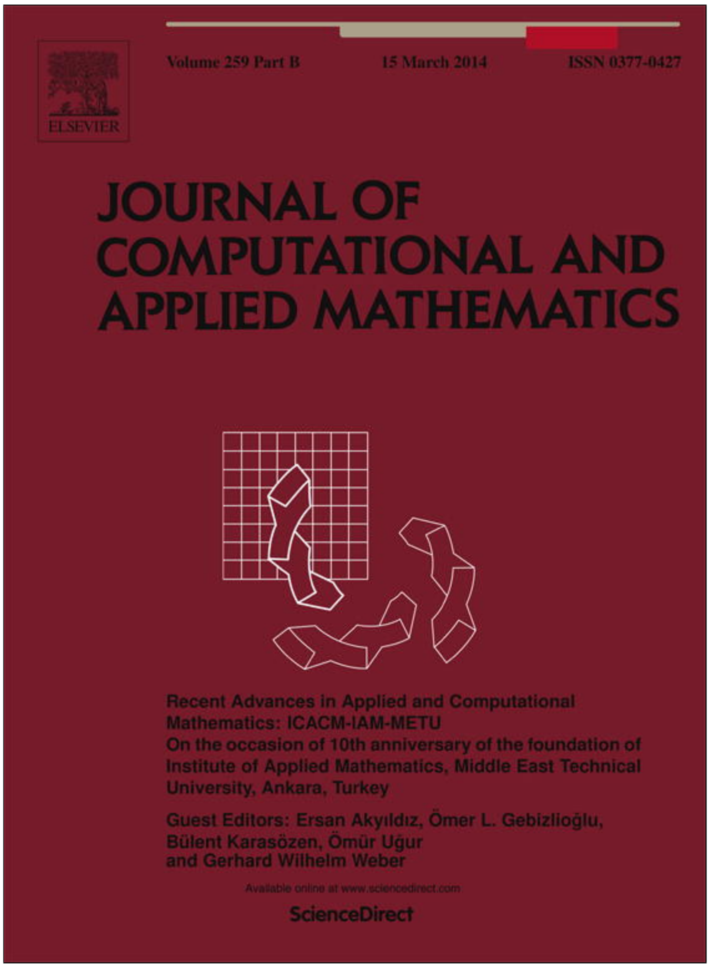
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A new approach for the optimal fuzzy linear time invariant controlled system with fuzzy coefficients

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HIGHLIGHTS

- The fuzzy control system and the optimal control system with fuzzy coefficients are solved.
- Using α-cuts and presentation of numbers in a more compact form, those systems are solved.
- This approach is very simple.

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ABSTRACT

In this article, a new technique for solving the linear time invariant dynamical system with fuzzy parameters is proposed. This approach is used to solve a fuzzy controlled system by using α-cuts and presentation of numbers in a more compact form by moving to the field of complex numbers. By considering a new feature of the Hamiltonian function and using the Pontryagin maximum principle, the solution of the fuzzy optimal controlled system is considered.

1. Introduction

Optimal control theory has been developed to find optimal ways to control dynamic systems. Uncertainty is inherent in most real-world systems. To deal with such systems, stochastic optimal control theories governed by Ito's stochastic differential equations are investigated (see [1]).

Many authors have studied several concepts of fuzzy systems. Zhu [2] applied Bellman's optimal principle to obtain the principle of optimality for fuzzy optimal control problems. Diamond and Kloeden [3] showed the existence of the fuzzy optimal control for the system $\dot{\tilde{x}}(t) = a(t) \odot \tilde{x}(t) \oplus \tilde{u}(t), \tilde{x}(0) = \tilde{x}_0$, where admissible pair $\tilde{p} = [\tilde{x}(.), \tilde{u}(.)]$ is a nonempty compact interval-valued function on E^1. Diamond and Kloeden interpreted the fuzzy differential equation as a family of differential inclusions $\dot{x}^\alpha(t) \in a(t)x^\alpha(t) + u^\alpha(t)$, $x^\alpha(0) \in x_0^\alpha$, $0 \le \alpha \le 1$. They used the H-derivative of a fuzzy-number-valued function, while it may lead to solutions which have an increasing length on their supports. Kwun, and Park [4] using the Kuhn–Tucker theorem, proved the existence of fuzzy optimal control for the nonlinear fuzzy differential system. Balasubramaniam, and Muralisankar [5] proved the existence and uniqueness of fuzzy solutions for the semilinear fuzzy integrodifferential equation with nonlocal initial conditions. Park et al. [6] found the sufficient conditions of nonlocal controllability for semilinear

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fuzzy integrodifferential equations. In [7] authors have studied a new approach for solving fuzzy controlled and optimal controlled systems with boundary conditions. In [8] a fuzzy optimal control model was formulated maximizing the expected discounted objective function subject to the fuzzy differential equation for the fuzzy control system. Agarwal et al. in [9] considered a differential equation of fractional order with uncertainty and presented the concept of the solution. Mazandarani and Kamyad in [10] defined the Caputo-type fuzzy fractional derivatives and applied the modified fractional Euler method for solving fuzzy fractional initial value problems.

We may refer the reader to some applications of the fuzzy control systems. In [11] authors studied the Nowak–May model with fuzzy variables and introduced the fuzzy control model to maximizing uninfected cells of HIV disease. Zarei et al. in [12] studied a fuzzy mathematical model of HIV infection consisting of linear fuzzy differential equations. Mazandarani, and Kamyad [13] considered a fuzzy model of diabetes mellitus type 2.

In [14], Xu et al. presented a new solution for fuzzy dynamical systems with crisp initial conditions and fuzzy matrices. Nieto et al. in [15] obtained an explicit solution for a linear fuzzy differential equation subject to impulsive behavior and periodic boundary value conditions.

In [16], Khastan and Nieto, using the generalized differentiability found new solutions for some fuzzy two-point boundary value problems, while by considering only the Hukuhara differentiability, no more solution exists. Georgiou et al. in [17] considered nth-order fuzzy differential equations with initial value conditions. They proved the existence and uniqueness of solutions for nonlinearities satisfying the Lipschitz condition. In [18] solutions of the first order linear fuzzy differential equations under the generalized differentiability concept have been presented, as well as in [19] the problem has been studied subject to periodic boundary conditions.

In this paper we obtain a new technique to find the solution of a fuzzy controlled system via α-cuts and use the field of complex numbers to present fuzzy numbers in a more compact form. By applying this presentation and considering a new form of the Hamiltonian function and using the Pontryagin Maximum Principle (PMP) (see Chapter 6 of [20]), solving the fuzzy linear time invariant controlled system with fuzzy coefficients is considered. Two numerical examples are simulated to show how easy this method can be used effectively.

2. Fundamental theorems for time invariant linear systems

In this section the solution of the linear controlled system with fuzzy coefficients would be considered. We begin by describing the following linear dynamical system:

$$\dot{\tilde{x}}(t) = A \odot \tilde{x}(t), \qquad \tilde{x}(t_0) = \tilde{x}_0, \tag{1}$$

where $\tilde{x} : [t_0, +\infty) \to E^n$ is a fuzzy function, E^n is the set of fuzzy numbers which is defined in Definition 1, $\tilde{x}_0$ is a fuzzy initial condition, $A = [a_{ij}]_{n\times n}$, $a_{ij} \in \mathbb{R}$ and $\dot{\tilde{x}}(t) = \frac{d\tilde{x}}{dt} = [\frac{d\tilde{x}_1}{dt} \dots \frac{d\tilde{x}_n}{dt}]^T$.

To describe our method for implementing (1), we need the following definitions, lemmas and theorems.

Definition 1. Denote $E^n = \{\tilde{w} : \mathbb{R}^n \to [0, 1] | \tilde{w}$ satisfies (i)–(iv)$\}$ where:

(i) $\tilde{w}$ is normal, i.e. there exists $t \in \mathbb{R}^n$, such that $\tilde{w}(t) = 1$,
(ii) $\tilde{w}$ is fuzzy convex, i.e. $\forall t, s \in \mathbb{R}^n$ and $\lambda \in [0, 1]$, $\tilde{w}(\lambda t + (1 - \lambda)s) \geq \min\{\tilde{w}(t), \tilde{w}(s)\}$,
(iii) $\tilde{w}$ is upper semicontinuous,
(iv) the closure of the set $\{t \in \mathbb{R}^n | \tilde{w}(t) > 0\}$, is compact in $\mathbb{R}^n$.

For $0 < \alpha \leq 1$, denote $[\tilde{w}]^\alpha = \tilde{w}^\alpha = \{t \in \mathbb{R}^n | \tilde{w}(t) \geq \alpha\}$, $\tilde{w}^\alpha$ is the α-level set.

If $\tilde{w} \in E^n$, then $\tilde{w}$ is fuzzy convex, so $\tilde{w}^\alpha$ is closed and bounded function in $\mathbb{R}^n$, i.e. $\tilde{w}^\alpha = [\underline{w}^\alpha, \overline{w}^\alpha]$, where $\underline{w}^\alpha = \inf\{t \in \mathbb{R}^n : \tilde{w}(t) \geq \alpha\} > -\infty$ and $\bar{w}^\alpha = \sup\{t \in \mathbb{R}^n : \tilde{w}(t) \geq \alpha\} < \infty$.

Lemma 1. *Denote $I = [0, 1]$; assume $a : I \to \mathbb{R}$ and $b : I \to \mathbb{R}$ satisfy the following conditions:*

(i) *$a : I \to \mathbb{R}$ is a bounded non-decreasing function,*
(ii) *$b : I \to \mathbb{R}$ is a bounded non-increasing function,*
(iii) *$a(1) \leq b(1)$,*
(iv) *for $0 < k \leq 1$, $\lim_{\alpha\to k^-} a(\alpha) = a(k)$ and $\lim_{\alpha\to k^-} b(\alpha) = b(k)$,*
(v) *$\lim_{\alpha\to 0^+} a(\alpha) = a(0)$ and $\lim_{\alpha\to 0^+} b(\alpha) = b(0)$,*

then $\tilde{\eta} : \mathbb{R} \to I$ defined by $\tilde{\eta}(t) = \sup\{\alpha | a(\alpha) \leq t \leq b(\alpha)\}$ is a fuzzy number with parametrization given by $\{(a(\alpha), b(\alpha), \alpha) | 0 \leq \alpha \leq 1\}$. Moreover, if $\tilde{\eta} : \mathbb{R} \to I$ is a fuzzy number with parametrization given by $\{(a(\alpha), b(\alpha), \alpha) | 0 \leq \alpha \leq 1\}$ then functions $a(\alpha)$ and $b(\alpha)$ satisfy conditions (i)–(v) of Lemma 1.

Proof. See [14]. □

For $\tilde{w}, \tilde{v} \in E^1$ and $\gamma \in \mathbb{R}$, the sum $\tilde{w} \oplus \tilde{v}$ and the product $\gamma \odot \tilde{w}$ are defined by $[\tilde{w} \oplus \tilde{v}]^\alpha = [\tilde{w}]^\alpha + [\tilde{v}]^\alpha$ and $[\gamma \odot \tilde{w}]^\alpha = \gamma \cdot [\tilde{w}]^\alpha$ for all $\alpha \in [0, 1]$.

The metric space is given by the Hausdorff distance

$$D : E^1 \times E^1 \longrightarrow \mathbb{R}^+ \cup \{0\}, \qquad D(\tilde{w}, \tilde{v}) = \sup_{\alpha \in [0,1]} \max\{|\underline{w}^\alpha - \underline{v}^\alpha|, |\bar{w}^\alpha - \bar{v}^\alpha|\},$$

where (E^1, D) is a complete metric space (see [18,21] for more details).

Definition 2. Let $\tilde{w}, \tilde{v} \in E^1$, if there exists $\tilde{z} \in E^1$ such that $\tilde{w} = \tilde{v} \oplus \tilde{z}$ then $\tilde{z}$ is called the H-difference of $\tilde{w}, \tilde{v}$ and it is denoted by $\tilde{w} -^H \tilde{v}$ (H refers to Hukuhara).

A triangular fuzzy number, denoted by $\tilde{w} = (a, b, c)$, where $a \le b \le c$, has α-cuts $[\tilde{w}]^\alpha = [b\alpha + a(1-\alpha), b\alpha + c(1-\alpha)]$, $\alpha \in [0, 1]$. The following lemma gives a sufficient condition for the existence of the H-difference of two triangular fuzzy numbers.

Lemma 2. *Let $\tilde{w}, \tilde{v} \in E^1$ be such that $w^1 - \underline{w}^0 > 0$, $\bar{w}^0 - w^1 > 0$, and $(\bar{v}^0 - \underline{v}^0) \le \min\{w^1 - \underline{w}^0, \bar{w}^0 - w^1\}$, then the H-difference $\tilde{w} -^H \tilde{v}$ exists (see [22]).*

Definition 3. Let $\tilde{f} : T \subset \mathbb{R} \to E^1$ and $t_0 \in T$ be fixed, then one can define (see more details in [21]):

1. $\tilde{f}(t)$ is (1)-differentiable at $t_0 \in T$ if for every sufficiently small $h > 0$, there exist the H-differences $\tilde{f}(t_0 + h) -^H \tilde{f}(t_0)$ and $\tilde{f}(t_0) -^H \tilde{f}(t_0 - h)$ and also exist the following limits (by the metric D):

$$\lim_{h \to 0^+} \frac{\tilde{f}(t_0 + h) -^H \tilde{f}(t_0)}{h} = \lim_{h \to 0^+} \frac{\tilde{f}(t_0) -^H \tilde{f}(t_0 - h)}{h} = \tilde{f}'(t_0),$$

2. $\tilde{f}(t)$ is (2)-differentiable at $t_0 \in T$ if for every sufficiently small $h > 0$, there exist the H-differences $\tilde{f}(t_0 - h) -^H \tilde{f}(t_0)$ and $\tilde{f}(t_0) -^H \tilde{f}(t_0 + h)$ and also exist the following limits (by the metric D):

$$\lim_{h \to 0^+} \frac{\tilde{f}(t_0 - h) -^H \tilde{f}(t_0)}{h} = \lim_{h \to 0^+} \frac{\tilde{f}(t_0) -^H \tilde{f}(t_0 + h)}{h} = \tilde{f}'(t_0).$$

If $\tilde{f}(t)$ is (n)-differentiable at t_0, we denote its first derivative by $D_n^{(1)}\tilde{f}(t_0)$, for $n = 1, 2$.

Theorem 1. *Let $\tilde{f} : T \to E^1$ be a fuzzy function, where $[\tilde{f}(t)]^\alpha = [\underline{f}^\alpha(t), \bar{f}^\alpha(t)]$;*

1. *if $\tilde{f}(t)$ is (1)-differentiable, then $\underline{f}^\alpha(t)$ and $\bar{f}^\alpha(t)$ are differentiable functions and*

$$[D_1^{(1)}\tilde{f}(t)]^\alpha = [\underline{f}'^\alpha(t), \bar{f}'^\alpha(t)],$$

2. *if $\tilde{f}(t)$ is (2)-differentiable, then $\underline{f}^\alpha(t)$ and $\bar{f}^\alpha(t)$ are differentiable functions and*

$$[D_2^{(1)}\tilde{f}(t)]^\alpha = [\bar{f}'^\alpha(t), \underline{f}'^\alpha(t)].$$

Proof. See [23]. □

Lemma 3. *Assume that each element of the vector $\tilde{x}(t)$ in (1) is a fuzzy number where*

$$\tilde{x}_k^\alpha(t) = [\underline{x}_k^\alpha(t), \bar{x}_k^\alpha(t)], \quad k = 1, \dots, n, \tag{2}$$

then, the evaluation of the system (1) can be described by the following $2n$ differential equations:

- *if $\tilde{x}(t)$ is (1)-differentiable then:*

$$\begin{cases} \dot{\underline{x}}_k^\alpha(t) = \min\{(Av)_k : v_j \in [\underline{x}_j^\alpha(t), \bar{x}_j^\alpha(t)],\ j = 1, 2, \dots, n\}, \\ \dot{\bar{x}}_k^\alpha(t) = \max\{(Av)_k : v_j \in [\underline{x}_j^\alpha(t), \bar{x}_j^\alpha(t)],\ j = 1, 2, \dots, n\}, \\ \underline{x}^\alpha(t_0) = \underline{x}_0^\alpha, \\ \bar{x}^\alpha(t_0) = \bar{x}_0^\alpha, \end{cases} \tag{3}$$

- *if $\tilde{x}(t)$ is (2)-differentiable then:*

$$\begin{cases} \dot{\underline{x}}_k^\alpha(t) = \max\{(Av)_k : v_j \in [\underline{x}_j^\alpha(t), \bar{x}_j^\alpha(t)],\ j = 1, 2, \dots, n\}, \\ \dot{\bar{x}}_k^\alpha(t) = \min\{(Av)_k : v_j \in [\underline{x}_j^\alpha(t), \bar{x}_j^\alpha(t)],\ j = 1, 2, \dots, n\}, \\ \underline{x}^\alpha(t_0) = \underline{x}_0^\alpha, \\ \bar{x}^\alpha(t_0) = \bar{x}_0^\alpha, \end{cases} \tag{4}$$

where $(Av)_k := \sum_{j=1}^n a_{kj} v_j$ is the kth row of Av, and $k = 1, 2, \dots, n$.

Proof. See [14,24,25], for details. □

Since the dynamical system (1) is linear, the following rules could be applied in (3) and (4): $\dot{\underline{x}}^{\alpha}_k(t) = \sum_{j=1}^{n} a_{kj} v_j,\ k = 1, 2, \dots, n$ where:

- if $\tilde{x}(t)$ is (1)-differentiable then: $v_j = \begin{cases} \underline{x}^{\alpha}_j(t), & a_{kj} \geq 0, \\ \bar{x}^{\alpha}_j(t), & a_{kj} < 0, \end{cases}$
- if $\tilde{x}(t)$ is (2)-differentiable then: $v_j = \begin{cases} \underline{x}^{\alpha}_j(t), & a_{kj} < 0, \\ \bar{x}^{\alpha}_j(t), & a_{kj} \geq 0, \end{cases}$

and $\dot{\bar{x}}^{\alpha}_k(t) = \sum_{j=1}^{n} a_{kj} v_j,\ k = 1, 2, \dots, n$ where:

- if $\tilde{x}(t)$ is (1)-differentiable then: $v_j = \begin{cases} \bar{x}^{\alpha}_j(t), & a_{kj} \geq 0, \\ \underline{x}^{\alpha}_j(t), & a_{kj} < 0, \end{cases}$
- if $\tilde{x}(t)$ is (2)-differentiable then: $v_j = \begin{cases} \bar{x}^{\alpha}_j(t), & a_{kj} < 0, \\ \underline{x}^{\alpha}_j(t), & a_{kj} \geq 0. \end{cases}$

Now we consider a class of fuzzy linear time invariant controlled systems with fuzzy coefficients as follows:

$$\begin{cases} \dot{\tilde{x}}(t) = \tilde{A} \odot \tilde{x}(t) \oplus \tilde{C} \odot \tilde{u}(t), \\ \tilde{x}(t_0) = x_0, \qquad \tilde{x}(t_f) = \tilde{x}_f, \end{cases} \tag{5}$$

where $\tilde{x}(t) \in E^n$, $\tilde{u}(t) \in E^m$, and x_0 is a crisp vector ($\tilde{x}_f$ refers to the final condition).

Let $\tilde{x}^{\alpha}(t) = \underline{x}^{\alpha}(t) + i\bar{x}^{\alpha}(t)$ be the solution of system (5), in which the entries of matrices $\tilde{A}$ and $\tilde{C}$ are fuzzy numbers. The system (5) can be rewritten as follows:

$$\begin{cases} \dot{\underline{x}}^{\alpha}(t) + i\dot{\bar{x}}^{\alpha}(t) = \tilde{A}^{\alpha}(\underline{x}^{\alpha}(t) + i\bar{x}^{\alpha}(t)) + \tilde{C}^{\alpha}(\underline{u}^{\alpha}(t) + i\bar{u}^{\alpha}(t)), \\ \underline{x}^{\alpha}(t_0) + i\bar{x}^{\alpha}(t_0) = x_0 + ix_0, \\ \underline{x}^{\alpha}(t_f) + i\bar{x}^{\alpha}(t_f) = \underline{x}^{\alpha}_f + i\bar{x}^{\alpha}_f. \end{cases} \tag{6}$$

Let $(\tilde{a}^{\alpha})_{ij} = [(\underline{a}^{\alpha})_{ij}, (\bar{a}^{\alpha})_{ij}]$, $(\tilde{c}^{\alpha})_{ij} = [(\underline{c}^{\alpha})_{ij}, (\bar{c}^{\alpha})_{ij}]$, $\tilde{A}^{\alpha} = [\underline{A}^{\alpha}, \bar{A}^{\alpha}]$ and $\tilde{C}^{\alpha} = [\underline{C}^{\alpha}, \bar{C}^{\alpha}]$, where $\underline{A}^{\alpha} = [(\underline{a}^{\alpha})_{ij}]_{n\times n}$, $\bar{A}^{\alpha} = [(\bar{a}^{\alpha})_{ij}]_{n\times n}$, $\underline{C}^{\alpha} = [(\underline{c}^{\alpha})_{ij}]_{n\times m}$ and $\bar{C}^{\alpha} = [(\bar{c}^{\alpha})_{ij}]_{n\times m}$. To solve (6), we need to prove the following two basic theorems.

Theorem 2. *Let $A(\mu, \alpha) = [a_{ij}(\mu, \alpha)]_{n\times n} = (1 - \mu)\underline{A}^{\alpha} + \mu\bar{A}^{\alpha}$ and $C(\lambda, \alpha) = [c_{ij}(\lambda, \alpha)]_{n\times m} = (1 - \lambda)\underline{C}^{\alpha} + \lambda\bar{C}^{\alpha}$ in which $\mu \in [0, 1]$ and $\lambda \in [0, 1]$. Then the following fuzzy controlled system*

$$\begin{cases} \dot{\underline{x}}^{\alpha}(t) + i\dot{\bar{x}}^{\alpha}(t) = \bigcup_{\mu=0}^{1} B(\mu, \alpha)(\underline{x}^{\alpha}(t) + i\bar{x}^{\alpha}(t)) + \bigcup_{\lambda=0}^{1} D(\lambda, \alpha)(\underline{u}^{\alpha}(t) + i\bar{u}^{\alpha}(t)), \\ \underline{x}^{\alpha}(t_0) + i\bar{x}^{\alpha}(t_0) = x_0 + ix_0, \end{cases} \tag{7}$$

has a solution given by

$$\begin{aligned} \underline{x}^{\alpha}(t) + i\bar{x}^{\alpha}(t) &= \exp\left((t - t_0)\bigcup_{\mu=0}^{1} B(\mu, \alpha)\right)(x_0 + ix_0) \\ &\quad + \int_{t_0}^{t}\left[\exp\left((t - s)\bigcup_{\mu=0}^{1} B(\mu, \alpha)\right)\bigcup_{\lambda=0}^{1} D(\lambda, \alpha)(\underline{u}^{\alpha}(s) + i\bar{u}^{\alpha}(s))\right] ds, \end{aligned}$$

where the entries of matrices $B(\mu, \alpha)$ and $D(\lambda, \alpha)$ are determined respectively from those of $A(\mu, \alpha)$ and $C(\lambda, \alpha)$ as follows:

- *if $\tilde{x}(t)$ is (1)-differentiable then:*

$$b_{ij}(\mu, \alpha) = \begin{cases} ea_{ij}(\mu, \alpha), & a_{ij}(\mu, \alpha) \geq 0, \\ ga_{ij}(\mu, \alpha), & a_{ij}(\mu, \alpha) < 0, \end{cases} \quad \text{and} \quad d_{ij}(\lambda, \alpha) = \begin{cases} ec_{ij}(\lambda, \alpha), & c_{ij}(\lambda, \alpha) \geq 0, \\ gc_{ij}(\lambda, \alpha), & c_{ij}(\lambda, \alpha) < 0, \end{cases} \tag{8}$$

- *if $\tilde{x}(t)$ is (2)-differentiable then:*

$$b_{ij}(\mu, \alpha) = \begin{cases} ga_{ij}(\mu, \alpha), & a_{ij}(\mu, \alpha) \geq 0, \\ ea_{ij}(\mu, \alpha), & a_{ij}(\mu, \alpha) < 0, \end{cases} \quad \text{and} \quad d_{ij}(\lambda, \alpha) = \begin{cases} gc_{ij}(\lambda, \alpha), & c_{ij}(\lambda, \alpha) \geq 0, \\ ec_{ij}(\lambda, \alpha), & c_{ij}(\lambda, \alpha) < 0. \end{cases} \tag{9}$$

For any complex number $a + bi$, we define:

$$e : a + bi \to a + bi, \qquad g : a + bi \to b + ai. \tag{10}$$

Proof. From (7) we have:

$$
\begin{aligned}
\underline{\dot{x}}^{\alpha}(t) + i\dot{\bar{x}}^{\alpha}(t) &= \frac{d}{dt}\left(\exp\left((t-t_0)\bigcup_{\mu=0}^{1} B(\mu,\alpha)\right)(x_0 + ix_0)\right) + \frac{d}{dt}\left(\int_{t_0}^{t}\left[\exp\left((t-s)\bigcup_{\mu=0}^{1} B(\mu,\alpha)\right)\right.\right. \\
&\quad \left.\left.\times \bigcup_{\lambda=0}^{1} D(\lambda,\alpha)(\underline{u}^{\alpha}(s) + i\bar{u}^{\alpha}(s))\right] ds\right) = \lim_{h\to 0} \frac{e^{(t+h-t_0)\bigcup_{\mu=0}^{1} B(\mu,\alpha)} - e^{(t-t_0)\bigcup_{\mu=0}^{1} B(\mu,\alpha)}}{h}(x_0 + ix_0) \\
&\quad + \lim_{h\to 0}\frac{1}{h}\left[\int_{t_0}^{t+h} e^{(t+h-s)\bigcup_{\mu=0}^{1} B(\mu,\alpha)} \bigcup_{\lambda=0}^{1} D(\lambda,\alpha)(\underline{u}^{\alpha}(s) + i\bar{u}^{\alpha}(s))ds\right. \\
&\quad \left. - \int_{t_0}^{t} e^{(t-s)\bigcup_{\mu=0}^{1} B(\mu,\alpha)} \bigcup_{\lambda=0}^{1} D(\lambda,\alpha)(\underline{u}^{\alpha}(s) + i\bar{u}^{\alpha}(s))ds\right] \\
&= e^{(t-t_0)\bigcup_{\mu=0}^{1} B(\mu,\alpha)} \lim_{h\to 0}\frac{e^{h\bigcup_{\mu=0}^{1} B(\mu,\alpha)} - I}{h}(x_0 + ix_0) \\
&\quad + \lim_{h\to 0}\frac{1}{h}\left[\int_{t_0}^{t} e^{(t+h-s)\bigcup_{\mu=0}^{1} B(\mu,\alpha)} \bigcup_{\lambda=0}^{1} D(\lambda,\alpha)(\underline{u}^{\alpha}(s) + i\bar{u}^{\alpha}(s))ds\right. \\
&\quad \left. - \int_{t_0}^{t} e^{(t-s)\bigcup_{\mu=0}^{1} B(\mu,\alpha)} \bigcup_{\lambda=0}^{1} D(\lambda,\alpha)(\underline{u}^{\alpha}(s) + i\bar{u}^{\alpha}(s))ds\right] \\
&\quad + \lim_{h\to 0}\frac{\int_{t}^{t+h} e^{(t+h-s)\bigcup_{\mu=0}^{1} B(\mu,\alpha)} \bigcup_{\lambda=0}^{1} D(\lambda,\alpha)(\underline{u}^{\alpha}(s) + i\bar{u}^{\alpha}(s))ds}{h} \\
&= e^{(t-t_0)\bigcup_{\mu=0}^{1} B(\mu,\alpha)} \lim_{h\to 0}\frac{e^{h\bigcup_{\mu=0}^{1} B(\mu,\alpha)} - I}{h}(x_0 + ix_0) \\
&\quad + \lim_{h\to 0}\frac{\int_{t_0}^{t} e^{(t-s)\bigcup_{\mu=0}^{1} B(\mu,\alpha)}\left(e^{h\bigcup_{\mu=0}^{1} B(\mu,\alpha)} - I\right)\bigcup_{\lambda=0}^{1} D(\lambda,\alpha)\left(\underline{u}^{\alpha}(s) + i\bar{u}^{\alpha}(s)\right)ds}{h} \\
&\quad + \lim_{h\to 0}\frac{\int_{t}^{t+h} e^{(t+h-s)\bigcup_{\mu=0}^{1} B(\mu,\alpha)} \bigcup_{\lambda=0}^{1} D(\lambda,\alpha)(\underline{u}^{\alpha}(s) + i\bar{u}^{\alpha}(s))ds}{h} \\
&= e^{(t-t_0)\bigcup_{\mu=0}^{1} B(\mu,\alpha)} \lim_{h\to 0}\left(\lim_{k\to+\infty}\left(\bigcup_{\mu=0}^{1} B(\mu,\alpha) + \frac{\left(\bigcup_{\mu=0}^{1} B(\mu,\alpha)\right)^2 h}{2!}\right.\right. \\
&\quad \left.\left. + \cdots + \frac{\left(\bigcup_{\mu=0}^{1} B(\mu,\alpha)\right)^k h^{k-1}}{k!}\right)\right)(x_0 + ix_0) \\
&\quad + \int_{t_0}^{t}\left[e^{(t-s)\bigcup_{\mu=0}^{1} B(\mu,\alpha)} \lim_{h\to 0}\left(\lim_{k\to+\infty}\left(\bigcup_{\mu=0}^{1} B(\mu,\alpha) + \frac{\left(\bigcup_{\mu=0}^{1} B(\mu,\alpha)\right)^2 h}{2!}\right.\right.\right.
\end{aligned}
$$

$$+\cdots+\frac{\left(\bigcup_{\mu=0}^{1}B(\mu,\alpha)\right)^{k}h^{k-1}}{k!}\Bigg)\Bigg)\bigcup_{\lambda=0}^{1}D(\lambda,\alpha)(\underline{u}^{\alpha}(s)+i\bar{u}^{\alpha}(s))\Bigg]ds$$

$$+\lim_{h\to 0}\frac{e^{(t+h-t)\bigcup_{\mu=0}^{1}B(\mu,\alpha)}\bigcup_{\lambda=0}^{1}D(\lambda,\alpha)(\underline{u}^{\alpha}(t)+i\bar{u}^{\alpha}(t))h}{h}$$

$$=\exp\left((t-t_0)\bigcup_{\mu=0}^{1}B(\mu,\alpha)\right)\bigcup_{\mu=0}^{1}B(\mu,\alpha)(x_0+ix_0)+\int_{t_0}^{t}\left[\exp\left((t-s)\bigcup_{\mu=0}^{1}B(\mu,\alpha)\right)\right.$$

$$\left.\times\bigcup_{\mu=0}^{1}B(\mu,\alpha)\bigcup_{\lambda=0}^{1}D(\lambda,\alpha)(\underline{u}^{\alpha}(s)+i\bar{u}^{\alpha}(s))\right]ds+\bigcup_{\lambda=0}^{1}D(\lambda,\alpha)(\underline{u}^{\alpha}(t)+i\bar{u}^{\alpha}(t)).$$

Since the matrix $\bigcup_{\mu=0}^{1}B(\mu,\alpha)$ commutes $\exp((t-t_0)\bigcup_{\mu=0}^{1}B(\mu,\alpha))$, then

$$\dot{\underline{x}}^{\alpha}(t)+i\dot{\bar{x}}^{\alpha}(t)=\bigcup_{\mu=0}^{1}B(\mu,\alpha)(\underline{x}^{\alpha}(t)+i\bar{x}^{\alpha}(t))+\bigcup_{\lambda=0}^{1}D(\lambda,\alpha)(\underline{u}^{\alpha}(t)+i\bar{u}^{\alpha}(t)),$$

where the right hand side of the above relation is the right hand side of (7), and this completes the proof. □

Theorem 3. *The fuzzy function $\underline{x}^{\alpha}(t)+i\bar{x}^{\alpha}(t)$ and the fuzzy control $\underline{u}^{\alpha}(t)+i\bar{u}^{\alpha}(t)$ are the solutions of fuzzy boundary valued controlled system (6), if and only if $\underline{x}^{\alpha}(t)+i\bar{x}^{\alpha}(t)$ and $\underline{u}^{\alpha}(t)+i\bar{u}^{\alpha}(t)$ are also the solutions of fuzzy controlled system (7).*

Proof. Let $a_{ij}\in\mathbb{R}$ with $a_{ij}\in[(\underline{a}^{\alpha})_{ij},(\bar{a}^{\alpha})_{ij}]$ and $c_{ij}\in\mathbb{R}$ with $c_{ij}\in[(\underline{c}^{\alpha})_{ij},(\bar{c}^{\alpha})_{ij}]$, then $A=[a_{ij}]_{n\times n}\in[\underline{A}^{\alpha},\bar{A}^{\alpha}]$ and also $C=[c_{ij}]_{n\times m}\in[\underline{C}^{\alpha},\bar{C}^{\alpha}]$. Let x and u be the solutions of following controlled system

$$\dot{x}(t)=Ax(t)+Cu(t),\qquad x(t_0)=x_0, \tag{11}$$

where $x(t)\in[\underline{x}^{\alpha}(t),\bar{x}^{\alpha}(t)]$ and $u(t)\in[\underline{u}^{\alpha}(t),\bar{u}^{\alpha}(t)]$, while $\underline{x}^{\alpha}(t)+i\bar{x}^{\alpha}(t)$ and $\underline{u}^{\alpha}(t)+i\bar{u}^{\alpha}(t)$ are the solutions of system (6). Then there exist $\mu_1,\mu_2\in[0,1]$ and also $\lambda_1,\lambda_2\in[0,1]$ such that $A\in[A(\mu_1,\alpha),A(\mu_2,\alpha)]\subset[\underline{A}^{\alpha},\bar{A}^{\alpha}]$, $C\in[C(\lambda_1,\alpha),C(\lambda_2,\alpha)]\subset[\underline{C}^{\alpha},\bar{C}^{\alpha}]$, thus:

$$Ax\subset[A(\mu_1,\alpha),A(\mu_2,\alpha)]\times[\underline{x}^{\alpha}(t),\bar{x}^{\alpha}(t)]\subset[\underline{A}^{\alpha},\bar{A}^{\alpha}]\times[\underline{x}^{\alpha}(t),\bar{x}^{\alpha}(t)],$$
$$Cu\subset[C(\lambda_1,\alpha),C(\lambda_2,\alpha)]\times[\underline{u}^{\alpha}(t),\bar{u}^{\alpha}(t)]\subset[\underline{C}^{\alpha},\bar{C}^{\alpha}]\times[\underline{u}^{\alpha}(t),\bar{u}^{\alpha}(t)].$$

Since the initial condition x_0 is a crisp number, then from (11) we have:

$$\begin{aligned}x(t)-x(t_0)&=\int_{t_0}^{t}(Ax(s)+Cu(s))ds\\&\subset\int_{t_0}^{t}\left(\bigcup_{\mu=\mu_1}^{\mu_2}B(\mu,\alpha)(\underline{x}^{\alpha}(s)+i\bar{x}^{\alpha}(s))+\bigcup_{\lambda=\lambda_1}^{\lambda_2}D(\lambda,\alpha)(\underline{u}^{\alpha}(s)+i\bar{u}^{\alpha}(s))\right)ds\\&\subset\int_{t_0}^{t}\left(\bigcup_{\mu=0}^{1}B(\mu,\alpha)(\underline{x}^{\alpha}(s)+i\bar{x}^{\alpha}(s))+\bigcup_{\lambda=0}^{1}D(\lambda,\alpha)(\underline{u}^{\alpha}(s)+i\bar{u}^{\alpha}(s))\right)ds\\&=\int_{t_0}^{t}(\dot{\underline{x}}^{\alpha}(s)+i\dot{\bar{x}}^{\alpha}(s))ds=(\underline{x}^{\alpha}(t)+i\bar{x}^{\alpha}(t))-(x_0+ix_0).\end{aligned}$$

Then $x(t)\in[\underline{x}^{\alpha}(t),\bar{x}^{\alpha}(t)]$, and $u(t)\in[\underline{u}^{\alpha}(t),\bar{u}^{\alpha}(t)]$ where $\underline{x}^{\alpha}(t)+i\bar{x}^{\alpha}(t)$ and $\underline{u}^{\alpha}(t)+i\bar{u}^{\alpha}(t)$ are the solutions of problem (7).

Now let x_1 and u_1 be the solutions of following system:

$$\dot{x}(t)=A(\mu,\alpha)x(t)+C(\lambda,\alpha)u(t),\qquad x(t_0)=x_0, \tag{12}$$

where $x_1(t) \in [\underline{x}^\alpha(t), \bar{x}^\alpha(t)]$, and $u_1(t) \in [\underline{u}^\alpha(t), \bar{u}^\alpha(t)]$ such that $\underline{x}^\alpha(t) + i\bar{x}^\alpha(t)$ and $\underline{u}^\alpha(t) + i\bar{u}^\alpha(t)$ are the solutions of system (7); there exist $\mu_1, \mu_2 \in [0, 1]$ and also $\lambda_1, \lambda_2 \in [0, 1]$ such that $A(\mu, \alpha) \in [A(\mu_1, \alpha), A(\mu_2, \alpha)] \subset [\underline{A}^\alpha, \bar{A}^\alpha]$, $C(\lambda, \alpha) \in [C(\lambda_1, \alpha), C(\lambda_2, \alpha)] \subset [\underline{C}^\alpha, \bar{C}^\alpha]$, so:

$$A(\mu, \alpha)x_1 \subset [A(\mu_1, \alpha), A(\mu_2, \alpha)] \times [\underline{x}^\alpha(t), \bar{x}^\alpha(t)] \subset [\underline{A}^\alpha, \bar{A}^\alpha] \times [\underline{x}^\alpha(t), \bar{x}^\alpha(t)],$$

$$C(\lambda, \alpha)u_1 \subset [C(\lambda_1, \alpha), C(\lambda_2, \alpha)] \times [\underline{u}^\alpha(t), \bar{u}^\alpha(t)] \subset [\underline{C}^\alpha, \bar{C}^\alpha] \times [\underline{u}^\alpha(t), \bar{u}^\alpha(t)].$$

Since the initial condition is a crisp number, then from (12) we have:

$$\begin{aligned} x_1(t) - x_1(t_0) &= \int_0^t (A(\mu, \alpha)x_1(s) + c(\lambda, \alpha)u_1(s))ds \\ &\subset \int_{t_0}^t \left(\bigcup_{\mu=\mu_1}^{\mu_2} B(\mu, \alpha)(\underline{x}^\alpha(s) + i\bar{x}^\alpha(s)) + \bigcup_{\lambda=\lambda_1}^{\lambda_2} D(\lambda, \alpha)(\underline{u}^\alpha(s) + i\bar{u}^\alpha(s)) \right) ds \\ &\subset \int_{t_0}^t ([\underline{A}^\alpha, \bar{A}^\alpha](\underline{x}^\alpha(s) + i\bar{x}^\alpha(s)) + [\underline{C}^\alpha, \bar{C}^\alpha](\underline{u}^\alpha(s) + i\bar{u}^\alpha(s)))ds \\ &= \int_0^t (\dot{\underline{x}}^\alpha(s) + i\dot{\bar{x}}^\alpha(s))ds = (\underline{x}^\alpha(t) + i\bar{x}^\alpha(t)) - (x_0 + ix_0). \end{aligned}$$

Then $x(t) \in [\underline{x}^\alpha(t), \bar{x}^\alpha(t)]$, and $u(t) \in [\underline{u}^\alpha(t), \bar{u}^\alpha(t)]$ where $\underline{x}^\alpha(t) + i\bar{x}^\alpha(t)$ and $\underline{u}^\alpha(t) + i\bar{u}^\alpha(t)$ are the solutions of system (6). □

3. Optimal control of the time invariant controlled system with fuzzy coefficients

Consider the following fuzzy optimal controlled system:

$$\begin{aligned} &\text{Min} \int_{t_0}^{t_f} \tilde{f}_0(t, \tilde{u}(t))dt, \\ &\text{s.t.} \\ &\quad \dot{\tilde{x}}(t) = \tilde{A} \odot \tilde{x}(t) \oplus \tilde{C} \odot \tilde{u}(t), \\ &\quad \tilde{x}(t_0) = x_0, \qquad \tilde{x}(t_f) = \tilde{x}_f. \end{aligned} \tag{13}$$

Recall that $\tilde{x}_f$ refers to the final condition. Similar to the fuzzy controlled system (5), the optimal fuzzy controlled system (13) changes to the following form:

$$\begin{aligned} &\text{Min} \int_{t_0}^{t_f} \underline{f}_0^\alpha(t, \underline{u}^\alpha(t), \bar{u}^\alpha(t)) + i\bar{f}_0^\alpha(t, \underline{u}^\alpha(t), \bar{u}^\alpha(t))dt, \\ &\text{s.t.} \\ &\quad \dot{\underline{x}}^\alpha(t) + i\dot{\bar{x}}^\alpha(t) = \bigcup_{\mu=0}^{1} B(\mu, \alpha)(\underline{x}^\alpha(t) + i\bar{x}^\alpha(t)) + \bigcup_{\lambda=0}^{1} D(\lambda, \alpha)(\underline{u}^\alpha(t) + i\bar{u}^\alpha(t)), \\ &\quad \underline{x}^\alpha(t_0) + i\bar{x}^\alpha(t_0) = x_0 + ix_0, \\ &\quad \underline{x}^\alpha(t_f) + i\bar{x}^\alpha(t_f) = \underline{x}_f^\alpha + i\bar{x}_f^\alpha, \end{aligned} \tag{14}$$

where $\mu, \lambda \in [0, 1]$, $\tilde{x}(t) \in E^n$, and $\tilde{u}(t) \in E^m$. The entries of matrices $B(\mu, \alpha)$ and $D(\lambda, \alpha)$ are determined from those of $A(\mu, \alpha)$ and $C(\lambda, \alpha)$, similar to (8) and (9) respectively. The functions e and g are defined as (10).

Let $(\underline{x}^{*\alpha}, \bar{x}^{*\alpha}, \underline{u}^{*\alpha}, \bar{u}^{*\alpha})$ be the optimal solution of (14), then the following theorem holds.

Theorem 4. *Define the Hamiltonian function as:*

$$\begin{aligned} H(\underline{x}^\alpha, \bar{x}^\alpha, \underline{u}^\alpha, \bar{u}^\alpha, \psi, t) &= -(\underline{f}_0^\alpha(t, \underline{u}^\alpha(t), \bar{u}^\alpha(t)) + i\bar{f}_0^\alpha(t, \underline{u}^\alpha(t), \bar{u}^\alpha(t))) \\ &\quad + \psi^T \left\{ \bigcup_{\mu=0}^{1} B(\mu, \alpha)(\underline{x}^\alpha(t) + i\bar{x}^\alpha(t)) + \bigcup_{\lambda=0}^{1} D(\lambda, \alpha)(\underline{u}^\alpha(t) + i\bar{u}^\alpha(t)) \right\}, \end{aligned}$$

then the necessary conditions for the quadruple $(\underline{x}^{*\alpha}, \bar{x}^{*\alpha}, \underline{u}^{*\alpha}, \bar{u}^{*\alpha})$ *to be the optimal solution of* (14) *is the existence of a costate vector function* ψ *which satisfies the following differential equations:*

$$\dot{\psi} = -\frac{\partial H}{\partial x^{\alpha}}, \tag{15}$$

where $x^{\alpha} = (\underline{x}^{\alpha}, \bar{x}^{\alpha})$*, and the Hamiltonian function is maximized in* $\underline{u}^{*\alpha}$ *and* $\bar{u}^{*\alpha}$*, i.e.*

$$\frac{\partial H}{\partial \underline{u}^{*\alpha}} = 0, \quad and \quad \frac{\partial H}{\partial \bar{u}^{*\alpha}} = 0. \tag{16}$$

Eqs. (15) *and* (16) *are called PMP conditions.*

Proof. Suppose $g(\underline{u}^{\alpha}, \bar{u}^{\alpha}) = \underline{f}_0^{\alpha}(\underline{u}^{\alpha}, \bar{u}^{\alpha}) + i\bar{f}_0^{\alpha}(\underline{u}^{\alpha}, \bar{u}^{\alpha})$ and $f_j(\underline{x}^{\alpha}, \bar{x}^{\alpha}, \underline{u}^{\alpha}, \bar{u}^{\alpha})$ is the jth row of $\bigcup_{\mu=0}^{1} B(\mu, \alpha)(\underline{x}^{\alpha}(t) + i\bar{x}^{\alpha}(t)) + \bigcup_{\lambda=0}^{1} D(\lambda, \alpha)(\underline{u}^{\alpha}(t) + i\bar{u}^{\alpha}(t))$ for $j = 1, 2, \ldots, n$, and $\mathcal{J} = \int_{t_0}^{t_f} g(\underline{x}^{\alpha}, \bar{x}^{\alpha}, \underline{u}^{\alpha}, \bar{u}^{\alpha})dt$. There is no loss of generality in taking $n = 2$ and $m = 1$. Let $\underline{u}^{*\alpha}(t)$ and $\bar{u}^{*\alpha}(t)$ be the optimal controls and $\underline{x}_1^{*\alpha}(t), \bar{x}_1^{*\alpha}(t), \underline{x}_2^{*\alpha}(t)$ and $\bar{x}_2^{*\alpha}(t)$ are the corresponding optimal states. Consider a small variation of $\underline{u}^{*\alpha}(t)$, as $\underline{u}^{\alpha}(t) = \underline{u}^{*\alpha}(t) + \delta\underline{u}^{\alpha}(t)$ and for $\bar{u}^{*\alpha}(t)$, as $\bar{u}^{\alpha}(t) = \bar{u}^{*\alpha}(t) + \delta\bar{u}^{\alpha}(t)$ with corresponding state $(\underline{x}_1^{*\alpha}(t) + \delta\underline{x}_1^{\alpha}(t), \bar{x}_1^{*\alpha}(t) + \delta\bar{x}_1^{\alpha}(t), \underline{x}_2^{*\alpha}(t) + \delta\underline{x}_2^{\alpha}(t), \bar{x}_2^{*\alpha}(t) + \delta\bar{x}_2^{\alpha}(t))$. Those states will not arrive at $\underline{x}_{1f}^{\alpha}, \bar{x}_{1f}^{\alpha}, \underline{x}_{2f}^{\alpha}$ and $\bar{x}_{2f}^{\alpha}$ at t_f but at a slightly different time $t_f + \delta t$. The final conditions give

$$\underline{x}_j^{\alpha}(t_f + \delta t) + i\bar{x}_j^{\alpha}(t_f + \delta t) = \underline{x}_{jf}^{\alpha} + i\bar{x}_{jf}^{\alpha}, \quad j = 1, 2.$$

As usual in variational arguments, in the first instance we are only interested in first-order effects, and from the end conditions we deduce that

$$(\delta\underline{x}_j^{\alpha}(t_f) + i\delta\bar{x}_j^{\alpha}(t_f)) + (\dot{\underline{x}}_j^{\alpha}(t_f)\delta t + i\dot{\bar{x}}_j^{\alpha}(t_f)\delta t) = 0, \quad j = 1, 2.$$

If we now use the state equations we obtain $\delta\underline{x}_j^{\alpha}(t_f) + i\delta\bar{x}_j^{\alpha}(t_f) = -f_j(t_f)\delta t$, where $f_j(t_f) = f_j(\underline{x}_1^{\alpha}(t_f), \bar{x}_1^{\alpha}(t_f), \underline{x}_2^{\alpha}(t_f), \bar{x}_2^{\alpha}(t_f), \underline{u}^{\alpha}(t_f), \bar{u}^{\alpha}(t_f))$. The consequent change $\Delta\mathcal{J}$ in $\mathcal{J}$ is

$$\begin{aligned}\Delta\mathcal{J} &= \int_{t_0}^{t_f+\delta t} g(\underline{u}^{*\alpha} + \delta\underline{u}^{\alpha}, \bar{u}^{*\alpha} + \delta\bar{u}^{\alpha})dt - \int_{t_0}^{t_f} g(\underline{u}^{*\alpha}, \bar{u}^{*\alpha})dt \\ &= \int_{t_0}^{t_f} \left\{\frac{\partial g}{\partial \underline{u}^{\alpha}}\delta\underline{u}^{\alpha} + \frac{\partial g}{\partial \bar{u}^{\alpha}}\delta\bar{u}^{\alpha}\right\} dt + g(t_f)\delta t + O((\delta\underline{u}^{*\alpha})^2) + O((\delta\bar{u}^{*\alpha})^2),\end{aligned}$$

where $f_0(t_f)$ is the value of f_0 at $t = t_f$ and the derivatives in the integrand are evaluated on the optimal trajectory. Let $\delta\mathcal{J}$ denote the first variation. If $\underline{u}^{*\alpha}$ and $\bar{u}^{*\alpha}$ are optimal, it is necessary that the first variation $\delta\mathcal{J}$ is zero, so

$$\int_{t_0}^{t_f} \left\{\frac{\partial g}{\partial \underline{u}^{\alpha}}\delta\underline{u}^{\alpha} + \frac{\partial g}{\partial \bar{u}^{\alpha}}\delta\bar{u}^{\alpha}\right\} dt + g(t_f)\delta t = 0,$$

on optimal states for all variations.

We simply need to introduce two Lagrange multipliers $\psi_1(t)$ and $\psi_2(t)$. Now consider the pair of integrals

$$\varphi_j = \int_{t_0}^{t_f} \psi_j(t)(\dot{\underline{x}}_j^{\alpha} + i\dot{\bar{x}}_j^{\alpha} - f_j(\underline{x}_1^{\alpha}, \bar{x}_1^{\alpha}, \underline{x}_2^{\alpha}, \bar{x}_2^{\alpha}, \underline{u}^{\alpha}, \bar{u}^{\alpha}))dt, \quad j = 1, 2.$$

A straightforward calculation (see Theorem 4.1 of [20]) gives:

$$\delta\varphi_j = \int_{t_0}^{t_f} \psi_j(t)\left\{-\frac{\partial f_j}{\partial \underline{x}_1^{\alpha}}\delta\underline{x}_1^{\alpha} - \frac{\partial f_j}{\partial \bar{x}_1^{\alpha}}\delta\bar{x}_1^{\alpha} - \frac{\partial f_j}{\partial \underline{x}_2^{\alpha}}\delta\underline{x}_2^{\alpha} - \frac{\partial f_j}{\partial \bar{x}_2^{\alpha}}\delta\bar{x}_2^{\alpha} - \frac{\partial f_j}{\partial \underline{u}^{\alpha}}\delta\underline{u}^{\alpha} - \frac{\partial f_j}{\partial \bar{u}^{\alpha}}\delta\bar{u}^{\alpha} + \frac{d}{dt}(\delta\underline{x}_j^{\alpha} + i\delta\bar{x}_j^{\alpha})\right\} dt.$$

Since $\varphi_j = 0$ for all $\underline{u}^{\alpha}$ and $\bar{u}^{\alpha}$, $\delta\varphi_j = 0$; therefore:

$$\int_{t_0}^{t_f} \psi_j(t)\left\{\frac{d}{dt}(\delta\underline{x}_j^{\alpha} + i\delta\bar{x}_j^{\alpha})\right\} dt = -f_j(t_f)\psi_j(t_f)\,\delta t - \int_{t_0}^{t_f} \left\{\dot{\psi}_j(\delta\underline{x}_j^{\alpha} + i\delta\bar{x}_j^{\alpha})\right\} dt.$$

The condition $\delta\mathcal{J} = 0$ can now be replaced by $\delta\mathcal{J} + \delta\varphi_1 + \delta\varphi_2 = 0$. If we introduce the Hamiltonian function as:

$$H = -g(\underline{u}^{\alpha}, \bar{u}^{\alpha}) + \psi_1 f_1(\underline{x}_1^{\alpha}, \bar{x}_1^{\alpha}, \underline{x}_2^{\alpha}, \bar{x}_2^{\alpha}, \underline{u}^{\alpha}, \bar{u}^{\alpha}) + \psi_2 f_2(\underline{x}_1^{\alpha}, \bar{x}_1^{\alpha}, \underline{x}_2^{\alpha}, \bar{x}_2^{\alpha}, \underline{u}^{\alpha}, \bar{u}^{\alpha}),$$

then the contention of the theorem can be found easily. □

By applying necessary PMP conditions in Theorem 4, and from (7), we can solve the fuzzy optimal control system (13). In the next section, we apply the mentioned strategy to solve two numerical examples.

4. Simulation and results

Example 1. Consider the fuzzy controlled system:

$$\begin{cases}\dot{\tilde{x}}_1(t) = \tilde{2} \odot \tilde{x}_1(t) \oplus \tilde{1} \odot \tilde{u}_1(t),\\ \dot{\tilde{x}}_2(t) = -\tilde{2} \odot \tilde{x}_2(t) \ominus \tilde{1} \odot \tilde{u}_2(t),\\ \tilde{x}_1(0) = \tilde{x}_2(0) = 1,\\ \tilde{x}_1(1) = \tilde{x}_2(1) = 0.\end{cases}$$

Let $\tilde{2} = (1, 2, 3)$ and $-\tilde{2} = (-3, -2, -1)$, $\tilde{1} = (0, 1, 2)$, $-\tilde{1} = (-2, -1, 0)$ now define:

$$a_1 = (1-\mu)(2\alpha + (1-\alpha)) + \mu(2\alpha + 3(1-\alpha)),$$

$$a_2 = (1-\mu)(-2\alpha - 3(1-\alpha)) + \mu(-2\alpha - (1-\alpha)),$$

$$b_1 = (1-\lambda)\alpha + \lambda(\alpha + 2(1-\alpha)),$$

$$b_2 = (1-\lambda)(-\alpha - 2(1-\alpha)) + \lambda(-\alpha),$$

where $\mu, \lambda \in [0, 1]$.

The vector functions $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$ are (1)-differentiable then:

$$\begin{bmatrix}\dot{\underline{x}}_1^\alpha\\ \dot{\bar{x}}_1^\alpha\\ \dot{\underline{x}}_2^\alpha\\ \dot{\bar{x}}_2^\alpha\end{bmatrix} = \begin{bmatrix}a_1 & 0 & 0 & 0\\ 0 & a_1 & 0 & 0\\ 0 & 0 & 0 & a_2\\ 0 & 0 & a_2 & 0\end{bmatrix}\begin{bmatrix}\underline{x}_1^\alpha\\ \bar{x}_1^\alpha\\ \underline{x}_2^\alpha\\ \bar{x}_2^\alpha\end{bmatrix} + \begin{bmatrix}b_1 & 0 & 0 & 0\\ 0 & b_1 & 0 & 0\\ 0 & 0 & 0 & b_2\\ 0 & 0 & b_2 & 0\end{bmatrix}\begin{bmatrix}\underline{u}_1^\alpha\\ \bar{u}_1^\alpha\\ \underline{u}_2^\alpha\\ \bar{u}_2^\alpha\end{bmatrix}.$$

The solution of the above system is:

$$\begin{bmatrix}\underline{x}_1^\alpha\\ \bar{x}_1^\alpha\\ \underline{x}_2^\alpha\\ \bar{x}_2^\alpha\end{bmatrix} = \begin{bmatrix}e^{a_1 t} & 0 & 0 & 0\\ 0 & e^{a_1 t} & 0 & 0\\ 0 & 0 & \dfrac{e^{a_2 t} + e^{-a_2 t}}{2} & \dfrac{e^{a_2 t} - e^{-a_2 t}}{2}\\ 0 & 0 & \dfrac{e^{a_2 t} - e^{-a_2 t}}{2} & \dfrac{e^{a_2 t} + e^{-a_2 t}}{2}\end{bmatrix}\begin{bmatrix}1\\1\\2\\2\end{bmatrix}$$

$$+ \int_0^t \left\{\begin{bmatrix}e^{a_1(t-s)} & 0 & 0 & 0\\ 0 & e^{a_1(t-s)} & 0 & 0\\ 0 & 0 & \dfrac{e^{a_2(t-s)} + e^{-a_2(t-s)}}{2} & \dfrac{e^{a_2(t-s)} - e^{-a_2(t-s)}}{2}\\ 0 & 0 & \dfrac{e^{a_2(t-s)} - e^{-a_2(t-s)}}{2} & \dfrac{e^{a_2(t-s)} + e^{-a_2(t-s)}}{2}\end{bmatrix}\right.$$

$$\left.\times \begin{bmatrix}b_1 & 0 & 0 & 0\\ 0 & b_1 & 0 & 0\\ 0 & 0 & 0 & b_2\\ 0 & 0 & b_2 & 0\end{bmatrix}\begin{bmatrix}\underline{u}_1^\alpha(s)\\ \bar{u}_1^\alpha(s)\\ \underline{u}_2^\alpha(s)\\ \bar{u}_2^\alpha(s)\end{bmatrix}\right\} ds.$$

We need to mention that the control functions $\underline{u}_1^\alpha(s)$, $\bar{u}_1^\alpha(s)$, $\underline{u}_2^\alpha(s)$ and $\bar{u}_2^\alpha(s)$ are assumed to be respectively a, b, c and d, (in fact, they are approximated by constant functions) where we find them such that to control the system from initial conditions $\tilde{x}_1(0) = \tilde{x}_2(0) = 1$ to final conditions $\tilde{x}_1(1) = \tilde{x}_2(1) = 0$.

Fig. 1 shows the graph of state and control functions, the red lines are centers of fuzzy functions and blue lines are the upper bounds and lower bounds of fuzzy functions for $\alpha = 0$ and $\mu, \lambda \in [0, 1]$.

Example 2. Consider the following fuzzy optimal controlled system:

$$\text{Min} \int_0^1 \{\tilde{u}_1^2(t) + \tilde{u}_2^2(t)\} dt,$$

s.t.

$$\dot{\tilde{x}}_1(t) = \tilde{2} \odot \tilde{x}_1(t) \oplus \tilde{1} \odot \tilde{u}_1(t),$$

$$\begin{aligned}
&\dot{\tilde{x}}_2(t) = -\tilde{2} \odot \tilde{x}_2(t) \ominus \tilde{1} \odot \tilde{u}_2(t),\\
&\tilde{x}_1(0) = 1,\\
&\tilde{x}_2(0) = 2,\\
&\tilde{x}_1(1) = \tilde{x}_2(1) = 0.
\end{aligned}$$

Let $\tilde{2} = (1, 2, 3)$ and $-\tilde{2} = (-3, -2, -1)$, $\tilde{1} = (0, 1, 2)$, $-\tilde{1} = (-2, -1, 0)$ now define:

$$\begin{aligned}
a_1 &= (1-\mu)(2\alpha + (1-\alpha)) + \mu(2\alpha + 3(1-\alpha)),\\
a_2 &= (1-\mu)(-2\alpha - 3(1-\alpha)) + \mu(-2\alpha - (1-\alpha)),\\
b_1 &= (1-\lambda)\alpha + \lambda(\alpha + 2(1-\alpha)),\\
b_2 &= (1-\lambda)(-\alpha - 2(1-\alpha)) + \lambda(-\alpha),
\end{aligned}$$

where $\mu, \lambda \in [0, 1]$.

Using (1)-differentiability the length of support of $u_2(t)$ is increasing, then we have to use (2)-differentiable and it leads to:

$$H = -(\underline{u}_1^{\alpha 2} + \bar{u}_1^{\alpha 2} + \underline{u}_2^{\alpha 2} + \bar{u}_2^{\alpha 2}) + \psi_1(a_1\bar{x}_1^\alpha + b_1\bar{u}_1^\alpha) + \psi_2(a_1\underline{x}_1^\alpha + b_1\underline{u}_1^\alpha) + \psi_3(a_2\underline{x}_2^\alpha + b_2\underline{u}_2^\alpha) + \psi_4(a_2\bar{x}_2^\alpha + b_2\bar{u}_2^\alpha).$$

By considering Theorem 4

$$\underline{u}_1^\alpha = \frac{b_1}{2}\psi_2, \qquad \bar{u}_1^\alpha = \frac{b_1}{2}\psi_1, \qquad \underline{u}_2^\alpha = \frac{b_2}{2}\psi_3, \qquad \bar{u}_2^\alpha = \frac{b_2}{2}\psi_4,$$

and

$$\begin{bmatrix}\dot{\psi}_1\\ \dot{\psi}_2\\ \dot{\psi}_3\\ \dot{\psi}_4\end{bmatrix} = \begin{bmatrix}0 & -a_1 & 0 & 0\\ -a_1 & 0 & 0 & 0\\ 0 & 0 & -a_2 & 0\\ 0 & 0 & 0 & -a_2\end{bmatrix}\begin{bmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{bmatrix}.$$

Therefore we will have:

$$\begin{aligned}
&\underline{u}_1^\alpha(t) = \frac{b_1}{4}[(e^{-a_1t} - e^{a_1t})c_1 + (e^{-a_1t} + e^{a_1t})c_2], \qquad \underline{u}_2^\alpha(t) = \frac{b_2}{2}e^{-a_2t}c_3,\\
&\bar{u}_1^\alpha(t) = \frac{b_1}{4}[(e^{-a_1t} + e^{a_1t})c_1 + (e^{-a_1t} - e^{a_1t})c_2], \qquad \bar{u}_2^\alpha(t) = \frac{b_2}{2}e^{-a_2t}c_4,
\end{aligned} \tag{17}$$

where c_i for $i = 1, \dots, 4$ are real numbers. The solution of the optimal controlled system is:

$$\begin{aligned}
\begin{bmatrix}\underline{x}_1^\alpha\\ \bar{x}_1^\alpha\\ \underline{x}_2^\alpha\\ \bar{x}_2^\alpha\end{bmatrix} &= \begin{bmatrix}\frac{e^{a_1t}+e^{-a_1t}}{2} & \frac{e^{a_1t}-e^{-a_1t}}{2} & 0 & 0\\ \frac{e^{a_1t}-e^{-a_1t}}{2} & \frac{e^{a_1t}+e^{-a_1t}}{2} & 0 & 0\\ 0 & 0 & e^{a_2t} & 0\\ 0 & 0 & 0 & e^{a_2t}\end{bmatrix}\begin{bmatrix}1\\1\\2\\2\end{bmatrix}\\
&+ \int_0^t \left\{\begin{bmatrix}\frac{e^{a_1(t-s)}+e^{-a_1(t-s)}}{2} & \frac{e^{a_1(t-s)}-e^{-a_1(t-s)}}{2} & 0 & 0\\ \frac{e^{a_1(t-s)}-e^{-a_1(t-s)}}{2} & \frac{e^{a_1(t-s)}+e^{-a_1(t-s)}}{2} & 0 & 0\\ 0 & 0 & e^{a_2(t-s)} & 0\\ 0 & 0 & 0 & e^{a_2(t-s)}\end{bmatrix}\right.\\
&\left.\times \begin{bmatrix}0 & b_1 & 0 & 0\\ b_1 & 0 & 0 & 0\\ 0 & 0 & b_2 & 0\\ 0 & 0 & 0 & b_2\end{bmatrix}\begin{bmatrix}\underline{u}_1^\alpha(s)\\ \bar{u}_1^\alpha(s)\\ \underline{u}_2^\alpha(s)\\ \bar{u}_2^\alpha(s)\end{bmatrix}\right\} ds.
\end{aligned}$$

Now by substituting (17) in the above equations and using the trapezoidal integration rule, the trajectories can be found straightforward. Fig. 2, shows the state and control functions for $\alpha = 0$ and $\mu, \lambda \in [0, 1]$, where the red lines are centers and the blue lines are the upper and lower bounds for the fuzzy state and control functions.

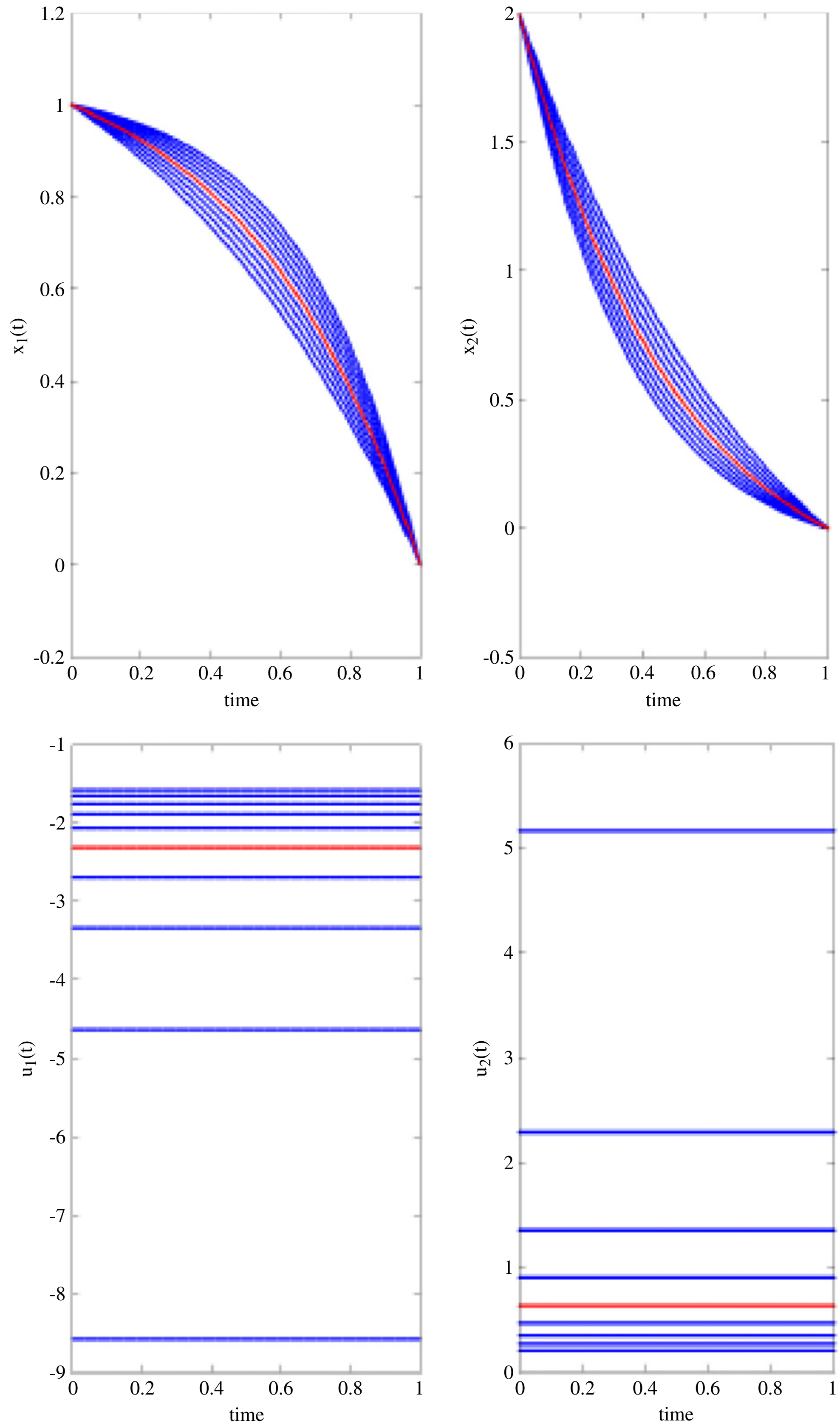

Fig. 1. The state and control functions for Example 1.

5. Conclusion

Using a new representation of the α-cuts of the fuzzy linear controlled system, we studied fuzzy linear time invariant control and optimal control systems with fuzzy coefficients. In this sequel, generalized differentiability has been considered.

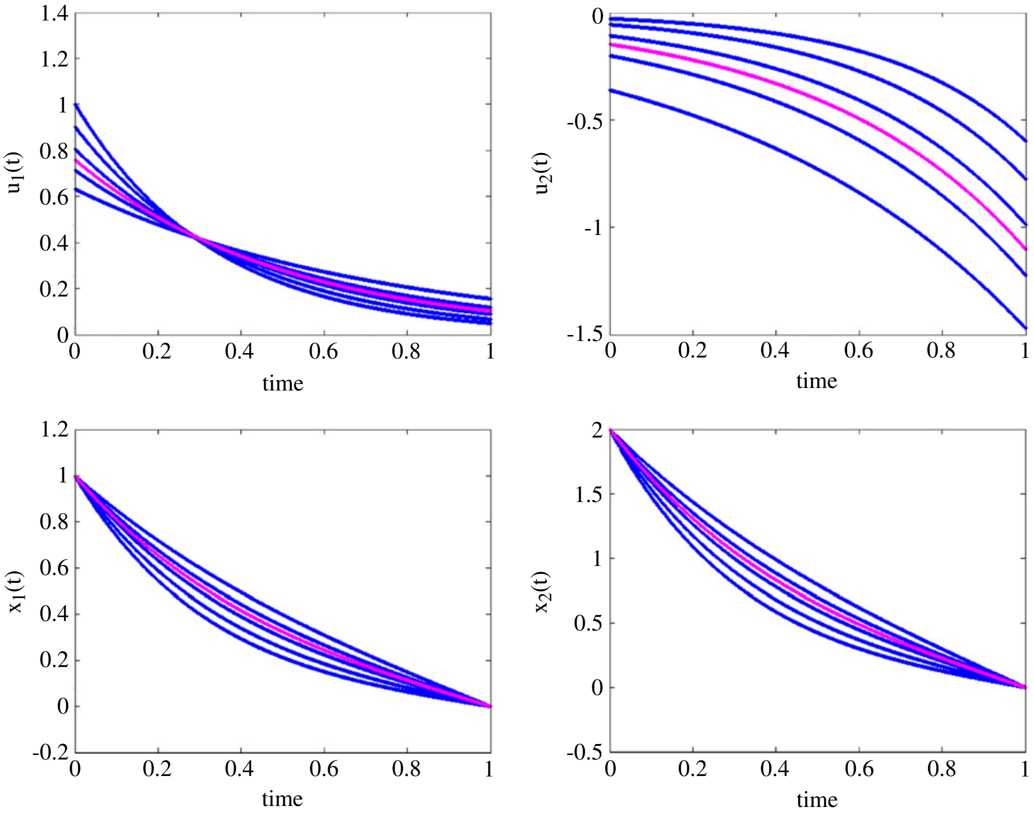

Fig. 2. The control and state functions for Example 2.

Since this kind of derivative is usually matching with control and optimal control problems, while using the Hukuhara derivative may result nonfuzzy solutions. Two numerical examples are simulated to show how easy this method can be used effectively. For further research, one can extend this approach for optimal fuzzy controlled systems with fuzzy initial conditions. Also, one can consider the optimal fuzzy time varying controlled system with fuzzy coefficients. We may suggest using this approach for solving fuzzy integrodifferential equations.

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