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ORIGINAL PAPER

# Damage detection by updating structural models based on linear objective functions

**Alireza Entezami · Hashem Shariatmadar · Mansour Ghalehnovi**

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**Abstract** The objective of this article is to detect the location and severity of structural damage according to direct model updating of physical properties by a Moore–Penrose inverse problem. The proposed method is based on expanding the dynamic orthogonality conditions in a damaged structure for attaining the difference between physical properties of undamaged and damaged structures. Hence, a two-stage damage detection process consisting of localization and quantification of damage is established by using linear objective functions which are applied in the expanded orthogonality conditions. Thus, an error matrix which is characterized as damage index is determined to identify the location of damage. Subsequently, damage extent is quantified by applying a linear objective function in the extended eigenproblem of the damaged structure. Eventually, two numerical examples are utilized to validate the proposed damage detection approach. In these examples, the modal data are considered to be incomplete and the inverses of rectangular matrices are accomplished by the Moore–Penrose technique while no multipliers are used. Furthermore, in all damage investigations, the predicted damage is compared with the preset values of induced damage. It can be concluded that the damage localization approach proposed in this study can precisely identify the location of damage through updating process. Moreover, the obtained results confirm this technique as being appropriate to predict the severity of damage.

A. Entezami (✉) · H. Shariatmadar · M. Ghalehnovi
Department of Civil Engineering, Faculty of Engineering, Ferdowsi University of Mashhad, Mashhad, Iran
e-mail: entezami@stu-mail.um.ac.ir

H. Shariatmadar
e-mail: shariatmadar@um.ac.ir

M. Ghalehnovi
e-mail: ghalehnovi@um.ac.ir

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## 1 Introduction

Throughout recent years, detection of structural damage using measured or simulated dynamic data has emerged as a new research area in the fields of civil, mechanical and aerospace engineering. Many methods have been recently presented to identify structural damage using the changes in modal parameters and also dynamic properties of structures including mass and stiffness. Generally, the process of damage detection is divided into two stages. These stages are characterized as damage localization and damage quantification. Modal analysis is performed to find modal parameters, i.e., natural frequencies, mode shapes and damping ratios, which detect damage that occurred in the structure. It should be noted that, in the engineering practice, only a few of modal frequencies and partial mode shapes can be obtained by a modal testing for large flexible structures. Hence, many practical methods have been presented to apply the incomplete modal data in any dynamical structure for identification of system and also detection of damage [1].

Although classical damage detection methods, in which alterations of both modal parameters and dynamic properties are used, yield beneficial results in the damage identification process, they have got some weaknesses. That is, computational errors due to deficiencies in defining structure's initial pattern, existence of incomplete measured modes and also contamination in the modal data (noisy modal data) will lead to inappropriate results in the damage detection process. Therefore, a new finite element model updating technique is proposed to overcome these shortcomings.

In most cases, finite element model updating method is a process by which an initial model of a dynamical system is modified or updated in order to minimize the discrepancy between system response measurements and model predictions [2–4]. Furthermore, in modern techniques for analysis of structural dynamics, much effort is devoted to the derivation of accurate models. These accurate models are widely applied in civil engineering structures for cases like damage detection, health monitoring, structural control, structural evaluation and assessment [3]. In addition, finite element model of a structure is constructed on the basis of highly idealized engineering aspects that may not truly represent all of the physical characteristics of an actual structure. Hence, dynamic tests such as modal testing are performed to validate the estimated analytical model. This process is usually defined as the system identification technique. Therefore, in this study, fundamental concepts of model updating are applied to propose a new damage detection algorithm based on incomplete modal data. Indeed, the proposed method attempts to obviate some of the flaws and ambiguities in the prior techniques and utilizes verified model updating method which is used in system identification. Concerning this, preliminary modal-based damage detection methods attempt to use changes in the modal frequencies for localizing damages sites. In an earlier work by Cawley and Adams [5], it was shown that the ratio of frequency changes in different modes is only a function of location of damage and not the magnitude of damage. Moreover, Salawu [6] reviewed different methods of structural damage detection considering changes in natural frequencies. Analysis of changes in mode shapes due to damage represents another subgroup of modal-based methods. Hence, Pandey et al. [7] introduced a new technique that used mode shape curvatures, which are more sensitive to damage. In another class of damage identification methods, updating of individual dynamic properties of structures is used. In this regard, Yan and Golinval [8] proposed a damage diagnosis technique based on changes in dynamically measured amounts of flexibility and stiffness in structures. They investigated the behavior of a cantilever beam and also a simulated three-span bridge using their technique. In connection with model updating of dynamic systems, Baruch [9] proposed a method to correct the stiffness matrix according to the measured mode shapes obtained from vibration tests. This was done by minimizing a cost function for using the positive definite symmetric mass matrix as the weighting matrix. Berman [10] described the required changes in the mass matrix to satisfy the orthogonality relationship using a minimum-weighted Euclidean norm and also the method of Lagrange multipliers. In addition, Yang and Chen [11] presented methods to update the mass and stiffness matrices of structural models for reproducing the frequencies obtained from the updated structures. Moreover, Lee and Eun [12] proposed an improved method for updating the corrected stiffness and mass matrices based on measured dynamic modal data. They presented analytical equations based on the updated physical properties in the satisfaction of such dynamic constraints. Furthermore, the applicable approach for damage detection in the shear building frame was presented by Shiradhonkar and Shrikhande [13]. They proposed a new method for detecting and also locating damage in beams with the aid of vibration-based system identification and finite element model updating method. They used limited number of responses recorded during strong earthquakes for their study.

In this article, a procedure for detecting location and quantity of damages in dynamical structures is presented by direct model updating technique. The proposed method is based on expanding the dynamic orthogonality conditions in damaged structure for obtaining the difference of physical properties in undamaged and damaged structures. Hence, a two-stage damage detection process consisting of damage localization and damage quantification is established by using linear objective functions which are applied in the expanded orthogonality conditions. According to presented algorithm, an error matrix is determined to identify the damage sites in the first stage. Subsequently, the damage severity is determined by applying a linear objective function in the extended eigenproblem of damaged structure. The applicability and effectiveness of the proposed method are verified by two numerical examples. In these examples, the modal data are considered to be incomplete, and inverse of rectangular matrices is accomplished by the Moore–Penrose inverse matrix without using any multipliers. Moreover, in all damage investigations, the predicted damages are compared with the preset values of induced damages. Eventually, it can be inferred that the accurate results of numerical models provide a reliable and simple algorithm for damage detection when the modal data are incomplete.

## 2 Theory

Measured and analytical data are unlikely to be equal due to the existence of noise in the measurement process, model inadequacies, existence of some damage, etc. Mode shapes and natural frequencies obtained from incomplete modal tests do not usually satisfy the eigenvalue problem and orthogonality conditions. Thus, the mass and stiffness matrices need to be modified for simulation and design studies [14]. Hence, the direct model updating method is applied by using the analytical undamaged mass and stiffness matrices $M_u$ and $K_u$, respectively. Generally, the finite element method (FEM) is utilized to generate the initial properties of dynamical structures [15]. Moreover, modal parameters such as natural frequencies and mode shapes can be determined by the eigenvalue problem for both undamaged and damaged structures. Thus, alterations of modal data in the structure may lead to adverse dynamic performance which is an indication of damage. As a result of adverse changes in the modal parameters, individual physical properties of structures will be altered. Thus, the proposed method attempts to use fundamental concepts of the modal updating technique and identifies the location and severities of damages by linear objective functions. These functions are defined as discrepancy between physical properties of structures before and after of damage as follows:

$$\Delta M = M_u - M_d \tag{1}$$

$$\Delta K = K_u - K_d \tag{2}$$

In all damage cases, effect of changes of mass on occurrence of damage is much less than that of stiffness. Therefore, in this study, changes of stiffness matrix are described as damage index. The mode shapes and natural frequencies obtained from vibration tests or simulated modal analysis are often incomplete and do not usually satisfy the dynamic equation and orthogonality requirements [16]. Thus, the desired mass and stiffness matrices must be modified to satisfy the eigenvalue equation and the orthogonality constraints using modal test data. Hence, initial constraint equations for damage detection process are the basic orthogonality condition and eigenvalue equation. These equations can be defined as follows:

$$\varphi^T K \varphi = \Lambda \tag{3}$$

$$K\varphi = M\varphi\Lambda \tag{4}$$

where $M$ and $K$ are individual mass and stiffness matrices of structure, respectively. Also, $\varphi$ and $\Lambda$ are the vibrational modal data that can be defined as eigenvector and eigenvalue, respectively. For damage detecting process, Eq. (4) is initially used to localize damage sites. This process will be implemented by a new direct model updating technique based on a linear objective function. In the present study, damage is considered to be directly related to a decrease in stiffness; hence the dynamic orthogonality conditions in damage state can be rewritten as:

$$\varphi_d^T K_d \varphi_d = \Lambda_d \tag{5}$$

$$K_d \varphi_d = M_u \varphi_d \Lambda_d \tag{6}$$

In these expressions subscripts, d and u, denote damaged and undamaged states of structure. Moreover, it is assumed that both undamaged and error stiffness matrices are utilized instead of calculating the stiffness of damaged structure. Since the undamaged stiffness matrix has previously been determined by finite element method, the stiffness error matrix is the only unknown parameter which must be calculated. Substituting Eq. (2) in Eq. (6) we have:

$$(K_u + \Delta K)\varphi_d = M_u \varphi_d \Lambda_d \tag{7}$$

or

$$\Delta K \varphi_d = M_u \varphi_d \Lambda_d - K_u \varphi_d \tag{8}$$

To determine the stiffness error matrix, an objective function of Berman and Nagy [17] is applied which can be expressed as:

$$J = \left\| M_u^{-1/2}(K_d - K_u)M_u^{-1/2} \right\| = \left\| M_u^{-1/2} \Delta K\, M_u^{-1/2} \right\| \tag{9}$$

Modification of constraint equation of Eq. (9), yields:

$$\begin{aligned} M_u^{-1/2}(\Delta K)M_u^{-1/2}M_u^{-1/2}\varphi_d = M_u^{-1/2}M_u\varphi_d\Lambda_d \\ - M_u^{-1/2}K_u\varphi_d \end{aligned} \tag{10}$$

As the mode shape matrix $\varphi$ is also rectangular, solution of Eq. (10) requires a pseudo-inverse technique. As discussed before, in this study, the Moore–Penrose inverse problem is applied to overcome the complicated mathematical solutions. Accordingly, Eq. (10) can be rewritten after being solved by Moore–Penrose approach:

$$\begin{aligned} M_u^{-1/2}(\Delta K)M_u^{-1/2} &= \left(M_u^{-1/2}M_u\varphi_d\Lambda_d - M_u^{-1/2}K_u\varphi_d\right) \\ &\times \left(M_u^{1/2}\varphi_d\right)^{+} + y_1\left[I - \left(M_u^{1/2}\varphi_d\right)\left(M_u^{1/2}\varphi_d\right)^{+}\right] \end{aligned} \tag{11}$$

where $y_1$ denotes an $(N \times N)$ arbitrary matrix, $I$ is an $(N \times N)$ identity matrix and "+" denotes inversing the rectangular matrices by Moore–Penrose method. It is clear that the left-hand sides of Eq. (11) is exactly similar to the right-hand sides of Eq. (9) and so the linear objective function is minimized in order to solve the arbitrary matrix as follows:

$$\begin{aligned} y_1 = &\left(-M_u^{-1/2}M_u\varphi_d\Lambda_d + M_u^{-1/2}K_u\varphi_d\right)\left(M_u^{1/2}\varphi_d\right)^{+} \\ &\times \left[I - \left(M_u^{1/2}\varphi_d\right)\left(M_u^{1/2}\varphi_d\right)^{+}\right] \\ &+ y_2\left(M_u^{1/2}\varphi_d\right)\left(M_u^{1/2}\varphi_d\right)^{+} \end{aligned} \tag{12}$$

With expanding and compacting Eq. (12), the arbitrary matrix $y_1$ will be solved to form:

$$y_1 = y_2\left(M_u^{1/2}\varphi_d\right)\left(M_u^{1/2}\varphi_d\right)^+ \tag{13}$$

where $y_2$ is an $(N \times N)$ arbitrary matrix. Substituting Eq. (13) into Eq. (11) the result can be expressed as:

$$M_u^{-1/2}(\Delta K)M_u^{-1/2} = \left(M_u^{-1/2}M_u\varphi_d\Lambda_d - M_u^{-1/2}K_u\varphi_d\right) \times \left(M_u^{1/2}\varphi_d\right)^+ \tag{14}$$

Pre-multiplying and post-multiplying both sides of Eq. (14) by $M_u^{-1/2}$, the change of stiffness error matrix can be defined as:

$$\Delta K = (M_u\varphi_d\Lambda_d - K_u\varphi_d)\left(M_u^{1/2}\varphi_d\right)^+ M_u^{1/2} \tag{15}$$

As can be seen, the damage index based on error stiffness matrix $\Delta K$ is formulated by both initial physical properties of undamaged structures and modal parameters before and after of damage. Indeed, Eq. (15) provides some general information regarding damage localization process based on change of stiffness matrix. In other words, stiffness damage localization index depends on the geometry and also combination of degrees of freedom for constructing discrete matrices. Hence, it is not possible to exactly detect locations of damage according to initial values of Eq. (15). The diagonals changes of stiffness discrepancy matrix are utilized to find its maximum values. This modification leads to a simpler damage localization process in comparison with Eq. (15). Hence, the stiffness damage localization indicator dR is presented as diagonal of the stiffness error matrix as follows:

$$\mathrm{dR} = \sum_{i=1}^{m}\left|\frac{\delta k}{k_u}\right| \tag{16}$$

where $\delta k$ and $k_u$ are the diagonals of discrepancy stiffness matrix, $\Delta K$ is the stiffness of undamaged structure $K_u$, respectively. This method detects location of damage readily and more precisely than Eq. (15). Indeed, the stiffness error matrix in dR equation has particularly obviated its data vagueness. Therefore, the changes in diagonals of $\Delta K$ divided by the correspondent quantities in undamaged stiffness matrix can determine maximum values of Eq. (16).

To attain damage severity, the eigenvalue problem of damaged structures is used which is almost similar to damage localization approach. Therefore, Eq. (6) is rewritten to find an efficient algorithm for determining damage quantity:

$$K_d\varphi_{dj} = \lambda_{dj}M_u\varphi_{dj} \tag{17}$$

where $\lambda$ is a component of diagonal eigenvalue matrix and $j$ is the available computed or measured number of modal parameters before and after the damage. Considering $\varphi$ as the mode shape of undamaged structure and pre-multiplying Eq. (8) by $\varphi_j^T$, it can be deduced as:

$$\varphi_j^T K_d\varphi_{dj} = \lambda_{dj}\varphi_j^T M_u\varphi_{dj} \tag{18}$$

As mentioned before, $K_d$ is an unknown matrix that can be defined according to the difference between the undamaged and error stiffness matrices. Accordingly, substituting the modified Eq. (2) in Eq. (18) will yield Eq. (19) as:

$$\varphi_j^T(K_u + \Delta K)\varphi_{dj} = \lambda_{dj}\varphi_j^T M_u\varphi_{dj} \tag{19}$$

By expanding Eq. (19), the basic equation for damage quantification can be expressed as:

$$\varphi_j^T\Delta K\varphi_{dj} = \lambda_{dj}\varphi_j^T M_u\varphi_{dj} - \varphi_j^T K_u\varphi_{dj} \tag{20}$$

The above equation cannot provide a unique and compressed equation for determining the damage quantity. As can be seen, this equation is derived by combining information of the structures and modal parameters before and after the damage. Thus, it is appropriate to change it and generate a simpler and more useful equation. In this regard, the modified mentioned expression can be rewritten as:

$$\varphi_{dj}^T\Delta K\varphi_j = \lambda_{dj}\varphi_{dj}^T M_u\varphi_j - \varphi_{dj}^T\lambda_j M_u\varphi_j \tag{21}$$

or

$$\varphi_{dj}^T\Delta K\varphi_j = \left(\lambda_{dj} - \lambda_j\right)\varphi_{dj}^T M_u\varphi_j \tag{22}$$

For damage quantification, the unknown stiffness error matrix should be defined by known parameters such as undamaged stiffness matrix and unknown damage index. Hence, in the proportional damage model $\Delta K$ are expressed by multiplying undamaged stiffness matrix by damage index as follows:

$$\Delta K = \sum_{e=1}^{N_d}(K_u)_e\beta_e \tag{23}$$

where $K_u$ is the stiffness matrix of undamaged structure, $\beta$ is the proportional stiffness damage modification factor (damage index) for element e and $N_d$ is number of damaged element, respectively. Therefore, Eq. (22) can be rewritten based on damage index $\beta$ as follows

$$\sum_{e=1}^{N_d}\left(\varphi_{dj}^T K_u\varphi_j\right)_e\beta_e = \left(\lambda_{dj} - \lambda_j\right)\varphi_{dj}^T M_u\varphi_j \tag{24}$$

It should be noted that both the undamaged and damaged modal parameters and also initial properties of structure can be obtained from a finite element modeling of the structure. Assuming that $N_d$ damaged element is available and also compressing Eq. (24) leads to a simpler equation as:

$$S \cdot \beta = P \tag{25}$$

where $S$ is a $(N_{\mathrm{d}} \times N_{\mathrm{d}})$ matrix and $\beta$ is $(N_{\mathrm{d}} \times 1)$ vector which can described as:

$$S = \varphi_{\mathrm{d}j}^{\mathrm{T}} K_{\mathrm{u}} \varphi_j \tag{26}$$

$$P = \left(\lambda_{\mathrm{d}j} - \lambda_j\right) \varphi_{\mathrm{d}j}^{\mathrm{T}} M_{\mathrm{u}} \varphi_j \tag{27}$$

For incomplete modal data, it is possible to solve Eq. (25) by defining a linear objective function that consists of both initial properties of undamaged structure and also undamaged and damaged modal data. This objective function can be determined as:

$$J = \left\| (S \cdot \beta - P)^{\mathrm{T}} \beta \ (S \cdot \beta - P) \right\| \tag{28}$$

or

$$J = \left\| \left( \left(\varphi_{\mathrm{d}}^{\mathrm{T}} K_{\mathrm{u}} \varphi\right) \beta - (\lambda_{\mathrm{d}} - \lambda) \varphi_{\mathrm{d}}^{\mathrm{T}} M_{\mathrm{u}} \varphi \right)^{\mathrm{T}} \beta \left( \left(\varphi_{\mathrm{d}}^{\mathrm{T}} K_{\mathrm{u}} \varphi\right) \beta - (\lambda_{\mathrm{d}} - \lambda) \varphi_{\mathrm{d}}^{\mathrm{T}} M_{\mathrm{u}} \varphi \right) \right\| \tag{29}$$

Making use of Moore–Penrose inverse problem, Eq. (25) is solved by minimizing aforementioned objective function similarly to damage localization approach. A general solution of Eq. (25) is as follows:

$$\beta = S^{+} P + z_1 (S^{+} S - I) \tag{30}$$

where $z_1$ denotes an $(N_{\mathrm{d}} \times N_{\mathrm{d}})$ arbitrary matrix and $I$ is an $(N_{\mathrm{d}} \times N_{\mathrm{d}})$ identity matrix. Considering the condition of minimizing objective function, the arbitrary matrix $z_1$ is solved as:

$$z_1 = S^{+} P (I - S S^{+})^{+} + z_2 (S^{+} P - I) \tag{31}$$

where $z_2$ is an $(N_{\mathrm{d}} \times N_{\mathrm{d}})$ arbitrary matrix. Once again, Eq. (31) is substituted into Eq. (30) and arranging the damage severity based on stiffness changes yields:

$$\beta = S^{+} P + (S^{+} S - P) \tag{32}$$

Equation (32) is a unique solution to sensitivity analysis of stiffness matrix which introduces as damage index. Moreover, the Moore–Penrose inversing method provides a simple procedure to use incomplete modal data and correct results of damage detection method.

In experimental modal tests, there may be some deviations in the results due to the existence of noise in measurements. In the numerical examples, this noise is simulated by adding a series of pseudo-random numbers to the theoretically calculated frequencies and mode shapes [18]. In other words, due to the complexity of the measurement process, an amount of noise may be inserted in measured data which contaminates the modal parameters. Thus, in order to investigate the effect of noise on the results obtained by proposed damage detection method a random noise is considered as follows:

$$\varphi_j^{*} = \varphi_j (1 + \xi_{\mathrm{r}}) \tag{33}$$

where $\varphi^{*}$ and $\varphi$ are the eigenvector components of the $j$th mode with and without noise, respectively. Moreover, $\xi_{\mathrm{r}}$ is a random number. In this study, a value equal to 1 % is applied to mode shapes and natural frequencies as proportional random noise. Accordingly, the damage detection algorithm will be repeated by a series of error-contaminated data created by Eq. (33).

## 3 Numerical investigation

### 3.1 A planner truss

A two-dimensional truss as shown in Fig. 1 was used to investigate the characteristics of damage detection algorithm proposed in this study. This structure's material has basic properties as Young modules $E = 200$ GPa and density $\rho = 7{,}850$ kg/m$^3$. The cross-section considered for all members of this planar truss are L-shaped double equal angles of 100 mm width and 5 mm thickness. Furthermore, each node of the truss has two degrees of freedom (DOF). The first three vibration modes of the structure are considered to study the damage.

This structure is a continuous dynamical system and its mass and stiffness matrices can be determined by basic concepts of finite element method [19]. After determining the physical parameters of intact truss structure, the generalized eigenvalue problem is used and so the modal parameters including natural frequencies and mode shapes are calculated. Main assumption considered here is that the proportional damping dominates the dynamic behavior of structure. Consequently the modal parameters are extracted as real data.

Four damage cases are imposed to elements of planner truss for damage detecting process and also exploring number of damaged elements in the results. In the first damage case, the stiffness of elements 2 and 14 is reduced by 15 %. In damage case number two, the stiffness of elements 6, 9 and 12 is decreased by 20, 25 and 30 %, respectively. In damage case number three; there is a 10, 15 and 20 % decline in the amount of stiffness in elements 3, 8 and 15, respectively. Finally, in damage case number four, the stiffness of elements 6 and 13 is decreased by 20 and 30 %, respectively.

To verify the formulation of proposed method, actual severities of damage in different cases that occur in the planner truss are theoretically inserted in initial properties of structure. This leads to a change in structure's physical properties and so occurrence of damage. As discussed before, changes characterised as damage cause adverse dynamical behavior. Thus, the damage quantities

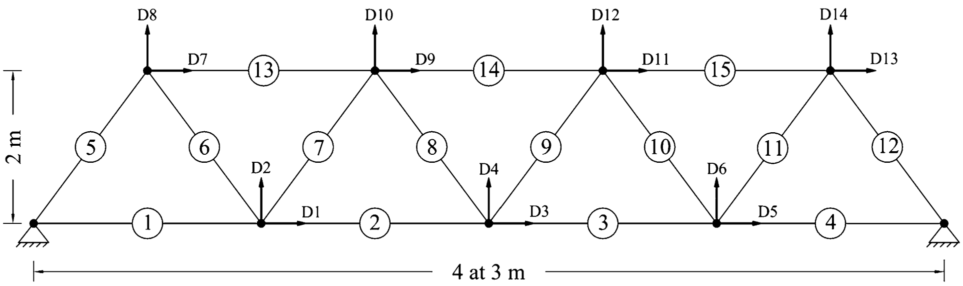



**Fig. 1** A planner truss

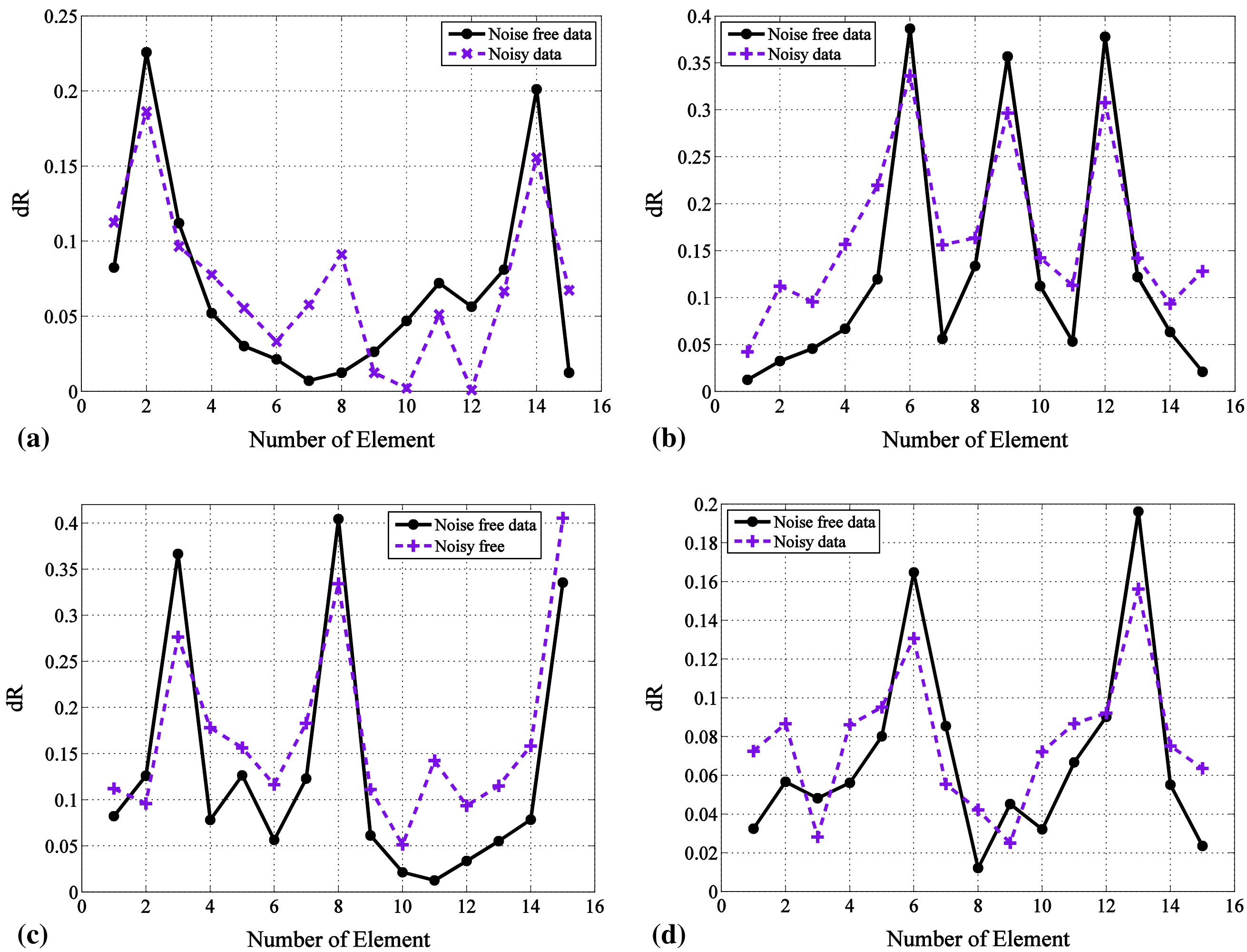



**Fig. 2** Damage localization of the planner truss, **a** damage case 1, **b** damage case 2, **c** damage case 3, **d** damage case 4

considered in the numerical study are inserted in stiffness matrix of undamaged structure. According to the measurement of incomplete modes and also contaminating the modal parameters with noise, locations of damages in different cases are identified by proposed direct model updating method. Figure 2a–d illustrates the location of induced damages in the cases of modal data contaminated and not contaminated with noise.

The location of damaged elements can be detected using direct model updating of stiffness matrix as the dR indicator. In the case of noise-free modal data, highest peaks in diagrams of Fig. 2 point to the exact location of damages. As a result of inserting noise in modal data, damage localization diagrams of the truss are altered to some extent. However, the amount of this alteration in dR is not considerable and there is no significant change in localization diagrams. In fact, location of damage is clear in the diagrams in the case of noisy data as well. Moreover, amount of dR in undamaged elements of the truss is altered. As this

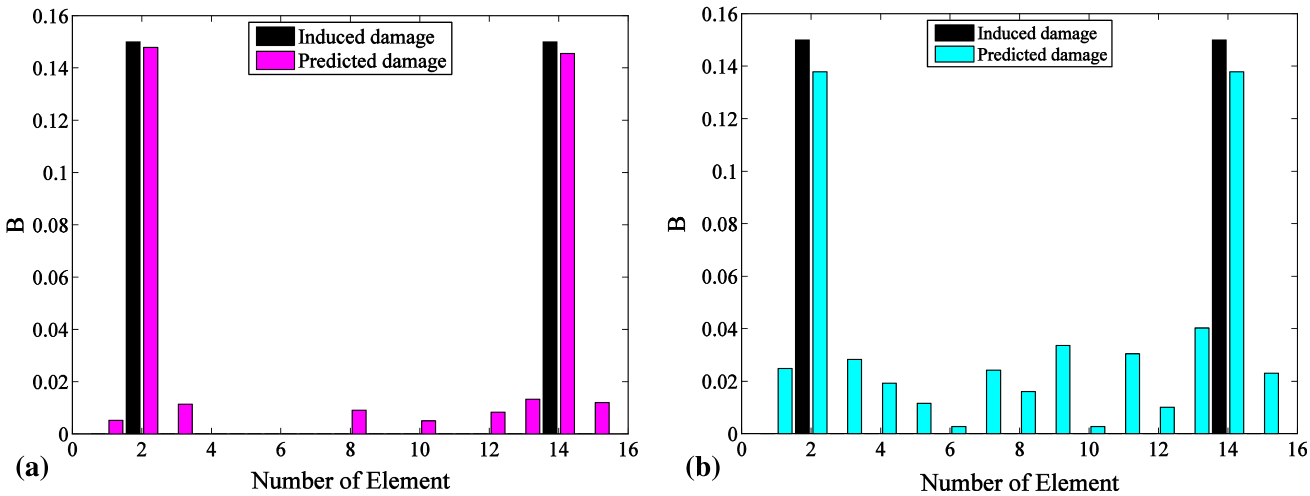



**Fig. 3** Damage quantification in planner truss in the damage case 1, **a** noise-free data, **b** noisy data

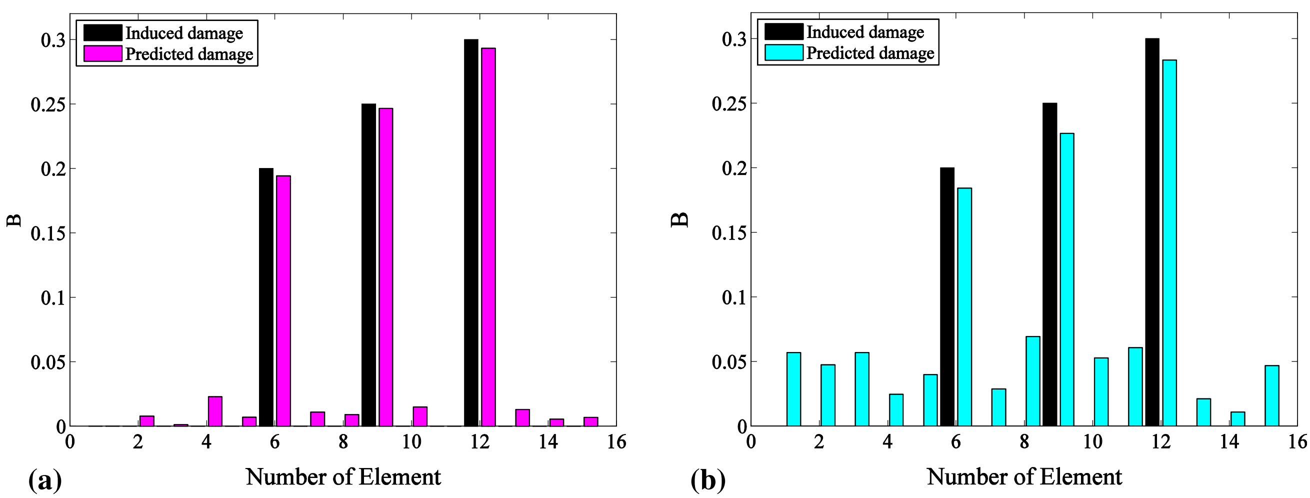



**Fig. 4** Damage quantification in the planner truss in the damage case 2, **a** noise-free data, **b** noisy data

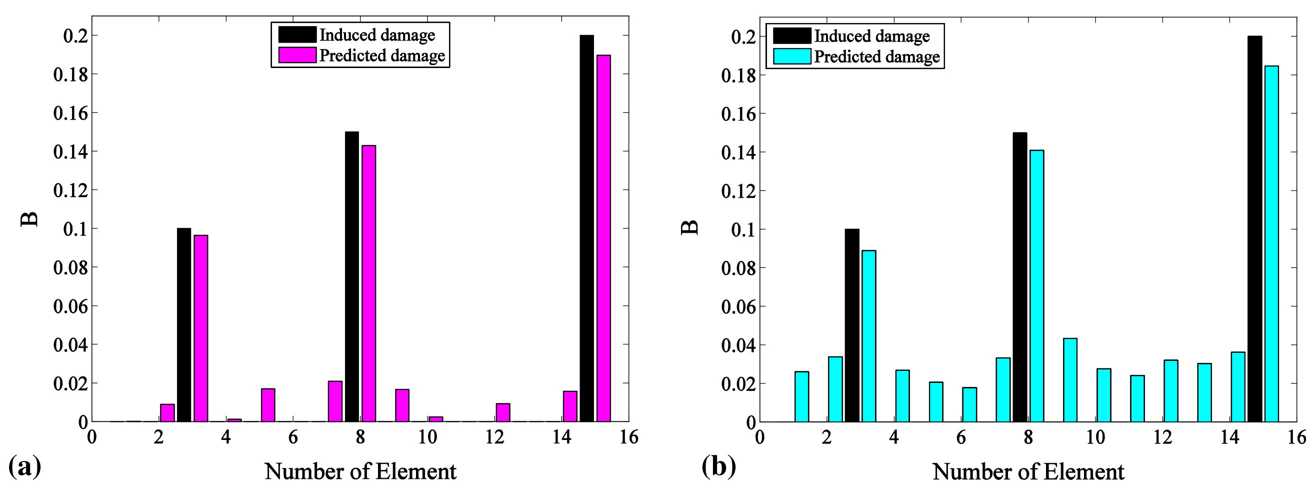



**Fig. 5** Damage quantification in the planner truss in the damage case 3, **a** noise-free data, **b** noisy data

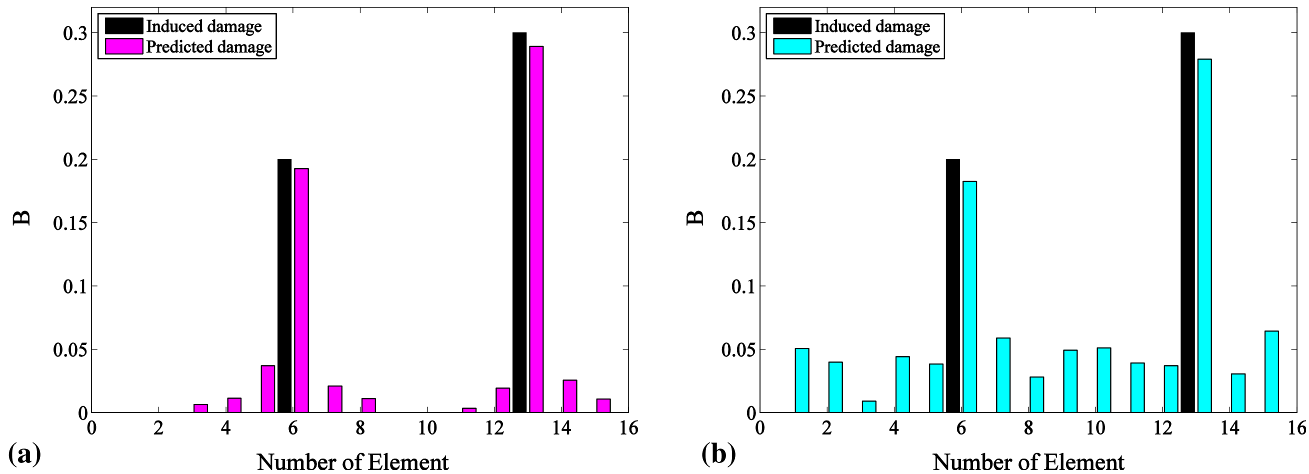



**Fig. 6** Damage quantification in the planner truss in the damage case 4, **a** noise-free data, **b** noisy data

alteration is negligible, there is no significant influence on the damage localization process. Accordingly, it can be deduced that limitations like incomplete modes and insertion of noise in modal data do not cause a failure in the damage localization process in which model updating method is utilized.

After determining locations of damage, Eq. (32) is used to get the severity of damages. Initially, it is assumed that the noise-free data have a commanding influence on damage detection algorithm. Figures 3, 4, 5 and 6 indicate the severity of damages that can be characterized as stiffness modification factor $\beta$. Then, a random amount of noise is inserted in modal data and the damage detection process is performed once again. Effect of inserting noise in modal data on damage detection can be observed in part (b) of Figs. 3, 4, 5 and 6.

As can be seen, in the first part of above figures, severities of damage have accurately been determined when the modal data are not contaminated by random noise. It is of great importance that the amounts of error in predicting the damage severity are <2 %. In addition, quantities of computational error in undamaged elements are negligible in comparison with the predicted severities of damage. In the case of noisy modal data, amounts of errors generated in prediction of damages are <5–10 %. These small amounts of error indicate that proposed damage quantification method is potentially able to determine any damage severity even in the case of contaminated modes. It is clear that quantities of computational error in undamaged elements are larger in comparison with the case of noise-free data. Nevertheless, these errors are insignificant taking into account the modal data as being incomplete and also susceptible to noise contamination. Thus, the proposed damage detection method is able to determine the damage extent to a satisfactory standard.

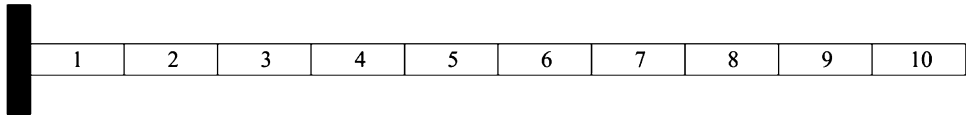
## 3.2 A cantilever beam

In this section, the damage detection algorithm described previously is used to identify the location and severity of stiffness reduction in a cantilever beam. The mentioned cantilever beam is illustrated in Fig. 7. Length, thickness and width of this beam are equal to 1.20, 0.05 and 0.1 m, respectively. Furthermore, beam's mass density and elasticity modulus are assumed to be 7,850 kg/m$^3$ and 210 GPa, respectively. In this example, the first three vibrating modes of structure are used for investigating the damage. It is considered that incomplete modal data are available accordingly.

Modal analysis is carried out to simulate the modal data using two-node beam elements [15]. Here, four damage cases are assumed to investigate capabilities of proposed method to detect damage in a flexural structure (beam). In the first damage case, the stiffness of element 2 is decreased by 20 %. In damage case number two; there is 15 % reduction in the stiffness of element 5. In the third damage case, the stiffness of elements 2 and 5 decreases by 15 % and 25 %, respectively. Finally, in the damage case number four, the stiffness of element 8 is declined by 25 %. Based on the proposed damage evaluation algorithm, location of induced damages is detected similarly to prior section. Figure 8a, b shows the location of induced damages in the cases consisting of noise-free and noisy data.

As shown in the above diagrams, induced damages are identified before and after the noise effect. It is clear that location of damage is identified correctly. Similarly to planar truss, dR index could precisely predict the damage locations only using the changes in the diagonal values of the stiffness error matrix even in the case of noisy modal data. Once again, the highest peaks of dR diagrams show the damage sites. It should be mentioned that, in this case,

**Fig. 7** A cantilever beam



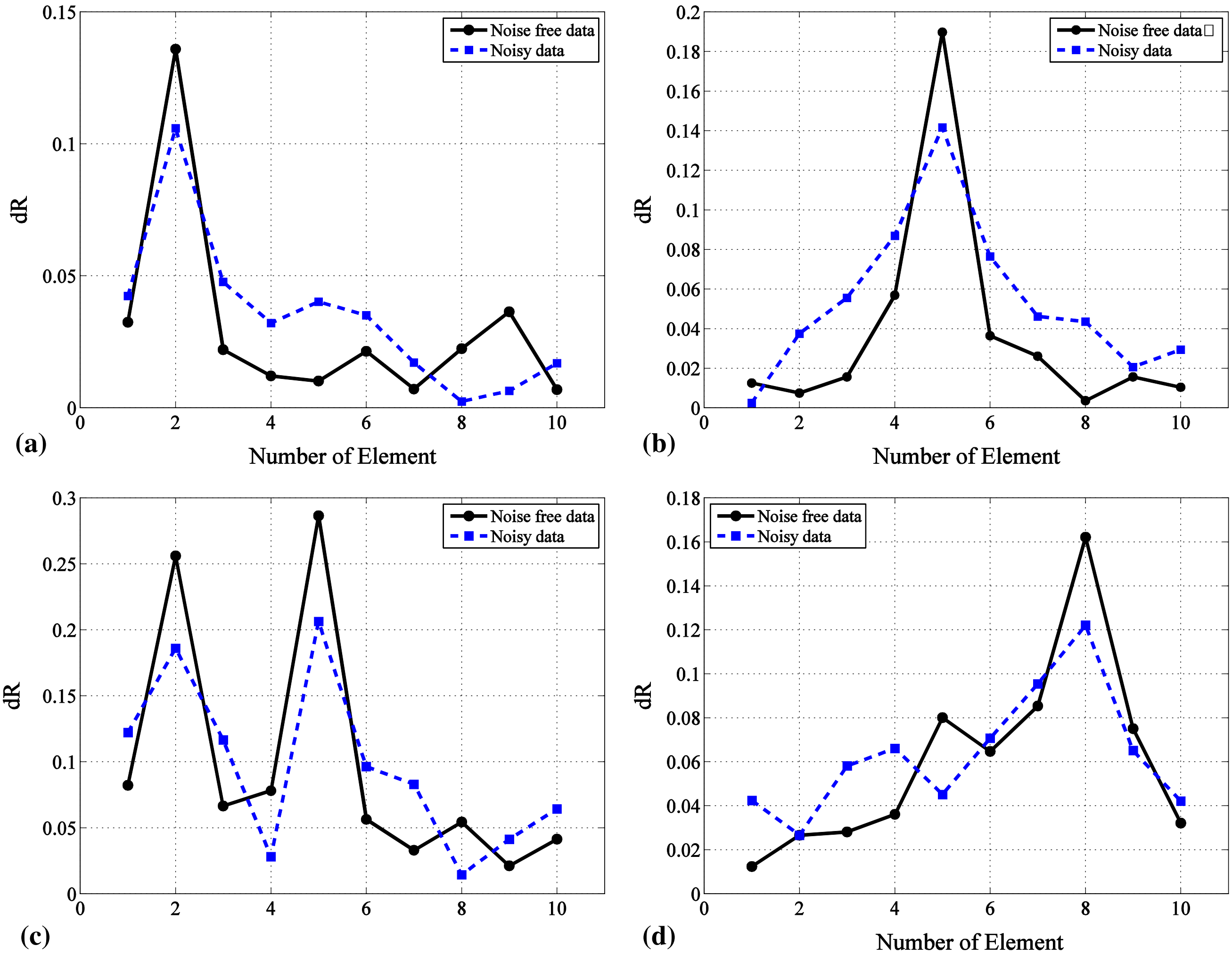




**Fig. 8** Damage localization in the cantilever beam, **a** damage case 1, **b** damage case 2, **c** damage case 3, **d** damage case 4

noisy modal data leads to an alteration in damage localization diagrams. In this regard, peaks of dR indices still indicate the damage locations. Therefore, the noisy data and incomplete measure modes do not corrupt the final results of damage localization. Indeed, merits of model updating technique have diminished the influences of some limitations such as noisy data, incomplete modes and errors in the structural modeling on the damage localization approach. Moreover, amounts of errors in the cantilever beam are less than the planner truss due to simplicity in structural modeling. Thus, it can be deduced that accurate modeling of structures lead to more precise results while carrying out damage detection process. For damage quantification, $S$ and $P$ matrices are initially calculated based on given information of both undamaged and damaged structures. Subsequently, the Moore–Penrose inverse problem is utilized to compute the pseudo-inverse of the $S$ matrix. As a final sequence, the vectors of damage quantity are determined for all damage cases using Eq. (32). Shown in Figs. 9, 10, 11 and 12 are amounts of $\beta$ in all damage cases in the cantilever beam for two states consisting of modal data without noise and noisy data.

As shown in Figs. 9a, 10, 11 and 12a, damage severities are predicted with an error <2 % in the case of noise-free data. For noisy data, amounts of error are <8 % when comparing predicted and induced damages. Quantity of damage in healthy elements is negligible in comparison with that in damaged elements. Hence, it can be inferred that in spite of errors being generated in the calculation process the proposed method can appropriately assess

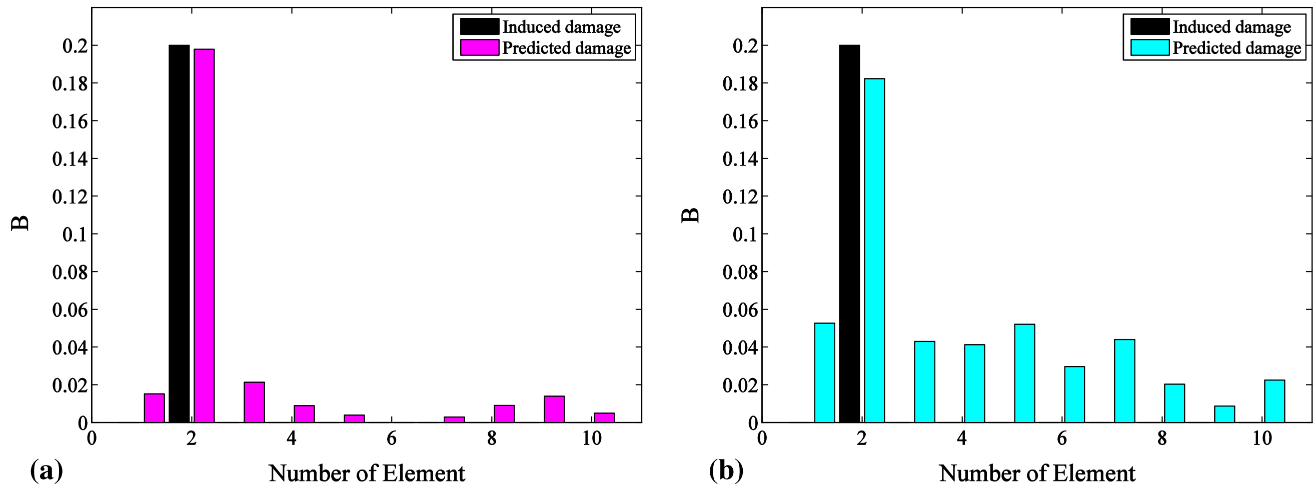



**Fig. 9** Damage quantification in the cantilever beam for damage case 1, **a** noise-free data, **b** noisy data

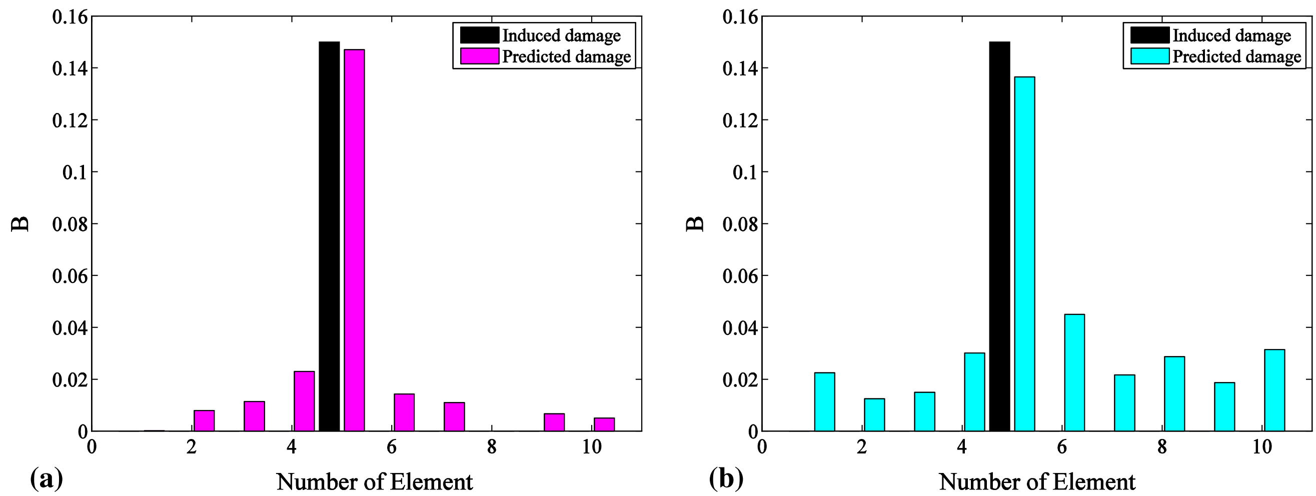



**Fig. 10** Damage quantification in the cantilever beam for damage case 2, **a** noise-free data, **b** noisy data

damage severity even when noise is inserted in modal data and also incomplete modes are applied.

## 4 Conclusion

In this paper, a direct model updating technique is presented to identify location as well as severity of structural damages. At each damage stage, a separation objective function is used which is related to updating of stiffness of structure. Therefore, the damage detection method proposed in this study makes use of fundamental concepts of model updating technique. To overcome the difficulty in identifying incomplete modal data, Moore–Penrose method is applied to invert the rectangular matrix of vibrational modes. Initially, the damage detection process identifies location of damage by determining the stiffness error matrix. This matrix is generally defined as the discrepancy between stiffness matrices of undamaged and damaged structures. It is not possible to detect the locations of damage exactly using original values of stiffness error matrix nonetheless. Therefore, changes in diagonal of this matrix are utilized to extract the maximum values which are characterized as damage sites. Having predicted the damage locations, a new linear objective function is introduced to compute damage severities. Accordingly, expansion of the orthogonality conditions and eigenvalue problem provide a correct and certain solution for damage quantification algorithm when incomplete modal data are present. In this regard, a stiffness modification factor is suggested to predict the damage severities. Furthermore, noisy modal parameters of both undamaged and damaged structures are utilized to indicate that the proposed method is potentially able to deal with any obstacle for achieving accurate results in damage

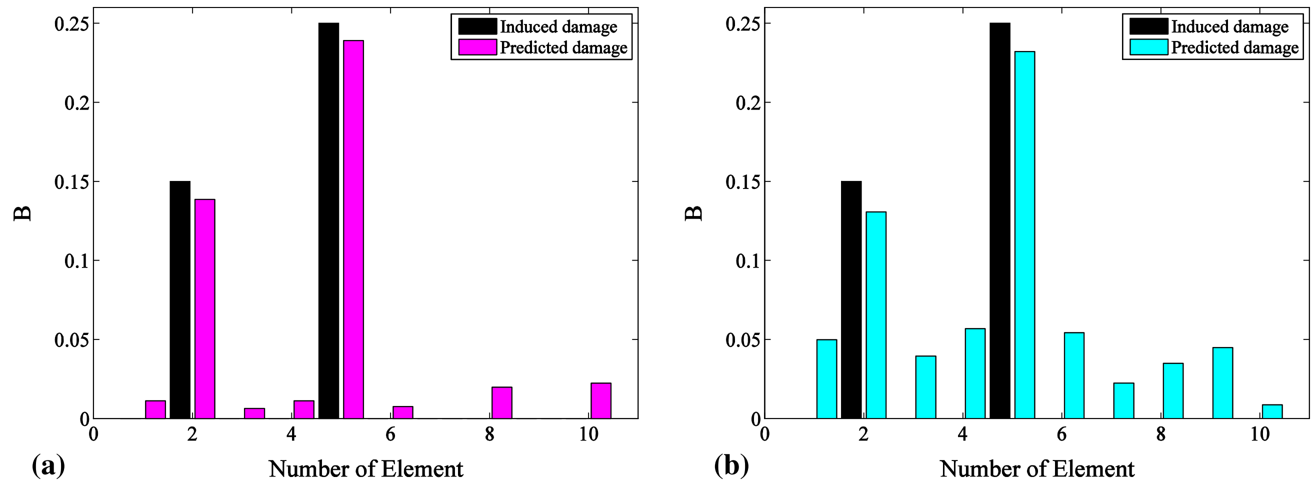



**Fig. 11** Damage quantification in the cantilever beam for damage case 3, **a** noise-free data, **b** noisy data

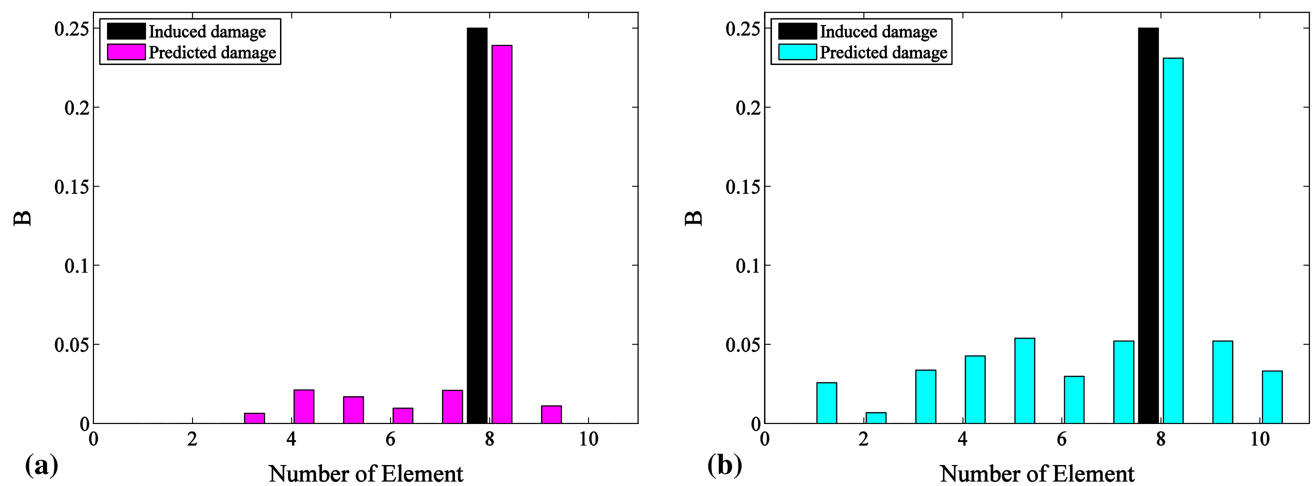



**Fig. 12** Damage quantification in the cantilever beam for damage case 4, **a** noise-free data, **b** noisy data

detection. The applicability and effectiveness of proposed method has numerically been verified by a planner truss and a cantilever beam. Initially, locations of damage are identified by damage localization factor dR before and after inserting noise in modal data. The obtained results show that locations of damage are precisely detected when the modal data is clear of any noise. The noisy data lead to an alteration in dR diagrams; however, still the highest peaks of dR for damaged elements indicate the locations of damage and insertion of noise in modal data does not cause failure in damage localization process. Eventually, damage severity is investigated in two mentioned numerical examples. Hence, results show that the damage quantities have accurately been determined even when the noisy and incomplete modal data are present. Thus, it can be concluded that proposed method provides a reliable, simple and correct algorithm for damage detection process.

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