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THE FRATTINI AND RESIDUALLY NILPOTENT PAIR OF GROUPS

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Abstract. A pair of groups (G, N) is said to be nilpotent if it has a central series. A detailed study of nilpotent pair of groups has been carried out recently. Related to, some further studies is done in this paper. Topics include the generalized Baer's theorem, Frattini of a pair of groups, residually nilpotent pair and Hopfian pair of groups.

1. Introduction

Pair of groups, or a group with a normal subgroup, newly find an important role in the literature of group theory. Ellis introduced the Schur multiplier [2] and also capability [1] for a pair of groups and attained some of their properties. Salemkar et al [5] obtained more properties of the Schur multiplier of a pair such as finding a covering

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pair for a pair of groups under some conditions. Also Hassanzadeh et al [3] verify a useful notion of nilpotency for pairs of groups which has some interesting results in the usual theory of nilpotency.
In this work, we define a new concept for a pair of groups called the Frattini of a pair. Using this concept, a necessary condition for the nilpotency of a pair is obtained. Moreover, we prove a generalization of Baer's theorem for nilpotent pair of groups. In the sequel, we define a residually nilpotent pair of groups and find a criterion for residually nilpotency of a pair.

Definition 1.1. Let (G, N) be a pair of groups. A normal series $1 = N_0 \leq N_1 \leq \ldots \leq N_t = N$ is called a central series of the pair (G, N) if each N_i is a normal subgroup of G and N_{i+1}/N_i is contained in the center of $(G/N_i, N/N_i)$, for all i. A pair of groups (G, N) is called nilpotent if it has a central series. The length of a shortest central series of the pair (G, N) is called nilpotent class of (G, N).

The lower central series for a pair of groups is also defined inductively as follows.

Definition 1.2. Put $\gamma_1(G, N) = (G, N)^{(0)} = N$, and let $\gamma_i(G, N)$ be defined inductively for $i \geq 1$. Then $\gamma_{i+1}(G, N)$ is defined as the subgroup $[\gamma_i(G, N), G]$. The obtained series

$$N = \gamma_1(G, N) \geq \gamma_2(G, N) \geq \ldots$$

is called the lower central series of (G, N).

Also for a pair (G, N), there is an ascending series which is dual to the lower central series in the sense that the center is dual to the first term of the derived series. This is the upper central series

$$1 = Z_0(G, N) \leq Z_1(G, N) \leq Z_2(G, N) \leq \ldots$$

which is defined by

$$\frac{Z_{n+1}(G,N)}{Z_n(G,N)} = Z\left(\frac{G}{Z_n(G,N)}, \frac{N}{Z_n(G,N)}\right).$$

Robinson showed how the first lower central factor $G_{ab} = G/G'$ exerts a very strong influence on subsequent lower central factors of a group G (for details see [4, Theorem 5.2.5]). Now, we state a main theorem of [3] which is a generalization of Robinson's theorem and it is applied for the proof of some theorems.

Theorem 1.3. ***(Generalized Robinson Theorem)*** *Let (G, N) be a pair of groups and let $F_i = \gamma_i(G, N)/\gamma_{i+1}(G, N)$. Then the map*

$$F_i \otimes \frac{G}{[N,G]} \to F_{i+1}$$

$$n\gamma_{i+1}(G, N) \otimes g[N, G] \mapsto [n, g]\gamma_{i+2}(G, N)$$

is a well-defined epimorphism.

2. Main results

Definition 2.1. Let (G, N) be a pair of groups. The Frattini of (G, N) which is denoted by $\phi(G, N)$ is defined as the intersection of $\phi(G)$ and N.

Theorem 2.2. *Let (G, N) be a nilpotent pair of groups. Then $[G, N] \leq \phi(G, N)$.*

Theorem 2.3. *Suppose that (G, N) is a pair of groups such that $\phi(G) \leq N$ and $(G/\phi(G), N/\phi(G))$ is a nilpotent pair of groups. Then (G, N) is nilpotent.*

It is a well known fact that finitely generated abelian groups satisfy the maximal condition. This result was generalized to nilpotent groups by Baer .In this section we prove a generalization of Baer Theorem to a pair of groups. A generalization of Mal'cev Theorem for a pair of groups shall also be stated.

Theorem 2.4. ***(Generalized Baer Theorem)*** *If (G, N) is a nilpotent pair of groups such that $(G/[N, G], N/[N, G])$ is a pair of finitely generated groups, then N satisfies the maximal condition.*

Theorem 2.5. *A nilpotent pair of finitely generated groups (G, N) has a central series whose factors are infinite cyclic or cyclic of a prime order.*

Theorem 2.6. ***(Generalized Mal'cev Theorem)*** *Let (G, N) be a pair of groups such that $Z(G, N)$ be torsion free. Then each upper central factor is torsion free.*

Corollary 2.7. *A nilpotent pair of finitely generated groups (G, N) with torsion free center has a central series whose factors are infinite cyclic.*

Theorem 2.8. *Let (G, N) be a nilpotent pair of groups.*
(1) *If $exp(Z(G, N)) = e$, then $exp(N)|e^c$, where c is the nilpotency class of (G, N).*
(2) *If (G, N) is a pair of infinite finitely generated groups, then its center has an element of infinite order.*

Definition 2.9. A pair of groups (G, N) is *residually nilpotent* if for every $1 \neq x \in N$, there exists a normal subgroup M_x of G contained in N such that $x \notin M_x$ and $(G/M_x, N/M_x)$ is nilpotent.

Note that the residually nilpotency property of G carries over to (G, N) and that of (G, N) forces N to be residually nilpotent.

Theorem 2.10. *A pair of groups (G, N) is residually nilpotent if and only if* $\cap_{n=1}^{\infty}\gamma_n(G, N) = 1$.

Corollary 2.11. *A pair of groups (G, G') is residually nilpotent if and only if G is.*

Definition 2.12. A pair of groups (G, N) is Hopfian if (G, N) is not isomorphic to $(G/K, N/K)$, for every $K \neq 1$.

It is obvious that for $N = G$, Definition 2.12 reduces to the usual definition of Hopfian for groups.

Theorem 2.13. *A residually nilpotent pair of finitely generated groups (G, N) is Hopfian.*

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