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THE CLASSIFICATION OF GROUPS VIA CAPABILITY; A REALITY TO DREAM

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Abstract. This talk is a survey article on the classification of groups. The classification of prime power groups of order at most p^6 was done using the notion of isoclinism and invoking a fundamental instrument, namely the capability of groups. Since the basic concepts of this classification i.e., isoclinism and capability were generalized to any variety of groups, therefore this talk intends to propound a basic question whether it is possible to define some suitable varieties that could play the key role for classifying some other families of groups.

1. Introduction

One of the most important problem in group theory is the classification of groups. The problem which has been always studied along with the age of group theory. There have been also various approaches to face the problem. Among several different approaches, one of the most classical notions is the concept of isomorphism between groups. However, this notion is too strong in many cases. For this reason, P. Hall in 1940 [5] introduced the notion of isoclinism between two groups (which is weaker than isomorphism) and could classify some groups of prime power order. Using his method, his student Easterfield [6] and then M. Hall and J. Senior in 1964 [6] and later R. James in 1988 [8] completed the classification of groups of order at most p^n, where p is a prime and n is at most 6.

We also know that the notion of isoclinism was generalized by P. Hall [5] to isologism. Isologism is in fact isoclinism with respect to a

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certain variety of groups. If one takes the variety of all trivial groups, one gets the notion of isomorphism back. The variety of all abelian groups yields isoclinism.

On the other hand, we know that the main strong tool in the P. Hall's classification is the notion of capability and this notion also was simultaneously generalized to varietal capability by J. Burns and G. Ellis [3] and a joint paper of the author [9] in 1997.

Now, using all these facts in hand, we just intend to propound a fundamental question; Is it possible to define some suitable varieties so that invoking them the classification of some other families of groups happens? Indeed, in the face of these facts, it seems that the classification of some other suitable families of groups will not be so far! By this sentence, we mean that it might probably exist some special kind of varieties such that their obtained classes caused by isologim can be considered as the first step of screening in the classification, though they might be so broad. Although we do understand that as usual, the stating such a problem is so easy while finding the answer may not!

2. Reality; The Classification of some Prime Power Groups

There are different approaches that can be considered for the description of finite p-groups. We used the word "description" rather than "classification" because we know that classifying p-groups is notoriously open and difficult. Some of the approaches are, for instance, the order, the coclass and the class of nilpotency. But each of them has some restrictions and difficulties in this way so that P. Hall in 1940 [5] introduced the notion of isoclinism for the classification of all groups, though he could classify only some prime power groups.

Definition 2.1. Two groups G_1 and G_2 are said to be isoclinic provided that there exist two isomorphisms $\alpha : G_1/Z(G_1) \to G_2/Z(G_2)$ and $\beta : \gamma_2(G_1) \to \gamma_2(G_2)$ such that if $\alpha(a_1Z(G_1)) = a_2Z(G_2)$ and $\alpha(b_1Z(G_1)) = b_2Z(G_2)$, then $\beta([a_1, b_1]) = [a_2, b_2]$. This notion is written by $G_1 \sim G_2$.

In the P. Hall's classification, regular group which was defined by him, plays the key role. There are many equivalent ways to define regular p-groups. One is that if a and b are any elements of the group, then

$$(ab)^{p^r} = a^{p^r} b^{p^r} {c_1}^{p^r} \ldots c_t^{p^r},$$

where c_i are elements of the commutator subgroup of $< a, b >$. In fact, a group is regular if the operation of taking pth powers interacts "well" with taking commutators. P. Hall [5] showed that in a regular p-group, one can define "type invariants" which are similar to the invariant factors for finite abelian groups. Though they do not completely determine the groups the way the invariant factors do for abelian groups, they are usually a very good first reduction towards the analysis. Note that if $p \geq n$, then a group of order p^n is necessarily regular (more generally, if the group is of class c and $p > c$, then the group is regular, in particular, since a group of order p^n is of class at most $n-1$, the observation just made follows). In fact, P. Hall mentioned in [4] that if we fix n, then "most" groups of order p^n are regular (since only those with $p < n$ may fail to be regular). This leads, classically, to a separation of p-groups into those of "small class" (when the class is smaller than p), and "the rest". In other words, this means that when classifying groups of order p^n, the analysis usually breaks into two different cases: when the group is regular (which includes all $p \geq n$), and when the group is irregular. The latter case leads to a case-by-case analysis for small primes. This occurs, for example, in the classification of groups of order p^3 (in which odd primes and 2 should be considered separately). Likewise Burnside's work on group of order p^4. Or similar to the work of R. James [8] in the classification of groups of order p^6 and E. A. O'Brien and M. R. Vaughan-Lee [10] for p^7. The latter separates $p \geq 7$ with the groups of order 3^7 and 5^7. Note that the latest work uses Lie rings and algebras as a starting point. There are algorithms that are known to produce and check isomorphism types (see [10]).

The above comments explains that how groups of order p^n for all primes p have been fully classified for $n \leq 7$, first (most of the times) by isoclinism and then up to isomorphism. Note, in particular, that the number of isomorphism classes increases with p when $n \geq 5$. One may know that the Higman PORC conjecture (polynomial on residue classes) is that, for each n, this number is a polynomial function of p and of $p \pmod{k}$ for a finite collection of values of k. Although a family of examples constructed by M. D. Sautoy and M. R. Vaughan-Lee [12] of order p^{10} show while not actually disproving the conjecture, suggests that it is very unlikely indeed that it is true. We should remind also that their construction depends on the geometry of elliptic curves. It will probably illustrate that how describing all groups of order p^{10} would be complicated.

Let us comeback to the isoclinism. The notion of isoclinism defines an equivalence relation on the class of all groups and has this trait that some other properties of groups are invariant under isoclinism. For instance, it is proved in [2] that, restricting ourselves to finite groups, we have the following hierarchy of classes of groups, invariant under isoclinism: abelian $<$ nilpotent $<$ supersolvable $<$ strongly monomial $<$ monomial $<$ solvable. For charactering the families of isoclinism, P. Hall [5] tried to find some properties which are invariant in each family. Accordingly, any quantity depending on a variable group and which is the same for any two groups of the same family is called a family invariant. For instance, it is easy to see that the members of the derived series and the central quotient groups are family invariants. It follows that the groups belonging to the same family have the same derived length and nilpotency class. Note that the commutator quotient group and the center are not family invariants, as is the minimal number of generators.

Now, the importance of central quotient groups for this classification may be seen. Such groups are called capable. More precisely;

Definition 2.2. A group G is called *capable if there exists a group* E *such that* $G \cong \frac{E}{Z(E)}$.

Among many different points on capability, there are two important tools for recognition the capability of a group. One of them is a necessary condition which was established by P. Hall [5]. He considered a generating system J for a group G and defined $\Delta_J = \cap_{x \in J} < x >$ and denoted the join of all subgroups Δ_J, where J varies over all generating systems of G, by $\Delta(G)$ and gave a necessary condition for the capability of G in such a way that a capable group G must satisfy $\Delta(G) = 1$.

The other tool for characterizing the capability is a criterion that was introduced by F. R. Beyl, U. Felgner and P. Schmid [1]. They showed that every group G possesses a uniquely determined central subgroup $Z^*(G)$ which is minimal subject to being the image in G of the center of some central extension of G. This $Z^*(G)$ is characteristic in G and is the image of the center of every stem cover of G.

Definition 2.3. The intersection of all subgroups of the form $\psi(Z(E))$, where $\psi : E \to G$ is a surjective homomorphism with $ker\psi \subseteq Z(E)$, is called the *precise center* subgroup of G and denoted by $Z^*(G)$.

F. R. Beyl et al. [1] proved that $Z^*(G)$ is the smallest central subgroup of G whose factor group is capable. Now using this fact the criterion for capability can be given as follows.

Theorem 2.4. *(F. R. Beyl, U. Felgner and P. Schmid 1979 [1]) G is capable if and only if* $Z^*(G) = 1$.

3. Dream; The Classification of Other Classes of Groups!

The works and attempts which were explained as the "Reality" in Section 1 for the classification of some prime power groups evidence that this way of classification might not be quite simple. In other words, determining finite p-groups of order p^n up to isomorphism, will be so complicated while n becomes large and larger. This shows the necessity of primary screening not only for p-groups, but also for other families of groups.

On the other hand, we should not forget that whatever, for instance, R. James [8] did was heavily indebted to the work of P. Hall [5] on isoclinism. That is why, we believe that as isoclinism could help us as the first step of classification of prime power groups of order p^n for small n, paying attention to its generalization may be helpful as the first step of classification of some other classes of groups! Therefore in this section, we remind all the notions and the tools in Section 1 which were generalized to any variety of groups. In fact, the primary definitions and the preliminary statements which provide the context of determining of the equivalence classes in isoclinism were generalized in two steps. First, the "isoclinism" and "capability" were transformed to "c-isoclinism" and "$c-$capability and then they were generalized to "isologism" and "varietal capability", respectively. In the following, the general case is provided.

Definition 3.1. Let $\mathfrak{V}$ be a variety of groups defined by the set of laws V. Two groups G and H are $\mathfrak{V}$- *isologic* if there exist isomorphisms

$$\alpha : \frac{G}{V^*(G)} \to \frac{H}{V^*(H)} \text{ and } \beta : V(G) \to V(H),$$

such that $\beta(v(g_1, g_2, \dots, g_n)) = v(h_1, h_2, \dots, h_n)$, where $g_i \in G,\ h_i \in \alpha(g_i V^*(G))$ for each $1 \leq i \leq n$. In this case, we write $G \sim_{\mathfrak{V}} H$. The pair (α, β) is said to be a $\mathfrak{V}$-*isologism* between G and H.

Likewise the isoclinism, for each variety $\mathfrak{V}$, isologism gives an equivalence relation on the class of all groups. The larger variety implies the weaker equivalence relation. If $\mathfrak{V}$ is the variety of all abelian groups, $\mathfrak{V}$-isologism coincides with isoclinism. The groups in a variety $\mathfrak{V}$ fall into one single equivalence class, they are actually $\mathfrak{V}$-isologic to the trivial group. (For more information about $\mathfrak{V}$-isologism see [7]).

On the other hand, it is observed from the definition of isologism that the marginal factor group can play an important role in this system of classification. Such a group is called varietal capable with respect to the variety $\mathfrak{V}$, or briefly $\mathfrak{V}$-capable [9].

Definition 3.2. Let $\mathfrak{V}$ be a variety of groups defined by the set of laws V. A group G is said to be $\mathfrak{V}$-*capable* , if there exists a group E such that $G \cong E/V^*(E)$.

Some properties of varietal capability are given in [9]. Specially, finding a criterion for recognition of varietal capability is illustrated in [9]. More precisely, it is shown that every group G possesses a uniquely determined subgroup $(V^*)^*(G)$ of the marginal subgroup $V^*(G)$, which is minimal subject to being the image in G of the marginal subgroup of some $\mathfrak{V}$-marginal extension of G. In fact, if $\psi : E \rightarrow G$ is a surjective homomorphism with $ker\psi \subseteq V^*(E)$, then $(V^*)^*(G)$ is defined to be the intersection of all subgroups of the form $\psi(V^*(E))$.

If $\mathfrak{V}$ is the variety of abelian groups then the subgroup $(V^*)^*(G)$ is $Z^*(G)$ and the $\mathfrak{V}$-capability coincides with the usual capability. If one takes $\mathfrak{V}$ to be the variety of nilpotent groups of class at most c, $c \geq 1$, then one gets $(V^*)^*(G)$ to be $Z_c^*(G)$ as J. Burns and G. Ellis introduced in [3] and the $\mathfrak{V}$-capability will be their c-capability.

$(V^*)^*(G)$ is characteristic and it is also proved in [9] that $(V^*)^*(G)$ is the smallest subgroup contained in the marginal subgroup of G for which the factor group $G/(V^*)^*(G)$ is $\mathfrak{V}$-capable. In other words;

Theorem 3.3. *(M. R. R. Moghaddam and S. Kayvanfar [9]) G is $\mathfrak{V}$-capable if and only if $(V^*)^*(G) = 1$.*

The above comments explain that for a classification of a family of groups, the notion of isologism might be helpful as a primary screening. The most important tool that can be considered for characterizing the families of isologism is the $\mathfrak{V}$-capable group. On the other hand, there are a few statements for recognition the varietal capability (for instances, see [3] and [9]) invoking them one can find out the $\mathfrak{V}$-capability of some groups. There is also another helpful tool for recognizing the $\mathfrak{V}$-capability; the Baer invariant of groups, which has been calculated for many varieties. Invoking the Baer invariant, the varietal capability of some types of groups has been characterized for some special kind of varieties (for example see [11]).

All these facts motivate us to think more to the dream of classification of groups by varietal isologism via the varietal capability.

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