties.

**Theorem 3.** Let $\lambda_j = \Re(\lambda_j) + i \Im(\lambda_j)$, and $\nu_j = \Re(\nu_j) + i \Im(\nu_j), i = 1, 2, \ldots, n$, be the eigenvalues of $G$ and $\bar{G}$, respectively. If $(1 - \Re(\lambda_i))^2 - (\Re(\lambda_i))^2 > 0$ (or $(1 - \Re(\lambda_i))^2 - (\Im(\lambda_i))^2 < 0$) for $i = 1, 2, \ldots, n$, then $\Re(\nu_j) > 1$ (or $\Re(\nu_j) < 1$) for $i = 1, 2, \ldots, n$.

Proof. The proof immediately follows from the fact that $\Re(\nu_j) = ((1 - \Re(\lambda_i))^2 - (\Im(\lambda_i))^2) + 1$.

The following corollary shows that by using the modified linear equation (23) and the homotopy analysis method with the corresponding $\bar{G}_h = (h + 1)I - h\bar{G}$, we can construct a convergent homotopy series for linear system (1).

**Corollary 4.** If $G$ has only real eigenvalues, then there exists $h \neq 0$ such that the series of vectors generated by

$$u_1 = h \left[ (I - \bar{G}) u_0 - \bar{e} \right], \quad u_i = \bar{G}_h u_{i-1}, \quad i = 2, 3, \ldots, \quad (24)$$

converges to the exact solution of (1).

Proof. The proof immediately follows from Theorems 2 and 3.

This corollary establishes that the series of vectors generated by (24) always converges if the iteration matrix $G$ is a symmetric matrix. When $A$ is symmetric with diagonal elements positive real numbers, (1) can be written as follows:

$$(D^{-1/2} AD^{-1/2})(D^{1/2} x) = (D^{-1/2} b),$$

where $D$ is the diagonal of $A$. Denoting again by $A$, $x$, and $b$ the expressions $(D^{-1/2} AD^{-1/2})$, $(D^{1/2} x)$, and $(D^{-1/2} b)$, respectively, it is obvious the new coefficient $A$ is still symmetric and can therefore be written in the form $A = I - L - L^T$. An immediate consequence of Corollary 4 and the above discussion is the following results.

(i) The series of vectors generated by (24) converges when $A$ is a symmetric matrix and the iterative method is the Richardson method

$$u^{(k+1)} = (I - A) u^{(k)} + b. \quad (25)$$

(ii) The series of vectors generated by (24) converges when $A = I - L - U$ is a symmetric matrix and the iterative method is the Jacobi method

$$u^{(k+1)} = (L + U) u^{(k)} + b. \quad (26)$$

(iii) If $A = I - L - U$ is a symmetric matrix and the iterative method is SAOR method

$$u^{(k+1)} = f_{r,w} u^{(k)} + c. \quad (27)$$
### Table 2: The basic method is convergent (Theorem 1).

<table>
<thead>
<tr>
<th>Matrix</th>
<th>Method</th>
<th>(\rho(G))</th>
<th>Convergence interval</th>
<th>(h_{opt})</th>
<th>(\rho(G_{h_{opt}}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>pde225</td>
<td>SOR ((w = r = 1))</td>
<td>0.9776</td>
<td>((-1, -0.0743))</td>
<td>-0.7423</td>
<td>0.7768</td>
</tr>
<tr>
<td>pde225</td>
<td>AOR ((r = 1, w = 0.8))</td>
<td>0.8053</td>
<td>((-1, -0.8096))</td>
<td>-0.9400</td>
<td>0.7739</td>
</tr>
<tr>
<td>pde225</td>
<td>SSOR ((w = r = 0.5))</td>
<td>0.8048</td>
<td>((-1.4441, -1))</td>
<td>-1.2941</td>
<td>0.7474</td>
</tr>
<tr>
<td>pde225</td>
<td>SAOR ((r = 0.5, w = 1))</td>
<td>0.7773</td>
<td>((-1, -0.6038))</td>
<td>-0.8920</td>
<td>0.6673</td>
</tr>
<tr>
<td>cage5</td>
<td>SOR ((w = r = 1))</td>
<td>0.3388</td>
<td>((-1.2814, -1))</td>
<td>-1.1591</td>
<td>0.2314</td>
</tr>
<tr>
<td>cage5</td>
<td>AOR ((r = 1.2, w = 1.9))</td>
<td>0.9240</td>
<td>((-1, -0.0589))</td>
<td>-0.68</td>
<td>0.3355</td>
</tr>
<tr>
<td>cage5</td>
<td>SSOR ((w = r = 0.5))</td>
<td>0.6427</td>
<td>((-1.7235, -1))</td>
<td>-1.5235</td>
<td>0.4557</td>
</tr>
<tr>
<td>cage5</td>
<td>SAOR ((r = 0.1, w = 0.7))</td>
<td>0.5590</td>
<td>((-1.5609, -1))</td>
<td>-1.3756</td>
<td>0.3890</td>
</tr>
<tr>
<td>pivtol</td>
<td>SOR ((w = r = 0.1))</td>
<td>0.9487</td>
<td>((-3.9226, -1))</td>
<td>-2.7526</td>
<td>0.8588</td>
</tr>
<tr>
<td>pivtol</td>
<td>AOR ((r = 0.3, w = 0.6))</td>
<td>0.9958</td>
<td>((-1, -0.0128))</td>
<td>-0.7300</td>
<td>0.7626</td>
</tr>
<tr>
<td>pivtol</td>
<td>SSOR ((w = r = 0.2))</td>
<td>0.8005</td>
<td>((-1.4850, -1))</td>
<td>-1.3450</td>
<td>0.7317</td>
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<tr>
<td>pivtol</td>
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<td>0.6943</td>
</tr>
<tr>
<td>bfwb782</td>
<td>SOR ((w = r = 1))</td>
<td>0.3732</td>
<td>((-1.2270, -1))</td>
<td>-1.1470</td>
<td>0.3024</td>
</tr>
<tr>
<td>bfwb782</td>
<td>AOR ((r = 1.2, w = 1.9))</td>
<td>0.9942</td>
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<td>-0.65</td>
<td>0.3988</td>
</tr>
<tr>
<td>bfwb782</td>
<td>SSOR ((w = r = 0.5))</td>
<td>0.5984</td>
<td>((-1.6665, -1))</td>
<td>-1.4704</td>
<td>0.4103</td>
</tr>
<tr>
<td>bfwb782</td>
<td>SAOR ((r = 0.1, w = 0.7))</td>
<td>0.5123</td>
<td>((-1.5123, -1))</td>
<td>-1.3423</td>
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</tr>
<tr>
<td>bfwb782</td>
<td>Richardson</td>
<td>0.999998</td>
<td>((-8.5201 \times 10^4, -1))</td>
<td>-8.0701 \times 10^4</td>
<td>0.8952</td>
</tr>
</tbody>
</table>

### Table 3: The basic method is divergent (Theorem 2).

<table>
<thead>
<tr>
<th>Matrix</th>
<th>Method</th>
<th>(\rho(G))</th>
<th>Convergence interval</th>
<th>(h_{opt})</th>
<th>(\rho(G_{h_{opt}}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>pde225</td>
<td>SOR ((r = \omega = 1.5))</td>
<td>2.4937</td>
<td>((-0.4318, 0))</td>
<td>-0.3</td>
<td>0.7980</td>
</tr>
<tr>
<td>pde225</td>
<td>AOR ((r = 1, \omega = 1.5))</td>
<td>1.5542</td>
<td>((-0.6816, 0))</td>
<td>-0.5</td>
<td>0.7745</td>
</tr>
<tr>
<td>pde225</td>
<td>SSOR ((r = 2.01, \omega = 2.01))</td>
<td>1.1112</td>
<td>((0, 17.9804))</td>
<td>12.0540</td>
<td>0.9388</td>
</tr>
<tr>
<td>pde225</td>
<td>SAOR ((r = 2, \omega = 3))</td>
<td>4.9844</td>
<td>((0, 0.3408))</td>
<td>0.2510</td>
<td>0.7741</td>
</tr>
<tr>
<td>cage5</td>
<td>SOR ((r = \omega = 2))</td>
<td>1.0150</td>
<td>((-0.9872, 0))</td>
<td>-0.4</td>
<td>0.7425</td>
</tr>
<tr>
<td>cage5</td>
<td>AOR ((r = 1.2, \omega = 3))</td>
<td>2.0370</td>
<td>((-0.6584, 0))</td>
<td>-0.43</td>
<td>0.3355</td>
</tr>
<tr>
<td>cage5</td>
<td>SSOR ((r = 2.2, \omega = 2.2))</td>
<td>1.8715</td>
<td>((0, 2.2948))</td>
<td>2.0280</td>
<td>0.7681</td>
</tr>
<tr>
<td>cage5</td>
<td>SAOR ((r = 2, \omega = 3))</td>
<td>4.0002</td>
<td>((0, 0.6666))</td>
<td>0.458</td>
<td>0.3748</td>
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<tr>
<td>pivtol</td>
<td>SOR ((r = \omega = 1))</td>
<td>4.3002</td>
<td>((-0.3773, 0))</td>
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<td>0.7778</td>
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<tr>
<td>pivtol</td>
<td>AOR ((r = 0.467, \omega = 0.981))</td>
<td>1.5280</td>
<td>((-0.7566, 0))</td>
<td>-0.58</td>
<td>0.6766</td>
</tr>
<tr>
<td>pivtol</td>
<td>SSOR ((r = \omega = 0.5))</td>
<td>2.1132</td>
<td>((-0.6424, 0))</td>
<td>-0.55</td>
<td>0.7306</td>
</tr>
<tr>
<td>pivtol</td>
<td>SAOR ((r = 0.1, \omega = 0.7))</td>
<td>2.6670</td>
<td>((-0.5454, 0))</td>
<td>-0.47</td>
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<tr>
<td>bfwb782</td>
<td>Jacobi</td>
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<td>bfwb782</td>
<td>AOR ((r = 1.2, \omega = 3))</td>
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</tr>
<tr>
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<td>SAOR ((r = 2, \omega = 3))</td>
<td>4.0000</td>
<td>((0, 0.6667))</td>
<td>0.45</td>
<td>0.3570</td>
</tr>
</tbody>
</table>

### Table 4: The basic method is divergent (Theorem 3).

<table>
<thead>
<tr>
<th>Matrix</th>
<th>Method</th>
<th>(\rho(G))</th>
<th>Convergence interval</th>
<th>(h_{opt})</th>
<th>(\rho(G_{h_{opt}}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>SI2</td>
<td>SOR ((r = w = 1.3))</td>
<td>1.0993</td>
<td>((0, 0.8415))</td>
<td>0.8</td>
<td>0.9974</td>
</tr>
<tr>
<td>SI2</td>
<td>AOR ((r = 1.5, w = 0.75))</td>
<td>1.0799</td>
<td>((0, 2.7964))</td>
<td>2.7</td>
<td>0.9942</td>
</tr>
<tr>
<td>SI2</td>
<td>SSOR ((r = w = 1.8))</td>
<td>1.24945</td>
<td>((0, 2.0007))</td>
<td>1.9</td>
<td>0.9909</td>
</tr>
<tr>
<td>SI2</td>
<td>SAOR ((r = 1.5, w = 0.75))</td>
<td>1.1380</td>
<td>((0, 2.0949))</td>
<td>2</td>
<td>0.9916</td>
</tr>
<tr>
<td>SI2</td>
<td>Jacobi</td>
<td>1.3233</td>
<td>((0, 0.3705))</td>
<td>0.7</td>
<td>0.9999</td>
</tr>
<tr>
<td>SI2</td>
<td>Richardson</td>
<td>40.3813</td>
<td>((0, 0.0012))</td>
<td>0.0011</td>
<td>0.9999</td>
</tr>
</tbody>
</table>
with
\[ J_{r,w} = (I - rU)^{-1} [(1 - \omega) I + (\omega - r) U + \omega L] \]
\[ \times (I - rL)^{-1} [(1 - \omega) I + (\omega - r) L + \omega U], \]
\[ c = \omega (I - rU)^{-1} [(2 - \omega) I - (\omega - r) (L + U)] (I - rL)^{-1} b, \]
(28)

then the series of vectors generated by (24) converges if
\[ \omega \in (0, 2), \quad \omega + \frac{2 - \omega}{\mu_{\text{min}}} < r < \omega + \frac{2 - \omega}{\mu_{\text{max}}} \]
(29)

where \(\mu_{\text{min}}\) and \(\mu_{\text{max}}\) stand for the minimum and the
maximum eigenvalues of \(B = L + U\), respectively. This result follows from the fact that in this case
the iteration matrix \(J_{r,w}\) of SAOR method has real

(iv) If \(A = I - L - U\) is a symmetric matrix and the
iterative method is SSOR method, then the series of
vectors generated by (24) converges if \(\omega \in (0, 2)\).
This result immediately follows from the fact that the
SAOR method reduces to the SSOR method for \(r = \omega\).

3. Numerical Examples

For numerical comparison we use some matrices from
the University of Florida sparse matrix Collection [20]. These
matrices with their properties are shown in Table 1. We
determined the spectral radii of iteration matrices of
the classical SOR, AOR, SSOR, SAOR, Jacobi, and Richardson
methods \((\rho(G))\) as well as those of the corresponding \(G_{k}\)
after the application HAM method with the experimentally
computed optimal value of \(h(\rho(G_{kopt})).\) In Tables 2–4, we list
\(\rho(G), \rho(G_{kopt}),\) the interval of convergence which introduced
in Theorems 1 and 2, the experimentally computed optimal
value of \(h(G_{kopt}),\) and the spectral radius of iteration matrix
\(\rho(G_{kopt})\) which introduced in Corollary 4.

In Table 2, we consider the convergent classical methods
\((\rho(G) < 1).\) It is easy to verify that the numerical results are
consistent with Theorem 1, and we observe that for the
convergent classical methods by choosing suitable convergence
control parameter \(h,\) the rate of the convergence of the HAM
method is faster than that of corresponding classical method.

In Table 3, we consider the divergent classical methods
when \(\text{Re} (\lambda_i) > 1\) \((\text{Re} (\lambda_j) < 1)\) for \(i = 1, 2, \ldots, n.\) We can
see that the numerical results are consistent with Theorem 2.
These results show that by applying the HAM to an iterative
scheme which is divergent, it is possible to construct a
convergent homotopy-series vectors when the iteration
matrix \(G\) has mentioned properties.

In Table 4, we report the results obtained for the
symmetric matrix \(S2\) which has positive diagonal elements.
For this example, the classical methods diverge and there exist \(i, j\)
such that \(\text{Re} (\lambda_i) < 1 \) and \(\text{Re} (\lambda_j) > 1.\) We observe that the
results are consistent with Theorem 3 and Corollary 4. The
results show that the HAM method is convergent but the rate
of the convergence is slow.

Finally, Tables 3 and 4 show that it is not necessary to
choose the the parameters \(r\) and \(\omega\) in the convergence interval of the classical methods. In the case of divergence, under
the assumptions of Theorems 2 and 3, the application of the
HAM can generate the convergent homotopy-series vectors
for linear system (1).

4. Conclusion

In this paper, we proposed to apply the homotopy analysis
method to the classical iterative methods for solving the linear
system of equations. The theoretical results show the HAM
can be used to accelerate the convergence of the basic iterative
methods. In addition, we show that by applying the HAM
to a divergent iterative scheme, it is possible to construct
a convergent homotopy-series solution when the iteration
matrix \(G\) of the iterative scheme has particular properties.
The numerical experiments confirm the theoretical results and
show the efficiency of the new method.

References

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