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A new generalized AOR iterative method for solving linear systems

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Abstract

In this paper, based on a block splitting of the coefficient matrix, we present a new generalized iterative method for solving the linear system $Ax = b$. This method is well-defined even when some elements on the diagonal of A are zero. Convergence analysis and comparison theorems of the proposed method are provided. Specially, the results show that our new generalized AOR iterative method also, converges when A is an H-matrix. And for L-matrices, our new generalized Jacobi iterative method is faster than the classical Jacobi. The Numerical examples are also given to illustrate our results.

Keywords: *AOR method; Generalized AOR method; M-matrix; $H-$matrix; L-matrix.*

1 Introduction

consider the linear system

$$Ax = b, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ is a known nonsingular matrix, $b \in \mathbb{R}^n$ is known, and $x \in \mathbb{R}^n$ is unknown. For the numerical solution of (1) the generalized AOR (GAOR) method is defined by

$$x^{(k+1)} = L_{\gamma,\omega}x^{(k)} + \omega(D - \gamma L)^{-1}b, \quad k = 0, 1, 2, \ldots \tag{2}$$

and

$$L_{\gamma,\omega} = (D - \gamma L)^{-1}[(1 - \omega)D + (\omega - \gamma)L + \omega U], \tag{3}$$

where γ and $\omega \neq 0$ are real parameters and D, L, and U which need not be diagonal and strictly lower triangular and upper triangular, respectively, are required to satisfy $A = D - L - U$. It is also assumed that $\det(D - \gamma L) \neq 0$. This method is well-defined even when some elements on the diagonal of A are zero. Some very interesting results concerning the GAOR method were given in [2], [3], [5], [6], [7], [11], [15]. Authors showed that with spacial conditions, the *GAOR* iterative method converges when A is an M-matrix or is a Hermitian positive definite matrix. The new method satisfies in these conditions and also, we prove our new method, converges when A is an H-matrix too. We note that the classical AOR [3] method is a special case of GAOR method, where D is the diagonal, $-L$ and $-U$ are strictly lower and upper triangular parts of A, respectively. As the classical AOR method, for certain values of γ and ω we have the generalized Jacobi (GJ), the generalized Gauss-seidel (GGS), and the generalized SOR (GSOR) methods. This is one of the benefits of our new method that the new generalized Jacobi iterative method is faster than the classical Jacobi method, for L-matrices. This new method is practical too.

In the following we are going to consider A as a block matrix in the form

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \text{ if } n = 2l, \qquad A = \begin{pmatrix} A_1 & -c_1 & A_2 \\ -d_1^T & a_{l+1l+1} & -d_2^T \\ A_3 & -c_2 & A_4 \end{pmatrix} \text{ if } n = 2l+1, \tag{4}$$

where $c_1, c_2, d_1, d_2 \in \mathbb{R}^l$ and $A_i \in \mathbb{R}^{l\times l}$, $i = 1, 2, 3, 4$. By splitting A_i, into $A_i = D_i - L_i - U_i$ where D_i, is the diagonal matrix, $-L_i$ and $-U_i$ are strictly lower and upper triangular parts of A_i, respectively. We split A into

$$A = V - L_V - U_V, \tag{5}$$

where V, L_V, and U_V are block matrices as follows:

for $n = 2l$

$$V = \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix}, \; L_V = \begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \end{pmatrix}, \; U_V = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix} \tag{6}$$

for $n = 2l + 1$

$$V = \begin{pmatrix} D_1 & 0 & D_2 \\ 0 & a_{l+1\ l+1} & 0 \\ D_3 & 0 & D_4 \end{pmatrix}, \; L_V = \begin{pmatrix} L_1 & 0 & L_2 \\ d_1^T & 0 & d_2^T \\ L_3 & 0 & L_4 \end{pmatrix}, \; U_V = \begin{pmatrix} U_1 & c_1 & U_2 \\ 0 & 0 & 0 \\ U_3 & c_2 & U_4 \end{pmatrix} \tag{7}$$

In the following we consider the case $n = 2l$, the case $n = 2l + 1$ can be discussed in a similar way.

By assuming that $D_1D_4 - D_2D_3$ is a nonsingular matrix, it is easy to see that

$$V^{-1} = \begin{pmatrix} (D_1D_4 - D_2D_3)^{-1} & 0 \\ 0 & (D_1D_4 - D_2D_3)^{-1} \end{pmatrix} \begin{pmatrix} D_4 & -D_2 \\ -D_3 & D_1 \end{pmatrix}, \tag{8}$$

and we have $\det(V - \gamma L_V) \neq 0$, where γ is a real parameter.

In this paper, based on the splitting (5), we define the GAOR method (called the $\mathrm{AOR_V}$ method) as follows:

$$x^{(k+1)} = \widetilde{L}_{\gamma,\omega} x^{(k)} + \omega(V - \gamma L_V)^{-1} b, \quad k = 0, 1, 2, \ldots \tag{9}$$

with the iteration matrix

$$\widetilde{L}_{\gamma,\omega} = (V - \gamma L_V)^{-1}[(1-\omega)V + (\omega - \gamma)L_V + \omega U_V], \tag{10}$$

where ω and γ are real parameters with $\omega \neq 0$.

As the *AOR* method for certain values of the parameter ω and γ, we can obtain the other iterative methods which are as follows:

1. J_V ($\mathrm{Jacobi_V}$) method for $\gamma = 0$ and $\omega = 1$.
2. $\mathrm{JOR_V}$ method for $\gamma = 0$.
3. $\mathrm{GS_V}$ ($\mathrm{Gauss-Seidel_V}$) method for $\gamma = \omega = 1$.
4. $\mathrm{EGS_V}$ method for $\gamma = 1$.
5. $\mathrm{SOR_V}$ method for $\gamma = \omega$.

By using (9) and (10), we have the following Algorithm:

Algorithm AOR_V

1. Given starting vector $x^{(0)}$
2. For $k = 0, 1, 2, \ldots$, until convergence Do:
3. $\quad z^{(k)} = [(1-\omega)V + (\omega - \gamma)L_V + \omega U_V]x^{(k)} + \omega b$

4. For $i = 1, 2, \ldots, l$ Do:

5. $y_1 = -\gamma\Sigma_{j=1}^{i-1}(a_{i,j}x_j^{(k+1)} + a_{i,j+l}x_{j+l}^{(k+1)})$

6. $y_2 = -\gamma\Sigma_{j=1}^{i-1}(a_{i+l,j}x_j^{(k+1)} + a_{i+l,j+l}x_{j+l}^{(k+1)})$

7. Solve the system 2×2, $\begin{bmatrix} a_{i,i} & a_{i,i+l} \\ a_{i+l,i} & a_{i+l,i+l} \end{bmatrix} \begin{bmatrix} x_i^{(k+1)} \\ x_{i+l}^{(k+1)} \end{bmatrix} = \begin{bmatrix} z_i^{(k)} + y_1 \\ z_{i+l}^{(k)} + y_2 \end{bmatrix}$.

8. End Do

9. End Do.

We observe that the method does not break down if the diagonal matrix $D_1D_4 - D_2D_3$ is a nonsingular matrix. In the following, in Section 2, we present the convergence analysis when A is a diagonally dominant, M-matrix or H-matrix, and Hermitian positive definite matrix. In section 3, comparison theorem is presented. In section 4, numerical examples are given to illustrate our results. Section 5 is devoted to concluding remarks.

2 Convergence Analysis

2.1 Diagonally dominant Matrices

In the sequel, we need the following.

Notation 2.1. *Let $A \in \mathbb{C}^{n\times n}$. Then $|A|$ denotes the matrix whose elements are the modula of the elements of A. The same notation applies to vectors $x \in \mathbb{C}^n$.*

Definition 2.2. *A matrix A is a strictly diagonally dominant matrix if*

$$|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}|, \qquad i = 1, \ldots, n.$$

Definition 2.3. [16]. *A matrix A is said to be irreducible if the directed graph associated with A is strongly connected.*

Definition 2.4. *A matrix A is irreducibly diagonally dominant if A is irreducible and*

$$|a_{ii}| \geq \sum_{j=1, j\neq i}^{n} |a_{ij}|, \qquad i = 1, \ldots, n$$

with strict inequality for at least one i.

Here we will assume that A is an $n \times n$ matrix with unit diagonal elements.

Theorem 2.5. *If A of (1) is a strictly diagonally dominant matrix and $\det(I - D_2D_3) \neq 0$, then $\rho(\widetilde{L}_{\gamma,\omega})$ satisfies the following:*

$$\min_i \frac{|1-\omega| - |\omega-\gamma|f_i - |\omega|g_i}{1 + |\gamma|f_i} \leq \rho(\widetilde{L}_{\gamma,\omega}) \leq \max_i \frac{|1-\omega| + |\omega-\gamma|f_i + |\omega|g_i}{1 - |\gamma|f_i}, \tag{11}$$

for $i = 1, 2, ..., n$, where $|\gamma| < \frac{1}{f_i}$, $f_i = \frac{e_i^T|L_V|e}{d_i}$, and $g_i = \frac{e_i^T|U_V|e}{d_i}$ with $e = (1, 1, \ldots, 1)^T \in \mathbb{R}^n$ and $d_i = \begin{cases} 1 - |a_{i,i+l}|, & i \leq l \\ 1 - |a_{i,i-l}|, & i > l \end{cases}$.

Proof. The proof is similar to that of Theorem 1 in [13]. Since the eigenvalues of $\widetilde{L}_{\gamma,\omega}$ are given from

$$\det(\widetilde{L}_{\gamma,\omega} - \lambda I) = 0, \tag{12}$$

after some manipulation, it is easy to verify that to solve (12) is equivalent to solving

$$\det(Q) = 0, \tag{13}$$

where Q is

$$Q = V - \frac{\gamma(\lambda-1)+\omega}{\lambda-1+\omega} L_V - \frac{\omega}{\lambda-1+\omega} U_V.$$

If we take the parameter $\gamma,\ \omega,\ \lambda$, in order that Q be strictly diagonally dominant, we get

$$d_i > \frac{|\gamma(\lambda-1)+\omega|}{|\lambda-1+\omega|} e_i^T |L_V| e + \frac{|\omega|}{|\lambda-1+\omega|} e_i^T |U_V| e.$$

Since A is a strictly diagonally dominant matrix, we have $d_i \neq 0$, for $i = 1, 2, \ldots, n$. So, we get

$$|\lambda - 1 + \omega| > |\gamma(\lambda-1)+\omega| f_i + |\omega| g_i, \qquad i = 1, 2, \cdots, n.$$

The rest of proof is similar to that of theorem 1 in [13]. □

Theorem 2.6. *If A of (1) is a strictly diagonally dominant matrix and* $\det(I - D_2 D_3) \neq 0$, $\omega > 0$, $0 \leq \gamma \leq \omega$, *then a sufficient condition for the convergence of the AOR_V method is*

$$0 < \omega < \frac{2}{1 + \max_i(f_i + g_i)}.$$

Proof. First, from the fact that $\frac{2}{1+f_i+g_i} < \frac{1}{f_i}$, we have $0 \leq \gamma < \frac{1}{f_i}$ when $0 < \gamma < \frac{2}{1+\max_i(f_i+g_i)}$. Now from (11), we see that $\rho(\widetilde{L}_{\gamma,\omega})$ will be less than one if

$$|\omega - \gamma| f_i + |\omega| g_i + |1 - \omega| + |\gamma| f_i < 1, \qquad \text{for } i = 1, 2, \cdots, n. \tag{14}$$

For $\omega \geq \gamma \geq 0$ and $0 < \omega \leq 1$, these conditions will be satisfied. For $\omega > \gamma \geq 0$, and $\omega > 1$, we observe that (14) will be satisfied if $0 < \omega < \frac{2}{1+\max_i(f_i+g_i)}$. □

Now from Theorem 2.6 and Theorem of Extrapolation [4] we can state the following theorem for J_V, JOR_V, GS_V, EGS_V, and SOR_V methods.

Theorem 2.7. *If A of (1) is a strictly diagonally dominant matrix and* $\det(I - D_2 D_3) \neq 0$ *then*

(i) $\rho(\widetilde{L}_{0,1}) \leq \max_i(f_i + g_i) < 1$.

(ii) $\rho(\widetilde{L}_{0,\omega}) \leq |\omega| \max_i(f_i + g_i) + |1 - \omega|$ and $\rho(\widetilde{L}_{0,\omega}) < 1$ for $0 < \omega < \frac{2}{1+\rho(\widetilde{L}_{0,1})}$.

(iii) $\rho(\widetilde{L}_{1,1}) \leq \max_i \frac{g_i}{1-f_i} < 1$.

(iv) $\rho(\widetilde{L}_{1,\omega}) \leq \frac{|\omega-1|f_i+|\omega|g_i+|1-\omega|}{1-f_i}$ and $\rho(\widetilde{L}_{1,\omega}) < 1$ for $0 < \omega < \frac{2}{1+\rho(\widetilde{L}_{1,1})}$.

(v) $\rho(\widetilde{L}_{\gamma,\gamma}) \leq \max_i \frac{|\gamma|g_i+|1-\gamma|}{1-|\gamma|f_i}$ and $\rho(\widetilde{L}_{\gamma,\gamma}) < 1$ for $0 < \gamma < \frac{2}{1+\max_i(g_i+f_i)}$.

Remark. In the case when A is only irreducibly diagonally dominant, Theorems 2.6 and 2.7 only show that $\rho(\widetilde{L}_{0,1}) \leq 1$, $\rho(\widetilde{L}_{1,1}) \leq 1$, and $\rho(\widetilde{L}_{\gamma,\gamma}) \leq 1$. As in [14], by contradiction, we can show that in fact the strict inequality also holds.

Finally, from the fact that AOR_V method is the extrapolated SOR_V method, when $\gamma \neq 0$ and its extrapolation parameter is $\frac{\omega}{\gamma}$, we can state the following theorem by using Theorems 2.6, 2.7, and Theorem of Extrapolation [4].

Theorem 2.8. *If A of (1) is a strictly diagonally dominant matrix, such that* $\det(I - D_2 D_3) \neq 0$, *then AOR_V method is convergent, i.e.,* $\rho(\widetilde{L}_{\gamma,\omega}) < 1$ *for:*

(i) $0 \leq \gamma \leq \omega$ and $0 < \omega < \frac{2}{1+\max_i(f_i+g_i)}$.

(ii) $0 < \gamma < \frac{2}{1+\max_i(f_i+g_i)}$ and $0 < \omega < \frac{2\gamma}{1+\rho(\widetilde{L}_{\gamma,\gamma})}$.

2.2 H-matrix, M-matrix, and L-matrix

In the sequel, we need the following.

Notation 2.9. *Let $A, B \in \mathbb{R}^{n\times n}$. If $a_{ij} \geq b_{ij}$ ($a_{ij} > b_{ij}$), $i, j = 1, 2, \ldots, n$, we write $A \geq B$ ($A > B$). The same notation applies to vectors $x, y \in \mathbb{R}^n$.*

Definition 2.10. [18]. *A matrix $A \in \mathbb{R}^{n\times n}$ is an L-matrix if $a_{ii} > 0$, $i = 1, 2, \ldots, n$, and $a_{ij} \leq 0$, for all $i, j = 1, 2, \ldots, n$; $i \neq j$.*

Definition 2.11. [16]. *A matrix $A \in \mathbb{R}^{n\times n}$ is said to be an M-matrix if $a_{ij} \leq 0$, $i \neq j = 1, 2, \ldots, n$, A is nonsingular and $A^{-1} \geq 0$.*

Theorem 2.12. [14]. *Let $A = (a_{ij})$ and $B = (b_{ij})$ satisfy $A \leq B$ and $b_{ij} \leq 0$, for $i \neq j$. If A is an M-matrix, then B is an M-matrix*

Definition 2.13. [16]. *A matrix $A \in \mathbb{C}^{n\times n}$ is said to be an H-matrix if its comparison matrix, that is, the matrix $\langle A\rangle$ with elements $\alpha_{ii} = |a_{ii}|$, $i = 1, 2, \ldots, n$, and $\alpha_{ij} = -|a_{ij}|$, $i \neq j = 1, 2, \ldots, n$, is an M-matrix.*

Lemma 2.14. [8]. *A is an H-matrix if and only if there exist a vector $r > 0$ such that $\langle A\rangle r > 0$.*

Definition 2.15. [17]. *Let $A \in \mathbb{R}^{n\times n}$. The splitting $A = M - N$ is called:*

1. *M-splitting if M is a nonsingular M-matrix and $N \geq 0$.*
2. *H-compatible splitting if $< A >=< M > -|N|$.*

Lemma 2.16. [16]. *Let $A = M - N$ be an M-splitting of A, then $\rho(M^{-1}N) < 1$ iff A is a nonsingular M-matrix.*

Lemma 2.17. [1]. *Let A be an H-matrix. If $A = M - N$ is an H-compatible splitting, then $\rho(M^{-1}N) < 1$, i.e., the splitting is convergent.*

For generalized AOR method (2) the two following theorems are given in [15] and [11].

Theorem 2.18. *If $L \geq 0$ and $U \geq 0$, $(I - L - U)^{-1} \geq 0$ and $\rho(\gamma L) < 1$, then $\rho(L_{\gamma,\omega}) < 1$ for $0 \leq \gamma < \frac{2}{1+\rho(L_{0,1})}$ and $0 < \omega < \max(\frac{2\gamma}{1+\rho(L_{\gamma,\gamma})}, \frac{2}{1+\rho(L_{0,1})})$.*

Theorem 2.19. *If $L \geq 0$ and $U \geq 0$, $0 \leq \gamma, \omega \leq 1$, $\omega > 0$ and $\rho(\gamma L) < 1$, then*

(a) $\rho(L_{0,1}) = 0$ *iff* $\rho(L_{\gamma,\omega}) = 1 - \omega$.

(b) $\rho(L_{0,1}) = 1$ *iff* $\rho(L_{\gamma,\omega}) = 1$.

(c) $0 < \rho(L_{0,1}) < 1$ *iff* $1 - \omega < \rho(L_{\gamma,\omega}) < 1$.

(d) $\rho(L_{0,1}) > 1$ *iff* $\rho(L_{\gamma,\omega}) > 1$.

Lemma 2.20. *Let $A = V - L_V - U_V$ be a nonsingular H-matrix with unit diagonal entries that partitioned as in (4). If $I - D_2D_3$ has positive diagonal entries, then $V^{-1}A$ is an H-matrix.*

Proof. Since A is an H-matrix, we have from Definition 2.13 that $\langle A\rangle^{-1} \geq 0$. Denote $r = \langle A\rangle^{-1}e$, where $e = (1, 1, \ldots, 1)^T \in \mathbb{R}^{2l}$. Then $r > 0$. Let $r = (r_1^T, r_2^T)^T$, where $r_1, r_2 \in \mathbb{R}^l$. By using the definition of comparison matrix (Definition 2.13), we have

$$\begin{aligned} \langle A\rangle r &= \begin{pmatrix} I - |A_1 - I| & -|A_2| \\ -|A_3| & I - |A_4 - I| \end{pmatrix} r \\ &= \begin{pmatrix} (I - |A_1 - I|)r_1 - |A_2|r_2 \\ (I - |A_4 - I|)r_2 - |A_3|r_1 \end{pmatrix} \\ &= \begin{pmatrix} e_1 \\ e_1 \end{pmatrix} \end{aligned} \tag{15}$$

where $e_1 = (1, 1, \ldots, 1)^T \in \mathbb{R}^l$. We now show that $\langle V^{-1}A\rangle r > 0$ which will be useful to show that $V^{-1}A$ is an H-matrix (Lemma 2.14). From (8), we have

$$V^{-1}A = D \begin{pmatrix} A_1 - D_2A_3 & A_2 - D_2A_4 \\ A_3 - D_3A_1 & A_4 - D_3A_2 \end{pmatrix},$$

where

$$D = \begin{pmatrix} (I - D_2D_3)^{-1} & 0 \\ 0 & (I - D_2D_3)^{-1} \end{pmatrix}.$$

From the assumption $I - D_2D_3$ has positive diagonal entries, we have $D \geq 0$. So, from the definition of comparison matrix, we have

$$\begin{aligned}
&\langle V^{-1}A\rangle \\
&= D\begin{pmatrix} |I - D_2D_3| - |(A_1 - D_2A_3) - (I - D_2D_3)| & -|A_2 - D_2A_4| \\ -|A_3 - D_3A_1| & |I - D_2D_3| - |(A_4 - D_3A_2) - (I - D_2D_3)| \end{pmatrix} \\
&= D\begin{pmatrix} |I - D_2D_3| - |(A_1 - I) - D_2(A_3 - D_3)| & -|(A_2 - D_2) - D_2(A_4 - I)| \\ -|(A_3 - D_3) - D_3(A_1 - I)| & |I - D_2D_3| - |(A_4 - I) - D_3(A_2 - D_2)| \end{pmatrix} \\
&\geq D\begin{pmatrix} I - |D_2D_3| - |A_1 - I| - |D_2(A_3 - D_3)| & -|A_2 - D_2| - |D_2(A_4 - I)| \\ -|A_3 - D_3| - |D_3(A_1 - I)| & I - |D_2D_3| - |A_4 - I| - |D_3(A_2 - D_2)| \end{pmatrix} \\
&\geq D\begin{pmatrix} I - |A_1 - I| - |D_2|(|D_3| + |A_3 - D_3|) & |D_2|(I - |A_4 - I|) - (|D_2| + |A_2 - D_2|) \\ |D_3|(I - |A_1 - I|) - (|D_3| + |A_3 - D_3|) & I - |A_4 - I| - |D_3|(|D_2| + |A_2 - D_2|) \end{pmatrix} \\
&= D\begin{pmatrix} I - |A_1 - I| - |D_2||A_3| & |D_2|(I - |A_4 - I|) - |A_2| \\ |D_3|(I - |A_1 - I|) - |A_3| & I - |A_4 - I| - |D_3||A_2| \end{pmatrix}
\end{aligned}$$

By using the vector $r = (r_1^T, r_2^T)^T > 0$ and the equation (8), we have

$$\begin{aligned}
\langle V^{-1}A\rangle r &\geq D\begin{pmatrix} (I - |A_1 - I|)r_1 - |A_2|r_2 + |D_2|((I - |A_4 - I|)r_2 - |A_3|r_1) \\ (I - |A_4 - I|)r_2 - |A_3|r_1 + |D_3|((I - |A_1 - I|)r_1 - |A_2|r_2) \end{pmatrix} \\
&= D\begin{pmatrix} e_1 + |D_2|e_1 \\ e_1 + |D_3|e_1 \end{pmatrix} > 0.
\end{aligned}$$

□

Theorem 2.21. *Let $A = V - L_V - U_V$ be a nonsingular H-matrix with unit diagonal entries that partitioned as in (4). If $I - D_2D_3$ has positive diagonal entries and $0 \leq \gamma \leq \omega \leq 1$ and $\omega \neq 0$ then $\rho(\widetilde{L}_{\gamma,\omega}) < 1$.*

Proof. From Lemma 2.26, $V^{-1}A$ is an H-matrix. Let $V^{-1}A = M_1 - N_1$ be a splitting of the H-matrix $V^{-1}A$, where

$$M_1 = \frac{1}{\omega}(I - \gamma V^{-1}L_V) \quad \text{and} \quad N_1 = \frac{1}{\omega}[(1-\omega)I + (\omega - \gamma)V^{-1}L_V + \omega V^{-1}U_V].$$

Then it is easy to see that

$$< V^{-1}A >=< M_1 > -|N_1|.$$

So, from Definition 2.15, the splitting $V^{-1}A = M_1 - N_1$ is an H-compatible splitting and from Lemma 2.17, we have $\rho(M_1^{-1}N_1) < 1$. On the other hand, we observe that

$$\widetilde{L}_{\gamma,\omega} = M_1^{-1}N_1$$

so, we have $\rho(\widetilde{L}_{\gamma,\omega}) < 1$ and the AOR_V method converges. □

In the following, we give the theoretical results of AOR_V method for L-matrix and M-matrix.

Lemma 2.22. *Let A be an L-matrix with unit diagonal elements that partitioned as in (4) and assume that $I - D_2D_3$ has positive diagonal entries. Then we have the following results:*

(a) $V^{-1} \geq 0$.

(b) $L_V \geq 0$, $U_V \geq 0$, $V^{-1}L_V \geq 0$, *and* $V^{-1}U_V \geq 0$.

(c) $(I - \gamma V^{-1}L_V)^{-1} \geq 0$ *and* $(V - \gamma L_V)^{-1} \geq 0$.

(d) $\rho(L_V) = 0$.

Proof. (a) From (8) and the assumption $I - D_2D_3$ has positive diagonal entries, we have $V^{-1} \geq 0$.

(b) Since A is an L-matrix, we have $L_V \geq 0$, $U_V \geq 0$. So, from part (a), it follows $V^{-1}L_V \geq 0$ and $V^{-1}U_V \geq 0$.

(c) It is easy to see that $(V^{-1}L_V)^l = 0$. So, $\rho(V^{-1}L_V) = 0$. From parts (a) and (b), we have

$$(I - \gamma V^{-1}L_V)^{-1} = (I + \gamma V^{-1}L_V + \gamma^2(V^{-1}L_V)^2 + \cdots + \gamma^{l-1}(V^{-1}L_V)^{l-1}) \geq 0$$

and

$$(V - \gamma L_V)^{-1} = (I - \gamma V^{-1}L_V)^{-1}V^{-1} \geq 0.$$

(d) Part (d) follows from the fact that $(L_V)^l = 0$, which completes the proof.

□

Theorem 2.23. *Let $A = V - L_V - U_V$ be a nonsingular M-matrix with unit diagonal elements. Assume that $I - D_2D_3$ has positive diagonal entries, then*

(i) $\rho(\widetilde{L}_{0,1}) < 1$

(ii) $\rho(\widetilde{L}_{\gamma,\gamma}) < 1$ *for* $0 < \gamma \leq 1$

(iii) $0 < \rho(\widetilde{L}_{\gamma,\omega}) < 1$ *for* $0 \leq \gamma \leq \frac{2}{1+\rho(\widetilde{L}_{0,1})}$ *and* $0 < \omega < \max(\frac{2\gamma}{1+\rho(\widetilde{L}_{\gamma,\gamma})}, \frac{2}{1+\rho(\widetilde{L}_{0,1})})$

Proof. (i) By Lemma 2.22, the splitting $A = M - N$ with $M = V$ and $N = L_V + U_V$ is an M-splitting. So by Lemma 2.16, we have $\rho(\widetilde{L}_{0,1}) < 1$.

(ii) By Lemma 2.22, the splitting $A = M - N$ with $M = \frac{1}{\gamma}(V - \gamma L_V)$ and $N = \frac{1}{\gamma}[(1-\gamma)V + \gamma U_V]$ is an M-splitting for $0 < \gamma \leq 1$. So by Lemma 2.16, we have $\rho(\widetilde{L}_{\gamma,\gamma}) < 1$ for $0 < \gamma \leq 1$

(iii) Let $A = V - L_V - U_V$ and $B = I - L_V - U_V$. Then we have $A \leq B$ and $b_{ij} \leq 0$, for $i \neq j$. So, by Theorem 2.12, B is an M-matrix and we have $(I - L_V - U_V)^{-1} \geq 0$. Now by using parts (b), (d) of Lemma 2.22 and Theorem 2.18, we obtain the result given in (iii).

□

Theorem 2.24. *Let $A = V - L_V - U_V$ be a nonsingular L-matrix with unit diagonal elements. Assume that $I - D_2D_3$ has positive diagonal entries and $0 \leq \gamma\ , \omega\ \leq 1$, $\omega > 0$, then*

(a) $\rho(\widetilde{L}_{0,1}) = 0$ *iff* $\rho(\widetilde{L}_{\gamma,\omega}) = 1 - \omega$

(b) $\rho(\widetilde{L}_{0,1}) = 1$ *iff* $\rho(\widetilde{L}_{\gamma,\omega}) = 1$

(c) $0 < \rho(\widetilde{L}_{0,1}) < 1$ *iff* $1 - \omega < \rho(\widetilde{L}_{\gamma,\omega}) < 1$

(d) $\rho(\widetilde{L}_{0,1}) > 1$ *iff* $\rho(\widetilde{L}_{\gamma,\omega}) > 1$

Proof. The proof follows immediately from Lemma 2.22 and Theorem 2.19.

□

2.3 Hermitian Positive Definite Matrix

For Hermitian positive definite coefficient matrix A, in [2] the author considered the splitting $A = D - E - E^H$ in which D is any Hermitian positive definite matrix and the generalized AOR method

$$x^{(k+1)} = L_{\gamma,\omega}x^{(k)} + \omega(D - \gamma E)^{-1}b, \quad k = 0, 1, 2, \ldots, \tag{16}$$

where the iterative matrix of (16) is

$$L_{\gamma,\omega} = (D - \gamma E)^{-1}[(1 - \omega)D + (\omega - \gamma)E + \omega E^H].$$

And they also proved the following Theorem.

Theorem 2.25. [2]. *Let $A = D - E - E^H$ be an $n \times n$ Hermitian positive definite matrix where D is Hermitian and positive definite. Then the eigenvalues μ_i, $i = 1, \ldots, n$ of generalized Jacobi matrix $B = D^{-1}(E + E^H) = I - D^{-1}A$ are real and less than 1. Moreover, if $\mu_m = \min_i \mu_i$ and $\mu_M = \max_i \mu_i$ and $\mu_m \leq 0 \leq \mu_M$, then for the values*

$$\omega \in (0, 2) \quad \text{and} \quad \omega + \frac{2 - \omega}{\mu_m} < \gamma \leq \omega + \frac{2 - \omega}{\mu_M}$$

provided that $det(D - \gamma E) \neq 0$, then AOR method converges.

For Hermitian positive definite matrix $A = V - L_V - L_V^H$ we can state the following results.

Lemma 2.26. *Let A be an Hermitian matrix with unit diagonal elements. Then V is an Hermitian positive definite matrix iff $1 - |a_{i,l+i}| > 0$ for $i = 1, \ldots, l$.*

Proof. The proof follows from the fact that $D_3 = D_2^H$ and the eigenpairs of V are $\lambda_i = 1 \pm |a_{i,l+i}|$ and $u_i = \begin{bmatrix} e_i \\ \pm \frac{\bar{a}_{i,l+i}}{|a_{i,l+i}|} |e_i \end{bmatrix}$, $i = 1, \ldots, l$, where $e_i \in \mathbb{R}^l$ is the ith column of identity matrix □

Theorem 2.27. *Let A be an Hermitian positive definite matrix with unit diagonal elements. If $1 - |a_{i,l+i}| > 0$ for $i = 1, \ldots, l$ then*

(i) The eigenvalues μ_i, $i = 1, 2, \ldots, n$ of $\widetilde{L}_{0,1}$ are real and less than 1. Moreover $\mu_m \le 0 \le \mu_M$, where $\mu_m = \min_i \mu_i$ and $\mu_M = \max_i \mu_i$.

(ii) For the values $\omega \in (0, 2)$ and $\omega + \frac{2-\omega}{\mu_m} < \gamma \le \omega + \frac{2-\omega}{\mu_M}$ the AOR_V method converges.

Proof. (i) By Lemma 2.26, V is an Hermitian positive definite matrix. So from Theorem 2.25, the J_V method converges and the eigenvalues μ_i, $i = 1, \cdots n$, are real and less than one. From the definition of L_V and (8), it is easy to verify that trace($\widetilde{L}_{0,1}$)=trace($V^{-1}(L_V + L_V^H)) = 0$. So we have $\mu_m \le 0 \le \mu_M$.

(ii) The proof of part (ii) follows immediately from Theorem 2.25, part (i), and the fact that $\det(V - \gamma L_V) \ne 0$. □

3 Comparison Theorem

In this Section we need the following.

Theorem 3.1. [16]. *Let $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$ satisfy $0 \le |A| \le B$, then $0 \le \rho(A) \le \rho(|A|) \le \rho(B)$.*

Lemma 3.2. [16]. *Let $A \ge 0$ then*

1. *A has nonnegative real eigenvalue equal to its spectral radius $\rho(A)$.*
2. *$\rho(A)$ does not decrease when any entry of A is increased.*

Here for comparing the asymptotic rate of convergence or equivalently the spectral radii of the iteration matrices of the Jacobi and the J_V methods, we suppose that A has unit diagonal elements and define

$$B = \widetilde{L}_{0,1} = L_V + U_V$$

and

$$S = I - V. \tag{17}$$

So, from the definition of the Jacobi matrix J, we have

$$J = B + S. \tag{18}$$

We now state the following theorem.

Theorem 3.3. *Let $A = V - B$ be a nonsingular L-matrix and partitioned as in (4). If $I - D_2D_3$ has positive diagonal entries, then*

(a) $\rho(V^{-1}B) < 1$ if and only if $\rho(J) < 1$ and $\rho(V^{-1}B) \le \rho(J) < 1$.

(b) $\rho(V^{-1}B) \ge 1$ if and only if $\rho(J) \ge 1$ and $\rho(V^{-1}B) \ge \rho(J) \ge 1$.

Proof. By using the assumptions, we have $V^{-1}B \ge 0$ and $J = B + S \ge 0$. Let $\bar{\lambda} = \rho(V^{-1}B)$ and $\bar{\mu} = \rho(J)$. By Lemma 3.2, $\bar{\lambda}$ is an eigenvalue of $V^{-1}B$ and for some $x \ne 0$, we have $V^{-1}Bx = \bar{\lambda}x$, which implies that

$$(\bar{\lambda}S + B)x = \bar{\lambda}x.$$

Since $\bar{\lambda}$ is an eigenvalue of $\bar{\lambda}S + B$, we have

$$\bar{\lambda} \le \rho(\bar{\lambda}S + B).$$

If $\bar{\lambda} \le 1$, then by Theorem 3.1, $\rho(\bar{\lambda}S + B) \le \rho(S + B) = \bar{\mu}$, which implies that $\bar{\lambda} \le \bar{\mu}$. So, we have

(i) If $\bar{\lambda} \leq 1$, then $\bar{\lambda} \leq \bar{\mu}$.

On the other hand, if $\bar{\lambda} \geq 1$, then by Theorem 3.1, we have

$$\bar{\lambda} \leq \rho(\bar{\lambda}S + B) \leq \rho(\bar{\lambda}S + \bar{\lambda}B) = \bar{\lambda}\bar{\mu},$$

which implies that $\bar{\mu} \geq 1$. So, we have

(ii) if $\bar{\lambda} \geq 1$, then $\bar{\mu} \geq 1$.

Assume that $\bar{\mu} \geq 1$. By the definition of S, $(I - \frac{1}{\bar{\mu}}S)$ is nonsingular for $\bar{\mu} \geq 1$. Since $J = B + S \geq 0$, it follows, by Lemma 3.2, $\bar{\mu}$ is an eigenvalue of J. Therefore for some $y \neq 0$, we have $(B + S)y = \bar{\mu}y$ and

$$(I - \frac{1}{\bar{\mu}}S)^{-1}By = \bar{\mu}y. \quad (19)$$

In addition for $\bar{\mu} \geq 1$, we have

$$0 \leq (I - \frac{1}{\bar{\mu}}S)^{-1} \leq (I - S)^{-1} = V^{-1}$$

and

$$0 \leq (I - \frac{1}{\bar{\mu}}S)^{-1}B \leq V^{-1}B.$$

This together with Theorem 3.1 and equation (19) implies that

$$\bar{\mu} \leq \rho((I - \frac{1}{\bar{\mu}}S)^{-1}B) \leq \rho(V^{-1}B) = \bar{\lambda}.$$

Therefore,

(iii) if $\bar{\mu} \geq 1$ then $\bar{\lambda} \geq \bar{\mu} \geq 1$.

Now, by (i) and (iii), we have (a); and by (ii) and (iii), we have (b). □

4 Numerical results

In this section we give the numerical examples to illustrate the results obtained in Sections 2 and 3. All numerical experiments are carried out using MATLAB 7.9. In all Tables, we report the spectral radii of the corresponding iteration matrices for the classical AOR and the AOR$_V$ methods associated with the given matrices. The parameters γ and ω , $\omega \neq 0$ are chosen in the convergence intervals. In the examples below n represents the dimension of matrices. For the classical AOR and the AOR$_V$ methods, the experimentally computed optimal value of γ and ω were also used and the corresponding spectral radii of the iteration matrices are represented by $\rho^*(L_{\gamma,\,\omega})$ and $\rho^*(\widetilde{L}_{\gamma,\,\omega})$ in the Tables, respectively.

Example 4.1. *The coefficient matrix A of (1) is given by*

$$A_1 = \begin{pmatrix} 1 & 0.1 & 0.2 & 0.0 & 0.2 & -0.5 \\ 0.2 & 1 & 0.3 & 0.0 & -0.4 & 0.1 \\ 0.0 & 0.2 & 1 & -0.6 & 0.2 & 0.0 \\ 0.2 & -0.3 & 0.1 & 1 & 0.1 & 0.3 \\ 0.0 & 0.3 & 0.2 & 0.1 & 1 & 0.2 \\ 0.2 & -0.3 & 0.0 & -0.3 & 0.1 & 1 \end{pmatrix}$$

where A_1 is an irreducibly diagonally dominant matrix. The numerical results of this example are given in Table 1.

Example 4.2. *(See [9].) The coefficient matrix A of (1) is given by*

$$A_2 = \begin{pmatrix} 1 & q & r & s & q & \cdots \\ s & 1 & q & r & \ddots & q \\ q & s & \ddots & \ddots & \ddots & s \\ r & \ddots & \ddots & \ddots & q & r \\ s & \ddots & q & s & 1 & q \\ \vdots & s & r & q & s & 1 \end{pmatrix},$$

where $q = \frac{-1}{n}$, $r = \frac{-1}{n+1}$, $s = \frac{-1}{n+2}$ *and* $n = 10$. A_2 *is an M-matrix. The numerical results of this example are given in Table 2.*

Example 4.3. *(See[10].) The H-matrix A is given by*

$$A_3 = \begin{pmatrix} 1 & c_1 & c_2 & \frac{2}{n} & c_1 & c_2 & c_3 & \cdots & \cdots & c_1 & c_2 & c_3 \\ c_3 & 1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & c_2 \\ c_2 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & c_1 \\ c_1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & c_1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ c_3 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \frac{2}{n} \\ c_2 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & c_2 \\ c_1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & c_1 \\ \frac{2}{n} & c_1 & c_2 & c_3 & \cdots & \cdots & \cdots & \cdots & c_1 & c_2 & c_3 & 1 \end{pmatrix},$$

where $c_1 = \frac{1}{n+1}$, $c_2 = \frac{1}{n}$, $c_3 = \frac{1}{n+1}$. *The numerical results of this example are given in Table 3.*

Example 4.4. *The Hermitian Positive Definite matrix A is given by*

$$A_4 = \begin{pmatrix} t_0 & t_1 & t_2 & \cdots & \cdots & t_{n-1} \\ t_1 & t_0 & t_1 & t_2 & \ddots & \vdots \\ t_2 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & t_2 \\ \vdots & \ddots & \ddots & \ddots & \ddots & t_1 \\ t_{n-1} & \ddots & \ddots & t_2 & t_1 & t_0 \end{pmatrix},$$

where $t_k = \begin{cases} 1 + \frac{\pi^2}{5}, & k = 0, \\ \frac{4(-1)^k}{k^2}(\pi^2 - \frac{6}{k^2}), & k \neq 0. \end{cases}$ *This is an Toeplitz matrix which generated by* $f(\theta) = \theta^4 + 1$. *For* $n = 50$, *the numerical results are given in Table 4.*

Example 4.5. *(See [12].) The coefficient matrix A of (1) is given by*

$$A_5 = \begin{pmatrix} 1 & \frac{1}{20} - \frac{1}{20} & \frac{1}{30} - \frac{1}{20} & \frac{1}{40} - \frac{1}{20} & \cdots & \frac{1}{10n} - \frac{1}{20} \\ \frac{1}{20} - \frac{1}{20} & 1 & \frac{1}{30} - \frac{1}{20} & \frac{1}{40} - \frac{1}{20} & \cdots & \frac{1}{10n} - \frac{1}{20} \\ \frac{1}{30} - \frac{1}{20} & \frac{1}{20} - \frac{1}{20} & 1 & \frac{1}{40} - \frac{1}{20} & \cdots & \frac{1}{10n} - \frac{1}{20} \\ \frac{1}{40} - \frac{1}{20} & \frac{1}{30} - \frac{1}{20} & \frac{1}{20} - \frac{1}{20} & 1 & \cdots & \frac{1}{10n} - \frac{1}{20} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{10n} - \frac{1}{20} & \frac{1}{10(n-1)} - \frac{1}{20} & \frac{1}{10(n-2)} - \frac{1}{20} & \frac{1}{10(n-3)} - \frac{1}{20} & \cdots & 1 \end{pmatrix}$$

This is an L-matrix. The numerical results of this example are given in Tables 5 and 6.

Example 4.6. *The coefficient matrix A of (1) is given by*

$$A_6 = \begin{pmatrix} 1 & 1 & 4 & 2 \\ 1 & 1 & -1 & 4 \\ 4 & 1 & 1 & -1 \\ 1 & 4 & 1 & 1 \end{pmatrix}$$

Table 1. The results for Example 4.1.

γ	0.0000	0.2500	0.1250	0.4550	0.3433	0.6730	
ω	1.0000	0.5000	0.7540	0.8970	0.9725	1.0000	
$\rho(\widetilde{L}_{\gamma,\ \omega})$	0.5539	0.5927	0.5050	0.4057	0.4530	0.3447	$\rho^*(\widetilde{L}_{\gamma,\ \omega})$=0.1930
$\rho(L_{\gamma,\ \omega})$	0.6513	0.6571	0.6046	0.5017	0.5516	0.4379	$\rho^*(L_{\gamma,\ \omega})$=0.2385

Table 2. The results for Example 4.2.

γ	0.0000	0.3628	0.9588	0.6921	0.1597	0.6513	
ω	1.0000	0.5568	0.9468	0.6588	0.6548	0.9851	
$\rho(\widetilde{L}_{\gamma,\ \omega})$	0.8070	0.8738	0.6877	0.8210	0.8649	0.7389	$\rho^*(\widetilde{L}_{\gamma,\ \omega})$=0.6580
$\rho(L_{\gamma,\ \omega})$	0.8254	0.8856	0.7168	0.8374	0.8777	0.7629	$\rho^*(L_{\gamma,\ \omega})$=0.6901

Table 3. The results for Example 4.3.

γ	0.0000	0.5497	0.2678	0.0128	0.3549	0.8451	
ω	1.0000	0.9546	0.8912	0.5469	0.8452	0.9549	
$\rho(\widetilde{L}_{\gamma,\ \omega})$	0.7563	0.3219	0.4057	0.5873	0.3275	0.1867	$\rho^*(\widetilde{L}_{\gamma,\ \omega})$=0.1545
$\rho(L_{\gamma,\ \omega})$	0.9322	0.3971	0.5190	0.5627	0.3785	0.2039	$\rho^*(L_{\gamma,\ \omega})$=0.1995

Table 4. The results for Example 4.4.

γ	0.3258	0.5687	0.3658	0.3698	0.8457	0.9812	
ω	0.2685	0.9546	0.1254	0.5874	0.9523	0.5461	
$\rho(\widetilde{L}_{\gamma,\ \omega})$	0.9845	0.9355	0.9926	0.9651	0.9205	0.9490	$\rho^*(\widetilde{L}_{\gamma,\ \omega})$=0.9053
$\rho(L_{\gamma,\ \omega})$	0.9843	0.9340	0.9925	0.9646	0.9175	0.9463	$\rho^*(L_{\gamma,\ \omega})$=0.9000

Table 5. Results for Example 4.5.

γ	ω	n	$\rho(\widetilde{L}_{\gamma,\ \omega})$	$\rho(L_{\gamma,\ \omega})$
0	1	10	0.2302	0.2571
		30	1.1575	1.1505
0.3715	0.9245	10	0.2564	0.2807
		30	1.1820	1.1726
0.1258	0.4598	10	0.6411	0.6534
		30	1.0777	1.0741
0.7569	0.9512	10	0.1869	0.2110
		30	1.2489	1.2334

For this example, we have $\rho(L_{0,1}) = 5.29$*,* $\rho(L_{1,1}) = 16.40$*, and* $\rho(\widetilde{L}_{0,\ 1}) = 0.5211$*. So, the Jacobi and the Gauss-Seidel methods diverge, while the* J_V *and the* GS_V *methods converge.*

Remark. From Tables 1-4, it is easy to verify that the numerical results are consistent with the Theorems in Section 2. We observe that, for strictly diagonally dominant matrix, M-matrix, L-matrix, H-matrix, the rate of convergence of the AOR_V method is faster than the rate of convergence of classical AOR method, while for Hermitian positive matrix, the AOR method is a little faster than the AOR_V method. From Table 5, we get that the results are in concord with Theorem 2.24 and the Theorem 3.3 in Section 3. In this table, we observe that in the case of L-matrix, when the Jacobi method converges, the J_V method also converges, and when the J_V method diverges the Jacobi method diverges too. We also observe that when both the Jacobi and J_V methods converge, the spectral radius of the J_V method is smaller than that of the Jacobi method. From Example 4.6, we see that, there is a coefficient matrix, A_6, for which the J_V and GS_V methods converge, but the Jacobi and the Gauss-Seidel methods diverge.

5 Conclusion

In this paper, we introduced a new basic iterative method, called AOR_V method, for solving linear systems. We showed that, for strictly diagonally dominant matrix, M-matrix, L-matrix, H-matrix, and Hermitian positive matrix, the new generalized AOR (AOR_V) method converges, provided that the parameters γ and ω take particular values. We have shown that, for L-matrix, the rate of convergence of the J_V method is faster than the rate of convergence of Jacobi method. This method is well-defined even when some elements on the diagonal of A are zero. The numerical tests presented in this paper show the effectiveness of the proposed method.

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