Burnside condition on some intersection subgroups

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Abstract

In this paper, first we present some preliminaries about graphs, core graphs, and combinatorial algebraic topology. Using these tools, and specially using immersions and covering maps, we establish our main theorem. Indeed, we can prove the Burnside condition for the intersection of those subgroups of free groups satisfying the Burnside condition.

1 Introduction

All our conceptions come from [1], [2] and [3]. A graph \(X\) consists of two sets \(E\) and \(V\) (edges and vertices), with three functions \(e^{-1} : E \to E\) and \(s, t : E \to V\) such that \((e^{-1})^{-1} = e, e^{-1} \neq e, s(e^{-1}) = t(e)\) and \(t(e^{-1}) = s(e)\). We say that the edge \(e \in E\) has initial vertex \(s(e)\) and terminal vertex \(t(e)\). The edge \(e^{-1}\) is the reverse of \(e\).

A map of graphs \(f : X \to Y\) is a function which maps edges to edges and vertices to vertices. Also we have \(f(e^{-1}) = f(e)^{-1}, f(s(e)) = s(f(e))\) and \(f(t(e)) = t(f(e))\).

A path \(p\) in \(X\) of length \(n = |p|\), with initial vertex \(u\) and terminal vertex \(v\), is an \(n\)-tuple of edges of \(X\) of the form \(p = e_1...e_n\) such that for \(i = 1, ..., n - 1\), we have \(t(e_i) = s(e_{i+1})\) and \(s(e_1) = u\) and \(t(e_n) = v\). For \(n = 0\), given any vertex \(v\), there

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is a unique path $\Lambda_v$ of length 0 whose initial and terminal vertices coincide and are equal to $v$. A path $p$ is called a circuit if its initial and terminal vertices coincide.

If $p$ and $q$ are paths in $X$ and the terminal vertex of $p$ equals the initial vertex of $q$, they may be concatenated to form a path $pq$ with $|pq| = |p| + |q|$, whose initial vertex is that of $p$ and whose terminal vertex is that of $q$.

A round-trip is a path of the form $ee^{-1}$. A reduced path is a path in $X$ containing no round-trip. An elementary reduction is insertion or deletion a round-trip in a path. Two paths $p$ and $q$ are in $X$ and the terminal vertex of $p$ equals the initial vertex of $q$, they may be concatenated to form a path $pq$ with $|pq| = |p| + |q|$, whose initial vertex is that of $p$ and whose terminal vertex is that of $q$.

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Let $v$ be a fix vertex in $X$, $\pi_1(X,v)$ is defined to be the set of all homotopy classes of closed paths with initial and terminal vertex $v$. Then $\pi_1(X,v)$ together with the product $[p][q] := [pq]$ forms a group with identity $[\Lambda_v]$ and inverse element $[w]^{-1} = [w]^{-1}$.

For a fix vertex $v$ in $X$, the star of $v$ in $X$ is defined as follows:

$$\text{St}(v,X) = \{e \in E : s(e) = v\}.$$  

A map $f : X \rightarrow Y$ yields, for each vertex $v \in X$, a function $f_v : \text{St}(v,X) \rightarrow \text{St}(f(v),Y)$. If for each vertex $v \in X$, $f_v$ is injective, we call $f$ an immersion. If each $f_v$ is bijective, we call $f$ a covering.

The theory of coverings of graphs is almost completely analogous to the topological theory of coverings. Immersions have some of the properties of coverings. One of them which is more important, and we also need it more, is the following one:

"For a given finite set of elements $\{\alpha_1, ..., \alpha_n\} \subseteq \pi_1(X,u)$, there is a connected graph $Y$ and an immersion $f : Y \rightarrow X$ such that $f_*(\pi_1(Y)) = S$, in which $S$ is the subgroup of $\pi_1(X,u)$ generated by $\{\alpha_1, ..., \alpha_n\}".

If $G$ is a group, a $G$-graph $X$ is a graph with an action of $G$ on the left on $X$ by maps of graphs, such that for all $g \in G$ and every edge $e$, $ge \neq e^{-1}$. In this case, the quotient graph $X/G$, and the quotient map of graphs $X \rightarrow X/G$ can be defined. It is easy to see that, in general $X \rightarrow X/G$ is locally surjective.

It is said that $G$ acts freely on $X$ when, whenever $v$ is a vertex of $X$, $g \in G$, and $gv = v$, then $g = 1$, the identity element of $G$. In this case, $X \rightarrow X/G$ is an immersion, and hence is a covering.

A translation of a map of graphs $f : X \rightarrow Y$ is a map $g : X \rightarrow X$ which is an isomorphism of graphs and for which $fg = f$. The set of all translations of $f$ forms a group $G(f)$ which acts on $X$. If $f$ is an immersion, and $X$ is connected, then $G(f)$ acts freely on $X$. 

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The universal cover $f : \tilde{X} \to X$, of a connected graph $X$, is a covering with ($\tilde{X}$ is connected and $\pi_1(\tilde{X})$ trivial. In this case, $G(f) \cong \pi_1(X)$ which acts freely, by covering translations, on $\tilde{X}$, and $f$ is isomorphic to the quotient map $\tilde{X} \to \tilde{X}$.

**Theorem 1.1.** [4] Let

\[
\begin{array}{c}
\xymatrix{ Z_3 & Z_1 \\
Z_2 & X \\
s_1 \ar[u] \ar[r] & f_1 \\
s_2 \ar[u] \ar[r] & f_2 \\
\end{array}
\]

be a pullback diagram of graphs, where $f_1$ and $f_2$ are immersions. Let $v_1$ and $v_2$ be vertices in $Z_1$ and $Z_2$ that $f_1(v_1) = f_2(v_2) = w$. Let $v_3$ be corresponding vertex in $Z_3$. Define $f_3 = f_1g_1 = f_2g_2 : Z_3 \to X$, and $S_i = f_i(\pi_1(Z_i, v_i))$, for $i = 1, 2, 3$. Then $S_3 = S_1 \cap S_2$.

**Theorem 1.2.** [4] Let $f : X \to Y$ be an immersion of graphs. Suppose that $Y$ has only one vertex and $X$ has only finitely many vertices. Then there exists a graph $\tilde{X}$ containing $X$ such that $\tilde{X} - X$ consists only of edges, and there exists a map $\tilde{f} : \tilde{X} \to Y$ extending $f$ such that $\tilde{f}$ is a covering.

### 2 Main results

In this section, we deduce our main result. Before it, we recall some notes from [4] which are essential in the proof of the main theorem. First, we note to the core graphs whose roles are more important.

A cyclically reduced circuit in a graph $X$ is a circuit $p = e_1...e_n$, which is reduced as a path and for which $e_1 \neq e_n^{-1}$. A graph $X$ is said to be a core-graph if $X$ is connected, has at least one edge and every edge belongs to at least one cyclically reduced circuit.

If $X$ is a connected graph with non-trivial fundamental group, an essential edge of $X$ is an edge belonging to some cyclically reduced circuit. The core of $X$ consists of all essential edges of $X$ and all initial vertices of essential edges.

If $X$ is a connected graph with non-trivial fundamental group and $\tilde{X}$ is the core of $X$, then $\tilde{X}$ is a core-graph. If $v$ is a vertex of $\tilde{X}$, then the inclusion $\pi_1(\tilde{X}, v) \to \pi_1(X, v)$ is an isomorphism.

Another notion, we are dealing with, is the Burnside condition for subgroups. If $S$ is a subgroup of a group $G$, we say that $S \subseteq G$ satisfies the Burnside condition when, for every $g \in G$, there exists some positive integer $n$ such that $g^n \in S$.  

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Lemma 2.1. [4] (a) Let $f : X \to Y$ be a finite-sheeted covering of connected graphs, $v$ a vertex of $X$. Then $f_*(\pi_1(X, v)) \subseteq \pi_1(Y, f(v))$ satisfies the Burnside condition. 

(b) Let $f : X \to Y$ be an immersion of connected graphs. Suppose that $Y$ is a core-graph; $v$ a vertex of $X$, $f_*(\pi_1(X, v)) \subseteq \pi_1(Y, f(v))$ satisfies the Burnside condition. Then $f$ is a covering.

Finally, using all the above notes, we establish the following theorem, which is our main result in this paper.

Theorem 2.2. Let $S_1$ and $S_2$ be finitely generated subgroups of a free group $F$. Suppose that $S_1 \cap S_2$ satisfies the Burnside condition both in $S_1$ and $S_2$. Then $S_1 \cap S_2$ satisfies the Burnside condition in the join $S_1 \vee S_2$, the subgroup generated by $A \cup B$.

References


